Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

EE2S11 SIGNALS AND SYSTEMS

Part 2 exam, 2 February 2023, 13:30–15:30

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of six questions (34 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important.

Question 1 (4 points)

a) Let $a(t)$ and $b(t)$ be two (possibly complex) signals. Let $A(\Omega)$ be the Fourier transform of $a(t)$, and similar for $B(\Omega)$. Show that

$$\int_{-\infty}^{\infty} a(t)b^*(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\Omega)B^*(\Omega)\mathrm{d}\Omega\,.$$

We call $\int a(t)b^*(t)\,\mathrm{d}t$ the *inner product* of the two signals. The two signals are called *orthogonal* if their inner product is zero.

b) Let $s(t)$ be a real-valued signal and let $s'(t)$ be its derivative. Show that $s(t)$ is orthogonal to $s'(t)$.

Solution

a) Use the Inverse FT to replace $a(t)$, then rearrange the integrals:

$$\begin{aligned}\int_{-\infty}^{\infty} a(t)b^*(t)\mathrm{d}t &= \frac{1}{2\pi}\int b^*(t)\int A(\Omega)e^{j\Omega t}\mathrm{d}\Omega\mathrm{d}t \\ &= \frac{1}{2\pi}\int A(\Omega)\left[\int b(t)e^{-j\Omega t}\mathrm{d}t\right]^*\mathrm{d}\Omega \\ &= \frac{1}{2\pi}\int A(\Omega)[B(\Omega)]^*\mathrm{d}\Omega\end{aligned}$$

b) The FT of $s'(t)$ is $j\Omega\, S(\omega)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} s(t)s'(t)\mathrm{d}t &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\Omega)(-j\Omega)S^*(\Omega)\mathrm{d}\Omega \\ &= \frac{-j}{2\pi}\int_{-\infty}^{\infty}\Omega|S(\Omega)|^2\mathrm{d}\Omega \\ &= 0\end{aligned}$$

using the fact that $|S(\omega)|^2$ is an even-valued function for real-valued signals.

Question 2 (7 points)

a) Given the signals $x[n] = \begin{cases} n, & 0 \le n \le 4, \\ 0, & \text{elsewhere} \end{cases}$ and $h[n] = [\cdots, 0, \boxed{2}, -1, 0, 0, \cdots]$.

Determine $y[n] = x[n] * h[n]$.

b) Given $X(z) = \dfrac{3\,z^{-1}}{(1 - \frac{1}{2}\,z^{-1})(1 - 2z^{-1})}$, $\quad z \in \text{ROC}$.

Determine $x[n]$ using the inverse z-transform if the ROC is $\frac{1}{2} < |z| < 2$.

c) Given the DTFT $X(e^{j\omega}) = e^{-j2\omega}\cos^2(\omega)$. Determine $x[n]$.

Hint: you could try to first determine the z-transform.

Solution

a) Compute $y[n] = x[n] * h[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$:

$h[0]x[n]$:	$\boxed{0}$	2	4	6	8	0	0	$0\cdots$
$h[1]x[n-1]$:	$\boxed{0}$	0	-1	-2	-3	-4	0	$0\cdots$
$y[n]$:	$\boxed{0}$	2	3	4	5	-4	0	$0\cdots$

b) First write in terms of z^{-1} (already the case), make 'proper' (already the case), then split (partial fraction expansion):

$$X(z) = -\frac{2}{1 - \frac{1}{2}\,z^{-1}} + \frac{2}{1 - 2z^{-1}}$$

We have poles at $z = \frac{1}{2}$ and at $z = 2$.

For this ROC, rewrite $X(z)$ as

$$X(z) = -\frac{2}{1 - \frac{1}{2}\,z^{-1}} - \frac{z}{1 - \frac{1}{2}z}\,.$$

The first term results in a causal response (ROC is $\{|z| > \frac{1}{2}\}$ and extends until $z \to \infty$), while the second term results in an anticausal response (ROC is $\{z < 2\}$ and includes $z = 0$). The factor z in the second term results in an "advance" $n \to n+1$. Hence,

$$x[n] = -2\left(\frac{1}{2}\right)^{n} u[n] - \left(\frac{1}{2}\right)^{-n-1} u[-n-1]$$

(Note that this series is absolutely summable; consistent with the ROC which encloses the unit circle.)

c) Rewrite (using $z = e^{j\omega}$) as a function of z:

$$X(z) = z^{-2}\frac{1}{4}(z + z^{-1})^2 = \frac{1}{4}(1 + 2z^{-2} + z^{-4})$$

Hence

$$x[n] = \frac{1}{4}\left(\delta[n] + 2\delta[n-2] + \delta[n-4]\right)$$

Question 3 (6 points)

The transfer function of a causal LTI system is given by $H(z) = \dfrac{z^2+2}{z^2+\frac{1}{2}}$.

a) Determine all poles and zeros of the system and draw a pole-zero diagram.

b) Specify the ROC.

c) Is the system BIBO stable? (Why?)

d) Determine the amplitude response $|H(e^{j\omega})|^2$ of the system.

Solution

a) Zeros for $z = \pm j\sqrt{2}$, poles for $z = \pm j\frac{\sqrt{2}}{2}$.

b) Causal hence the ROC is the outside of a circle; this gives $|z| > \frac{\sqrt{2}}{2}$.

c) Unit circle is in the ROC, hence BIBO stable.

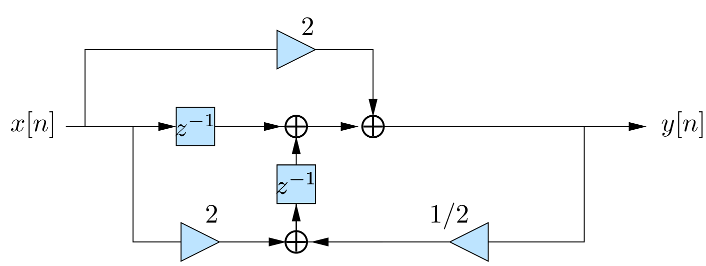
d)

$$|H(e^{j\omega})|^2 = \frac{e^{j2\omega}+2}{e^{j2\omega}+\frac{1}{2}} \cdot \frac{e^{-j2\omega}+2}{e^{-j2\omega}+\frac{1}{2}} = \frac{5+4\cos(2\omega)}{\frac{5}{4}+\cos(2\omega)} = 4$$

The amplitude response is constant (hence $H(z)$ is an all-pass function).

Question 4 (5 points)

a) Determine the transfer function $H(z)$ of the following system:



b) Is this a minimal realization?

c) Draw the "Direct Form II" realization.

Solution

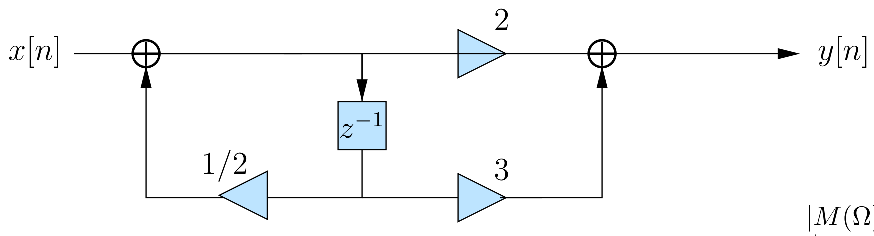
a) Insert extra parameters at each delay:

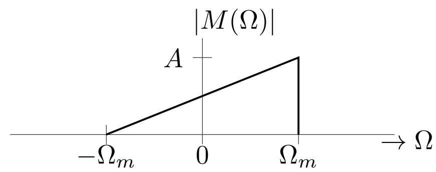
$$\begin{cases} Y(z) &= 2X(z) + Q_1(z) + Q_2(z) \\ Q_1(z) &= z^{-1}X(z) \\ Q_2(z) &= z^{-1}(2X(z) + \frac{1}{2}Y(z)) \end{cases}$$

Substitution results in

$$Y(z) = X(z)\frac{2+3z^{-1}}{1-\frac{1}{2}z^{-1}} \quad \Rightarrow \quad H(z) = \frac{2+3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

b) Not minimal (2 delays used for a first order system) c)





Question 5 (6 points)

Given the continuous-time signal $x_a(t) = m(t)\cos(1000t)$.

Here, $m(t)$ is a signal with spectrum $M(\Omega)$ as shown in the figure above, where $\Omega_m = 200$ rad/s and A is a certain amplitude.

The signal $x_a(t)$ is sampled at $\Omega_s = 1500$ rad/s, resulting in a discrete-time signal $x[n]$.

a) Compute the Fourier Transform $X_a(\Omega)$ of $x_a(t)$. You don't need to expand $M(\Omega)$.

b) Draw the amplitude spectrum $|X_a(\Omega)|$. Accurately indicate the frequencies and amplitudes. Also mark the sample frequency.

c) Do we satisfy the Nyquist criterion?

d) Let $X(\omega)$ denote the DTFT of $x[n]$. What is the relation between ω and Ω?

e) Make a drawing of the amplitude spectrum $|X(\omega)|$. Accurately indicate the frequencies (also draw the corresponding Ω-axis).

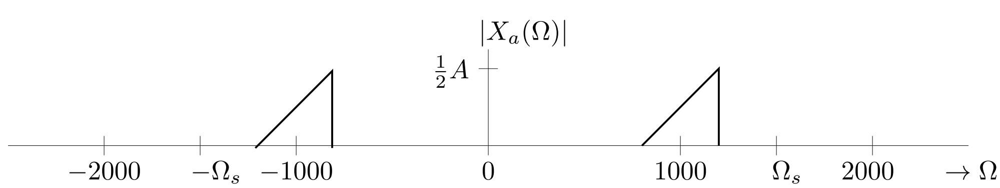
Solution

a) Directly from the table:

$$X_a(\Omega) = \frac{1}{2}M(\Omega - \Omega_0) + \frac{1}{2}M(\Omega + \Omega_0)$$

with $\Omega_0 = 1000$.

b)

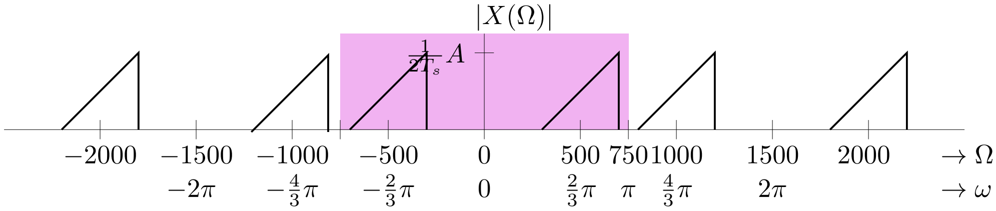


c) No. The highest frequency in the signal is 1200 rad/s and the sample frequency is less than two times as high.

d) The sample frequency Ω_s corresponds to $\omega = 2\pi$, hence $\omega = \frac{2\pi}{\Omega_s}\Omega$.

e) In terms of Ω: the spectrum of X has components at $\Omega = 1000$ and -1000. Because of the sampling, these return at $1000 + k \cdot 1500$, e.g. -500 and 2500, and at $-1000 + k \cdot 1500$, e.g. 500 and 2000.

The pink box indicates the fundamental interval (from $-\pi$ until π, outside the spectrum is periodic).



Question 6 (6 points)

We design an analog 3rd order *high-pass* Chebyshev filter with a stop-band frequency of 5 rad/s, a pass-band frequency of 10 rad/s and maximal damping in the pass-band of 2 dB.

The amplitude response of a prototype n-th order low-pass Chebyshev filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\, T_n^2(\Omega)}\,.$$

Also, the 3rd order Chebyshev polynomial is given by $T_3(\Omega) = 4\Omega^3 - 3\Omega$.

a) We first design a 3rd order Chebyshev low-pass filter with a pass-band frequency of 1 rad/s and a maximal damping in the pass-band of 2 dB. Determine all unknown parameters.

b) Which frequency transformation is needed to transform this lowpass filter into the desired high-pass filter?

c) Give the expression for the amplitude response $|G(\Omega)|^2$ of the resulting high-pass filter $G(\Omega)$.

d) What is the minimal damping in the stop-band of this high-pass filter?

e) Carefully draw the amplitude response of the high-pass filter; indicate the stop-band and pass-band frequencies, and also denote the corresponding dampings.

Solution

a)

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2[T_3(\Omega)]^2} = \frac{1}{1+\epsilon^2(4\Omega^3-3\Omega)^2}\,.$$

The only unknown parameter is ϵ.

For $\Omega = 1$ we require a damping of 2 dB, hence $|H(1)|^2 = 10^{-2/10}$.

$$|H(1)|^2 = \frac{1}{1+\epsilon^2} = 10^{-2/10} \qquad \Rightarrow \qquad \epsilon = \sqrt{10^{2/10}-1} = 0.7648\,.$$

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2(4\Omega^3-3\Omega)^2} = \frac{1}{1+(0.5849)(4\Omega^3-3\Omega)^2}\,.$$

b) $\Omega \to \frac{10}{\Omega}$.

c)

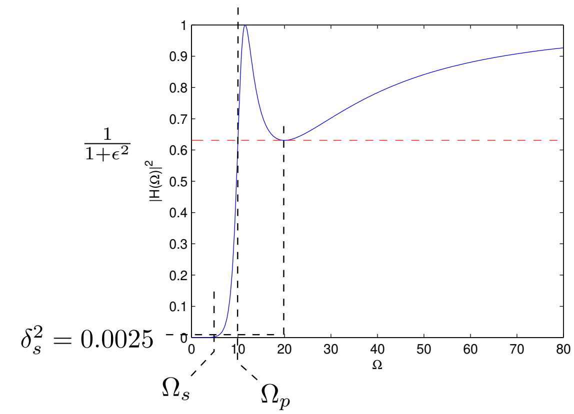
$$|G(\Omega)|^2 \;=\; \frac{1}{1+(0.5849)[4\left(\frac{10}{\Omega}\right)^3-3\left(\frac{10}{\Omega}\right)]^2} = \cdots$$

d) Using $\Omega_s = 5.0$, we obtain

$$|G(\Omega_s)|^2 = \frac{1}{1+(0.5849)(4(2.0)^3-3(2.0))^2} = 0.0025\,.$$

This corresponds to a damping of 26.0 dB.

e)



The shape of the wiggles is determined from the behavior of $T_3(\Omega)$ at the interval $\Omega \in [0,1]$. In particular, $T_3(0) = 0$, so $|H(0)|^2 = 1$ and then $|G(\infty)|^2 = 1$. There is one other frequency where $T_3(\Omega) = 0$ and the response becomes 1. We could also compute that $T_3(\Omega) = -1$ for $\Omega = 0.5$, so after the transformation, we find that $|G(20)|^2 = 1/(1+\epsilon^2)$.