Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science

EE2S1 (or EE2S11) SIGNALS AND SYSTEMS

Part 2 exam, 7 November 2024, 13:30–15:30

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of five questions (27 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important.

Question 1 (6 points)

(a) Given the signals $x[n] = \delta[n+1] - 2\delta[n-2]$ and $h[n] = [\cdots, 0, \boxed{1}, 3, 2, 0, \cdots]$.

Determine $y[n] = x[n] * h[n]$ using the convolution sum (in time-domain).

(b) Given $x[n] = -(\frac{1}{2})^{-n-1}u[-n-1]$. Determine $X(z)$ and also specify the ROC.

(c) Given $X(z) = \dfrac{z^{-2}}{(1-z^{-1})(1+0.25z^{-1})}$, ROC $= \{|z| > 1\}$.

Determine $x[n]$ using the inverse z-transform.

(d) Suppose the DTFT of a signal $x[n]$ is $X(e^{j\omega})$. What is the DTFT of $x[n-3]$?

(e) Let $h[n]$ be the impulse response of an ideal low-pass filter with cut-off frequency at 0.4π. Let the impulse response of a new filter be $h_1[n] = (-1)^n h[n]$.

Determine the frequency response $H_1(e^{j\omega})$ in terms of $H(e^{j\omega})$, and give a sketch of it.

Solution

1p (a) Compute $y[n] = x[n] * h[n] = \sum_{k=0}^{\infty} x[k]h[n-k]$: the only nonzero terms occur for $k = -1$ and $k = 2$.

$x[-1]h[n+1]$:	[$\cdots$ 0	1	$\boxed{3}$	2	0	0	0	0 $\cdots$]
$x[2]h[n-2]$:	[$\cdots$ 0	0	$\boxed{0}$	0	−2	−6	−4	0 $\cdots$]
$y[n]$:	[$\cdots$ 0	1	$\boxed{3}$	2	−2	−6	−4	0 $\cdots$]

1.5p (b)

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{\infty} (\frac{1}{2})^{-n-1}u[-n-1]z^{-n} = -\sum_{n=-\infty}^{\infty} (\frac{1}{2})^{n-1}u[n-1]z^{n} \\ &= -\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}z^{n} = -z\sum_{n=0}^{\infty} (\frac{1}{2})^{n}z^{n} = \frac{-z}{1-\frac{1}{2}z} = \frac{2z}{z-2} \end{aligned}$$

ROC: $|z| < 2$ (the signal is anti-causal)

1.5p (c)

$$X(z) = \frac{z^{-2}}{(1-z^{-1})(1+0.25z^{-1})} = \frac{4/5z^{-2}}{1-z^{-1}} + \frac{1/5z^{-2}}{1+0.25z^{-1}}$$

Check the ROC: both terms are causal. The factor z^{-2} in the numerators results in a delay: $n \to n-2$. This gives

$$x[n] = \frac{4}{5}u[n-2] + \frac{1}{5}(-\frac{1}{4})^{n-2}u[n-2]$$

Various other, equivalent, expressions are possible depending on how you carried out the partial fraction decomposition. All require $x[0] = 0, x[1] = 0$. Some options are:

$$x[n] = -4\delta[n] + \frac{16}{5}(-\frac{1}{4})^{n}u[n] + \frac{4}{5}u[n]$$

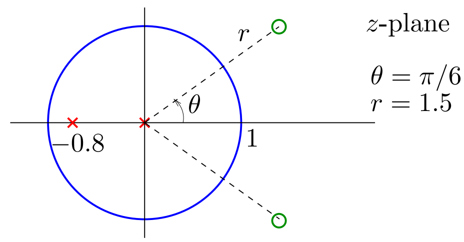
$$x[n] = \frac{4}{5}u[n-1] - \frac{4}{5}(-\frac{1}{4})^{n-1}u[n-1]$$

1p (d) This corresponds to a delay of 3 samples. The z-transform would be $z^{-3}X(z)$, hence the answer is $e^{-3j\omega}X(e^{j\omega})$.

1p (e) Use the modulation property: $H_1(e^{j\omega}) = H(e^{j(\omega-\pi)})$. This shift of the spectrum by π transforms a low-pass to a highpass (with cut-off at 0.6π).

Question 2 (6 points)

A causal system $H(z)$ has the following pole-zero diagram:



(a) What does the fact that $H(z)$ is a causal system tell you on the ROC of $H(z)$?

(b) Specify $H(z)$, up to an arbitrary gain c.

(c) Is this a stable system?

(d) Sketch the amplitude spectrum $|H(e^{j\omega})|$, also indicate values on the frequency axis.

(e) Give a pole-zero diagram of the inverse system, $G(z) = [H(z)]^{-1}$. Is this a causal stable system?

Solution

1p (a) Causal implies that the ROC extends from a circle containing the "largest" pole until infinity, i.e., ROC: $|z| > 0.8$.

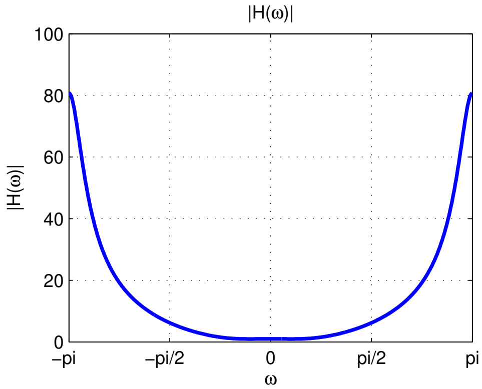
1p (b)

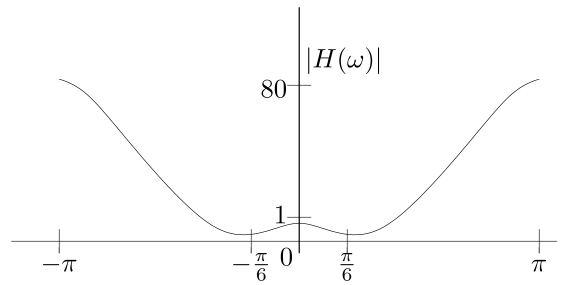
$$H(z) = c\,\frac{(z - re^{j\theta})(z - re^{-j\theta})}{z(z+0.8)} = \cdots$$

[The original question didn't contain the pole at $z = 0$, leading to an inconsistency regarding the stated causality.]

1p (c) Unit circle contained in the ROC (for a causal system equal to: all poles contained in the unit circle): stable

1p (d) Here is a matlab plot, and a sketch that better shows the essential features.

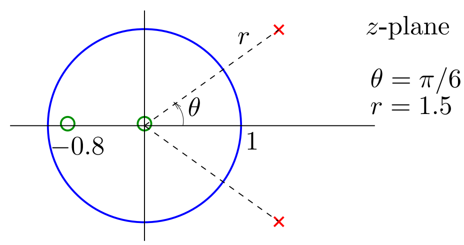




The sketch should show: for $\omega = 0$ the amplitude response is $|H(e^{j\omega})| = 1$; for $\omega = \pm\pi$, the amplitude response is maximal (with flat derivative); for approximately $\omega = \pi/6$, the response is minimal (but not quite zero).

2p (e)

$$G(z) = H^{-1}(z) = \frac{1}{c}\frac{z(z+0.8)}{(z - re^{j\theta})(z - re^{-j\theta})} = \cdots$$

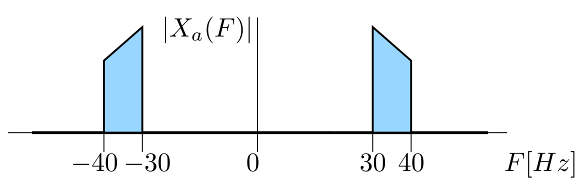


For $G(z)$, we can select an ROC $|z| > 1.5$. In that case, $G(z)$ is causal but not stable (ROC does not contain the unit circle). Alternatively, we select ROC $|z| < 1.5$. In that case, $G(z)$ is anti-causal but stable (ROC contains unit circle).

Question 3 (5 points)

A continuous-time signal $x_a(t)$ has frequencies in the range 30 to 40 Hz. The signal is sampled with period T_s so that we obtain a discrete-time signal $x[n] = x_a(nT_s)$.

The amplitude spectrum $|X_a(F)|$ of $x_a(t)$ is as follows (with F in hertz, using $\Omega = 2\pi F$):



(a) What is the Nyquist frequency at which $x_a(t)$ should be sampled to avoid aliasing?

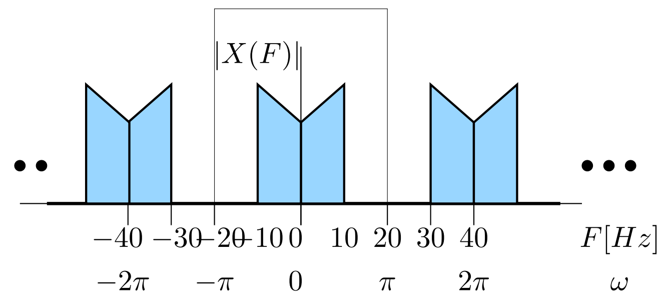
(b) We sample at a rate of 40 Hz. Give a drawing of the amplitude spectrum $|X(\omega)|$ of the signal $x[n]$. Also indicate the frequency axis for ω and relate it to the corresponding frequencies in Hz.

(c) Is it possible to reconstruct $x_a(t)$ from $x[n]$? If not, why not? If yes, indicate how this could be done. (Assume ideal D/A converters and ideal filters.)

Solution

1p (a) $F_s = 80$ Hz.

2p (b) With $F_s = 40$ Hz we obtain after sampling the original spectrum, shifted by multiples of ± 40 Hz. The part of the spectrum at 30-40 Hz will also return at -10-0 Hz (in the fundamental interval). The part of -40--30 Hz will also return at 0-10 Hz.



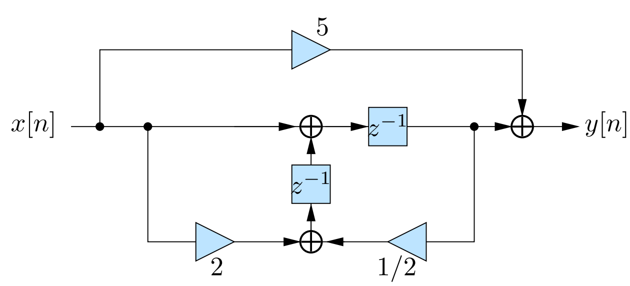
2p (c) Although there was aliasing, it was not destructive. We can first take an ideal DAC: this will return an analog signal (consisting of delta-spikes) with the same spectrum as the digital signal. Then apply an ideal analog bandpass filter which passes 30-40 Hz.

Note that we cannot first apply the bandpass filter and then do the DAC, because the spectrum of a digital signal is periodic and we cannot filter to keep the band corresponding to 30-40 Hz in the digital domain without also keeping the band from -10 to 10 Hz.

Note that for reconstruction, sampling at Nyquist rate is sufficient but not always necessary.

Question 4 (4 points)

(a) Determine the transfer function $H(z)$ of the following realization:



(b) Is this a minimal realization?

(c) Draw the "Direct form no. II" realization and also specify the coefficients.

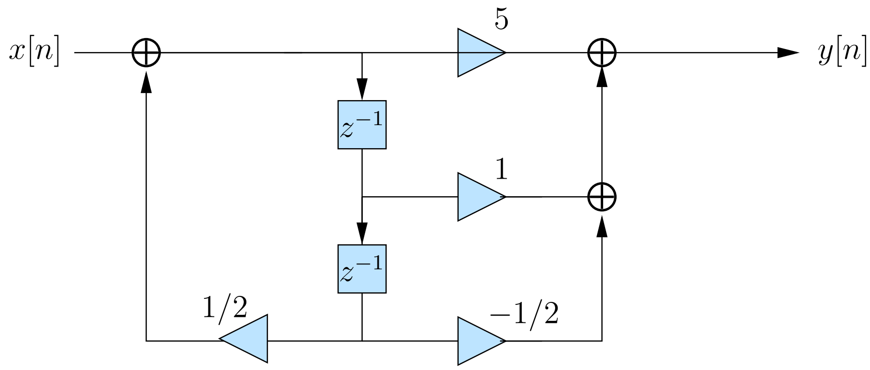
Solution

2p (a)

$$H(z) = 5 + \frac{z^{-1} + 2z^{-2}}{1 - 1/2z^{-2}} = \frac{5 + z^{-1} - 1/2z^{-2}}{1 - 1/2z^{-2}}$$

1p (b) Yes: 2 delays used for a 2nd order filter.

1p (c)



Question 5 (6 points)

We would like to design a *digital high*-pass filter with the following specifications:

Pass-band: starting at 5.0 kHz
Stop-band: below 3.0 kHz
Sample rate: 20 kHz
Ripple in the pass-band : ≤ 0.5 dB
Stop-band damping: ≥ 30 dB

The digital filter is designed by applying the bilinear transform to an analog transfer function.

(a) What are the pass-band and stop-band frequencies (in rad) in the digital time-domain?

(b) What are the filter specifications in the analog time-domain?

(c) Give a template expression for the amplitude response of an N-th order analog low-pass Butterworth filter.

(d) What frequency transformation is needed to transform this into an analog high-pass filter? What is the resulting template expression for the analog high-pass filter, $|G(j\Omega)|^2$?

(e) Use the design specifications to compute the unknown parameters of $|G(j\Omega)|^2$.

(f) It is known that the poles of a low-pass Butterworth filter are all located on a (semi-)circle in the complex s-plane. What can you say about the poles and zeros of the high-pass filter, as a result of the lowpass-to-highpass transformation?

Solution

1p (a)

$$f_p = \frac{5}{20} = \frac{1}{4} \;\Rightarrow\; \omega_p = \frac{\pi}{2} \text{ rad}$$

$$f_s = \frac{3}{20} \;\Rightarrow\; \omega_s = \frac{6\pi}{20} = 0.3\pi \text{ rad}$$

1p (b) Use the bilinear transform: $\omega = 2\arctan(\Omega)$, $\Omega = \tan(\frac{\omega}{2})$:

$$\Omega_p = \tan(\frac{\omega_p}{2}) = 1$$

$$\Omega_s = \tan(\frac{\omega_s}{2}) = 0.5095$$

For the ripples: $\delta_p = 10^{-0.5/20} = 0.9441$, $\delta_s = 10^{-30/20} = 0.0316$.

1p (c) The Butterworth template is: $|H(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$.

1p (d) $s \to \frac{\Omega_p^2}{s}$, $\Omega \to \frac{\Omega_p^2}{\Omega}$:

$$|G(j\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega_p/\Omega)^{2N}}$$

(In this case, $\Omega_p = 1$ which can be inserted to simplify the expressions.)

1p (e) For the passband (evaluate at $\Omega = \Omega_p$), we have:

$$|G(\Omega_p)|^2 = \frac{1}{1+\epsilon^2} = \delta_p^2 \quad \Rightarrow \quad \epsilon = \sqrt{\frac{1}{\delta_p^2} - 1} = 0.3493$$

For the filter order (evaluate at $\Omega = \Omega_s$):

$$|G(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\Omega_p/\Omega_s)^{2N}} = \delta_s^2 \quad \Rightarrow \quad \left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \frac{\frac{1}{\delta_s^2} - 1}{\epsilon^2} =: \frac{\delta^2}{\epsilon^2} \quad \Rightarrow \quad N \geq \frac{\log(\delta/\epsilon)}{\log(\Omega_p/\Omega_s)}$$

Substitution results in $\delta = 31.6070$ and $N \geq 6.6815$, i.e., the filter order is $N \geq 7$.

1p (f) Denote the poles by $s_1, \cdots, s_N$ and let $a = \Omega_p^2$ and $c = 1/(s_1 \cdots s_N)$, then

$$H(s) = \frac{1}{(s-s_1)\cdots(s-s_N)} \qquad \Rightarrow \qquad G(s) = \frac{1}{(\frac{a}{s}-s_1)\cdots(\frac{a}{s}-s_N)} = c\,\frac{s^N}{(\frac{a}{s_1}-s)\cdots(\frac{a}{s_N}-s)}$$

There are N zeros at $s = 0$ (which ensure that the filter response is exactly zero at $\Omega = 0$), and the poles are still on a semicircle in the left-hand plane. (To see this, use a polar notation for the s_i.)

Certainly not correct to say that the poles and zeros swap places: we do not have $G(s) = [H(s)]^{-1}$.