EE2S11 Signals and Systems

Chapter 9: Discrete-time signals and systems - LTI systems

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20 December 2023

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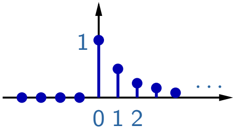

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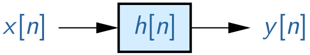


Discrete-time signal

A discrete-time signal is a series of real or complex numbers:

$$n \in \mathbb{Z} \qquad \Rightarrow \qquad x[n] \in \mathbb{R} \ \text{ or } \mathbb{C}$$

The sample period is not mentioned (but sometimes present implicitly).

Notation

- as series: $x = [\cdots, 0, 0, \boxed{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots]$, the square indicates $x[0]$
- as explicit expression:

$$x[n] = \begin{cases} 0\,, & n < 0 \\ 2^{-n}\,, & n \geq 0 \end{cases}$$

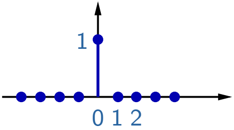
- as implicit expression (recursion):

$$x[n] = \begin{cases} 0\,, & n < 0 \\ 1\,, & n = 0 \\ \frac{1}{2}x[n-1]\,, & n > 0 \end{cases}$$

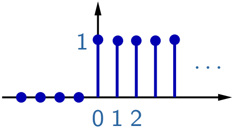
Examples of signals

- Unit pulse: $\delta[n] = \begin{cases} 1\,, & n = 0 \\ 0\,, & \text{elsewhere} \end{cases}$

  Note that this is not a degenerated function.



- Unit step: $u[n] = \begin{cases} 1\,, & n \geq 0 \\ 0\,, & n < 0 \end{cases}$



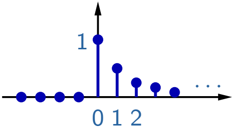
We can also write:

$$\delta[n] = u[n] - u[n-1]\,, \qquad \text{(discrete differential)}$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m]\,, \qquad \text{(discrete integral)}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Examples of signals



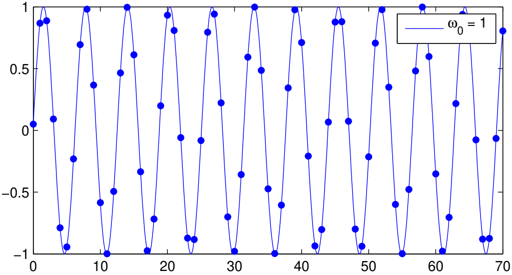
- Exponential series: $x[n] = A\,\alpha^n\,u[n]$
- Complex exponential series: $x[n] = A\,e^{j\omega n}$.

- $x[n]$ is periodic with period N if $x[n] = x[n+N]\;\forall n$.
  This is only possible if $\omega = \frac{2\pi}{N}k$, for $k \in \mathbb{Z}$ (else "quasi-periodic").
  And if N can be divided by k, the actual period is smaller than N.
- If $\omega_2 = \omega_1 + 2\pi$, then $x_2[n] = e^{j\omega_2 n}$ is equal to $x_1[n] = e^{j\omega_1 n}$.
  Therefore, it is sufficient to take $\omega \in \langle -\pi, \pi]$.
  The frequency response of a digital system is periodic!

- Is the sum of two periodic signals also periodic? Which period?

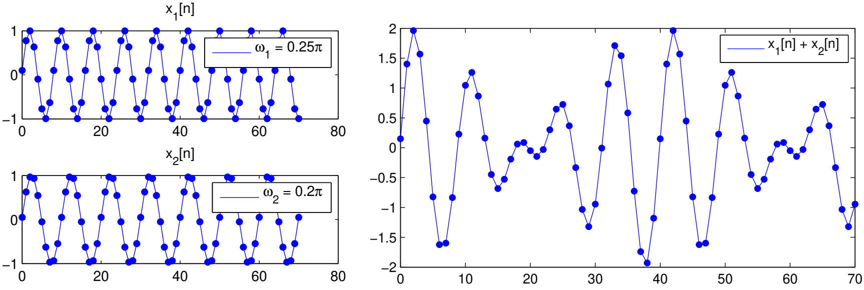
Example: quasi-periodic signal

$$x[n] = \cos(\omega_0 n + \theta_0) \qquad \text{with} \quad \omega_0 = 1$$



If $\omega_0 \neq \frac{2\pi}{N} k$ for integers N and k, then the signal is not periodic. But because every real number can be approximated by a ratio $\frac{k}{N}$, such a signal will be approximately periodic.

Sum of two periodic signals



- $x_1[n] = \sin(\omega_1 n + \theta_1)$ with $\omega_1 = \frac{\pi}{4}$: period is $T_1 = 2\pi/\omega_1 = 8$
- $x_2[n] = \sin(\omega_2 n + \theta_2)$ with $\omega_2 = \frac{\pi}{5}$: period is $T_2 = 2\pi/\omega_2 = 10$
- $x_1[n] + x_2[n]$ has period 40 samples: least common multiple of T_1 and T_2.

Energy and signal space

- The energy in a discrete-time signal $x[n]$ is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- The set of discrete-time signals for which $E < \infty$ is called ℓ_2:

$$\ell_2 = \{x : \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty\}$$

  This is a "Hilbert space", with pleasant properties

- Similar:

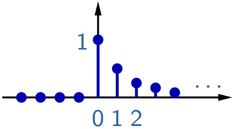
$$\ell_1 = \{x : \sum_{n=-\infty}^{\infty} |x[n]| < \infty\} \qquad \text{absolutely summable}$$

$$\ell_\infty = \{x : \max_n |x[n]| < \infty\} \qquad \text{absolutely bounded}$$

Example

$x[n] = (\frac{1}{2})^n u[n]$

$$E = \sum_{n=0}^{\infty} (\frac{1}{2})^{2n} = 1 + \frac{1}{4} + (\frac{1}{4})^2 + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$



Power

Not all signals have finite energy (e.g. $x[n] = 1 \ \forall n$).

The power of a signal $x[n]$ is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Example

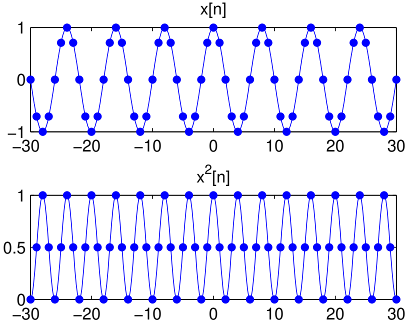
- Determine the power of $x[n] = \cos(\omega_0 n)$ with $\omega_0 \neq 0 \bmod \pi$.

$$\begin{aligned} P &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2(\omega_0 n) \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2}[1+\cos(2\omega_0 n)] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{2} = \frac{1}{2} \end{aligned}$$

because $\sum \cos(2\omega_0 n) \to 0$ als $\omega_0 \neq 0 \bmod \pi$.

Power

$$x[n] = \cos(\omega_0 n) \quad \text{with } \omega_0 = \frac{\pi}{4}$$



From the plot of $x^2[n]$ we see that the power of $x[n]$ is equal to $P = \frac{1}{2}$: the "average" of $x^2[n]$.

Systems

A system $\mathcal{S}$ is a mapping of the signal space ℓ onto itself:

$$x \in \ell \quad \rightarrow \quad y = \mathcal{S}\{x\} \in \ell$$

Generally, $y[n]$ at some moment n depends on $x[k]$ for all $k \in \mathbb{Z}$

Elementary systems

- Time reversal: $y[n] = (\mathcal{R}x)[n] := x[-n]$
  This can be used to split a signal into an even and odd part:

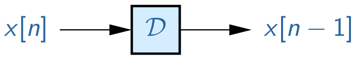
  $$x[n] = x_e[n] + x_o[n] \text{ with } x_e[n] = \frac{1}{2}(x[n] + x[-n]) \, , \; x_o[n] = \frac{1}{2}(x[n] - x[-n])$$

  Note: the energy of $x[n]$ is the sum of energies of $x_e[n]$ and $x_o[n]$.
  (Does this generally hold for the sum of two signals?)

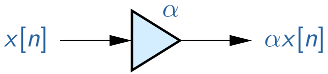
Elementary systems

- Time delay over k samples:

$$y[n] = (\mathcal{D}_k x)[n] := x[n-k]$$



- Memoryless system:
  $y[n]$ is only a function of $x[n]$ (also called a static system, in contrast to a dynamic system)



- Causal system:
  $y[n]$ only depends on $x[k]$ for $k \leq n$.

Linear time-invariant system (LTI)

- Linear: $\mathcal{S}\{ax_1 + bx_2\} = a\mathcal{S}\{x_1\} + b\mathcal{S}\{x_2\}$: superposition

- Time invariant: $\mathcal{S}\{\mathcal{D}_k\{x\}\} = \mathcal{D}_k\{\mathcal{S}\{x\}\}$
  Or: $\mathcal{S}\{x[n]\} = y[n] \quad \Rightarrow \quad \mathcal{S}\{x[n-k]\} = y[n-k]$.

$$x[n] \longrightarrow \boxed{\mathcal{S}} \longrightarrow y[n]$$

$$\Rightarrow \quad x[n-1] \longrightarrow \boxed{\mathcal{S}} \longrightarrow y[n-1]$$

Fundamental property

Suppose that $\mathcal{S}$ is an LTI system, and $y[n] = \mathcal{S}\{x[n]\}$ for an arbitrary signal $x[n]$. Then

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]\,, \qquad \text{in which} \quad h[n] = \mathcal{S}\{\delta[n]\}$$

$h[n]$ is the impulse response of the system. Notation: $y[n] = (x * h)[n]$.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Proof

Earlier, we saw $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Apply $\mathcal{S}$ and use the LTI properties:

$$\begin{aligned} y[n] = \mathcal{S}\{x[n]\} &= \mathcal{S}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} x[k]\mathcal{S}\{\delta[n-k]\} \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

Exercise

The unit-step response of a discrete-time LTI system is

$$s[n] = 2[(0.5)^n - 1]u[n]$$

Find the impulse response $h[n]$.

Exercise

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$$s[n] = 2[(0.5)^n - 1]u[n]$$

Find the impulse response $h[n]$.

For an LTI system, the response to $\delta[n] = u[n] - u[n-1]$ is

$$\begin{aligned} h[n] &= s[n] - s[n-1] \\ &= [2(0.5)^n - 2]u[n] \; - \; [2(0.5)^{n-1} - 2]u[n-1] \\ &= 0 \cdot \delta[n] \; + \; [2(0.5)^n - 2]u[n-1] \; - \; [2(0.5)^{n-1} - 2]u[n-1] \\ &= [(0.5)^{n-1} - 2(0.5)^{n-1}]u[n-1] \\ &= -(0.5)^{n-1}u[n-1] \end{aligned}$$

Discrete convolution

$$(x * y)[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

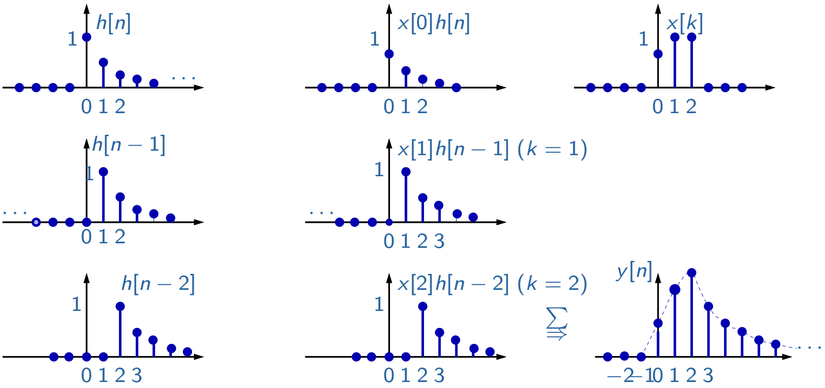
[The notation $x[n] * y[n]$ is common, but not quite right.]

Properties (cf. multiplication):

- linear (distributive):
  $h[n] * (\alpha_1 x_1[n] + \alpha_2 x_2[n]) = \alpha_1\, h[n] * x_1[n] + \alpha_2\, h[n] * x_2[n]$
- commutative: $x * y = y * x$
- associative: $(x * y) * z = x * (y * z)$
- $\delta[n]$ is the identity element: $x * \delta = x$

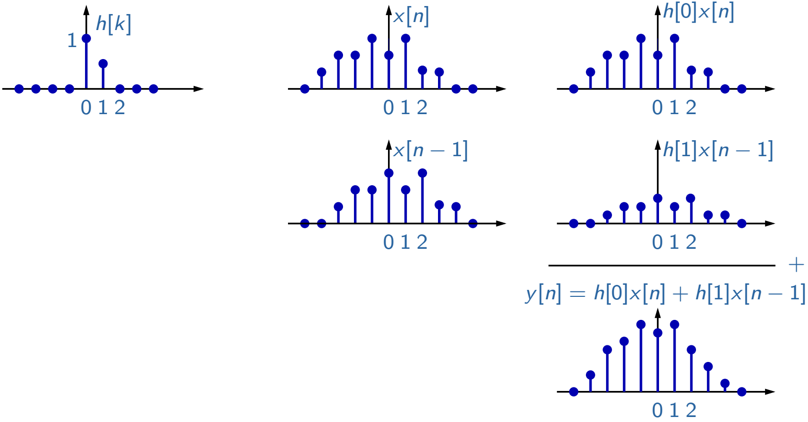
Computing the convolution (1)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \; = \; \cdots + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$



Computing the convolution (2): short impulse responses

Because $x * h = h * x$, also $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$



Exercise [exam January 2023]

Given the signals

$$\begin{aligned} x[n] &= [\cdots, 0, \boxed{0}, 1, 2, 3, 4, 0, \cdots] \\ h[n] &= [\cdots, 0, \boxed{2}, -1, 0, 0, \cdots]. \end{aligned}$$

Determine $y[n] = x[n] * h[n]$.

Exercise [exam January 2023]

Given the signals

$$\begin{aligned} x[n] &= [\cdots, 0, \boxed{0}, 1, 2, 3, 4, 0, \cdots] \\ h[n] &= [\cdots, 0, \boxed{2}, -1, 0, 0, \cdots]. \end{aligned}$$

Determine $y[n] = x[n] * h[n]$.

Compute $y[n] = x[n] * h[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$:

$h[0]x[n]$:	$\boxed{0}$	2	4	6	8	0	0	$0 \cdots$
$h[1]x[n-1]$:	$\boxed{0}$	0	-1	-2	-3	-4	0	$0 \cdots$
$y[n]$:	$\boxed{0}$	2	3	4	5	-4	0	$0 \cdots$

Properties of LTI systems

An LTI system is **causal** iff $h[n] = 0$ for $n < 0$

Proof

$$y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots$$

Note that $y[n]$ should not depend on $x[n+1], x[n+2], \cdots$. Therefore, we need $h[-1] = 0, h[-2] = 0, \cdots$.

Properties of LTI systems

Description in matrix-vector notation (strictly speaking only for $\mathcal{S} : \ell_2 \to \ell_2$)

$$
\begin{bmatrix} \vdots \\ y[-2] \\ y[-1] \\ \boxed{y[0]} \\ y[1] \\ y[2] \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
\ddots & & & & & & \\
\cdots & h[0] & & & & 0 & \\
\cdots & h[1] & h[0] & & & & \\
\cdots & h[2] & h[1] & \boxed{h[0]} & & & \\
\cdots & h[3] & h[2] & h[1] & h[0] & & \\
\cdots & h[4] & h[3] & h[2] & h[1] & h[0] & \\
 & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{bmatrix}
\begin{bmatrix} \vdots \\ x[-2] \\ x[-1] \\ \boxed{x[0]} \\ x[1] \\ x[2] \\ \vdots \end{bmatrix}
$$

linear $\leftrightarrow$ matrix-vector; causal $\leftrightarrow$ lower triangular

time-invariant $\leftrightarrow$ constant along diagonals ("Toeplitz")

Stability

A system $\mathcal{S} : x \to y$ is called **"BIBO" stable** (bounded-input bounded-output) if for every $x : \ |x[n]| \le M_x < \infty$ there is an $M_y < \infty$ such that $y : \ |y[n]| \le M_y$.

Equivalently: $\mathcal{S} : \ \ell_\infty \to \ell_\infty$

Stability

An LTI system is BIBO stable iff $h[n]$ is absolutely summable:
$\sum |h[n]| < \infty$

Equivalently: $h \in \ell_1$

Proof

- Sufficient:

$$|y[n]| \;=\; |\sum_{-\infty}^{\infty} h[k]x[n-k]| \;\leq\; \sum_{-\infty}^{\infty} |h[k]|\,|x[n-k]| \;\leq\; M_x \sum_{-\infty}^{\infty} |h[k]|$$

- Necessary: Suppose $\sum_{-\infty}^{\infty} |h[k]| = \infty$. Consider $x[n] = \frac{h^*[-n]}{|h[-n]|}$. Then

$$M_x = 1 < \infty \quad \text{while}$$

$$y[0] \;=\; \sum_{-\infty}^{\infty} h[k]x[0-k] \;=\; \sum_{-\infty}^{\infty} h[k]\frac{h^*[k]}{|h[k]|} \;=\; \sum_{-\infty}^{\infty} |h[k]| = \infty$$

Example

$h[n] = \alpha^n u[n]$

This system is causal. Is it stable?

- If $|\alpha| < 1$, then

$$\sum_{0}^{\infty} |h[n]| = \sum_{0}^{\infty} |\alpha|^n = \frac{1}{1 - |\alpha|} < \infty \qquad : \text{stable}$$

- If $|\alpha| \geq 1$, then the sum diverges: not stable.

FIR and IIR

An LTI system is **FIR** (Finite Impulse Response) if

$$h[n] = 0 \qquad \text{for} \quad n < N_1 \quad \text{and} \quad n > N_2$$

and else it is called **IIR** (Infinite Impulse Response).

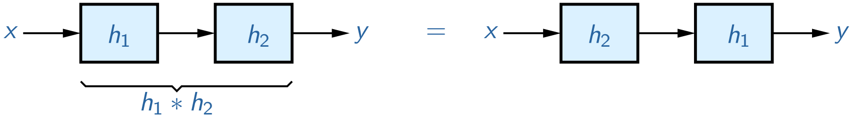
Examples

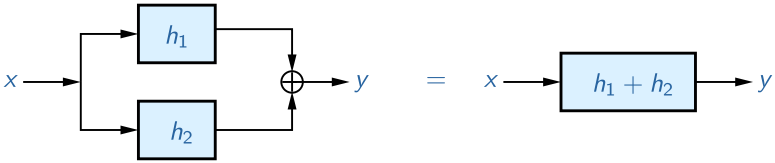
- $h[n] = u[n] - u[n-3] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & \text{elsewhere} \end{cases}$ is FIR.
- $h[n] = \alpha^n u[n]$ is IIR.

FIR systems are automatically stable (always summable)

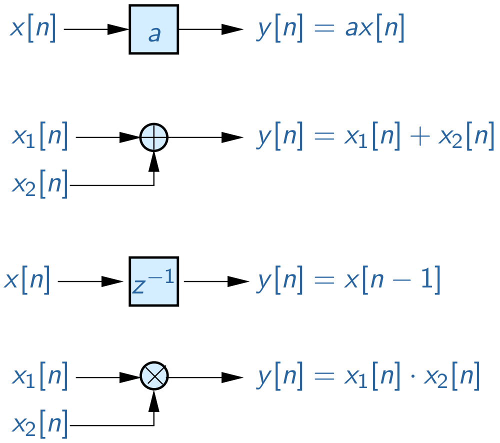
Interconnections of LTI systems

- Cascade connection: $y = (x * h_1) * h_2 \equiv (x * h_2) * h_1$
- Parallel connection: $y = x * h_1 + x * h_2 = x * (h_1 + h_2)$





Elementary discrete-time building blocks



Here, the notation "z^{-1}" is purely formal (corresponds to the delay operator $\mathcal{D}$)

LTI system described by a Linear Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] \quad = \quad \sum_{k=0}^{M} b_k x[n-k]\,, \qquad n = \cdots, -1, 0, 1, \cdots$$

This is an N-th order system (assuming $a_0 \neq 0$ and $a_N \neq 0$).

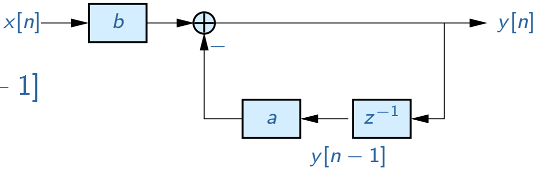
There are N initial conditions (if the recursion starts at $n = 0$).

The implementation is recursive:

$$y[n] = -\frac{1}{a_0}\sum_{k=1}^{N} a_k y[n-k] \quad + \quad \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k]$$

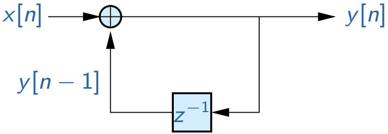
Example: first-order system

$$y[n] + ay[n-1] = bx[n]$$
$$\Rightarrow \quad y[n] = bx[n] - ay[n-1]$$



Example: accumulator

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{n} x[k] \\
&= \sum_{k=-\infty}^{n-1} x[k] \quad + \quad x[n] \\
&= y[n-1] + x[n]
\end{aligned}
$$



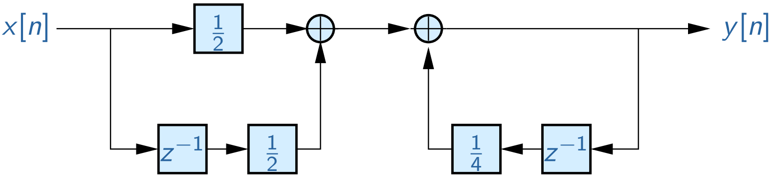
This implementation requires only one adder and memory element (=delay). The delay remembers everything from the past that is needed for the future (=the state).

Is this a stable system?

Example: 1st order system

$$y[n] - \frac{1}{4}y[n-1] \;=\; \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

$$y[n] = \tfrac{1}{4}y[n-1] + \tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-1]$$



We will see later that there also exists a realization that uses only 1 delay element.