Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE2S11 SIGNALS AND SYSTEMS

Part 2 exam, 27 January 2022, 13:30–15:30

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of five questions (35 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each answer sheet.

Question 1 (9 points)

(a) Let $x[n] = u[n]$, a unit step function, and let $h[n] = [\cdots, 0, \boxed{0}, 1, -1, 0, 0, \cdots]$, where the 'box' denotes the value for $n = 0$. Determine the convolution $y[n] = x[n] * h[n]$.

(b) Determine the z-transform for the following discrete-time signal, also specify the ROC:

$$x[n] = u[n] + \left(\tfrac{1}{2}\right)^n u[n-2]\,.$$

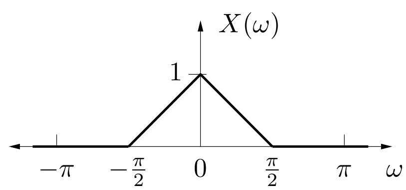
(c) Given the transfer function

$$H(z) = \frac{z^{-1}(1 - z^{-1})}{1 + 2\,z^{-1}}\,.$$

Assume the system is stable. Specify the ROC and determine $h[n]$.

(d) Determine the frequency response for $H(z)$ in (c).

(e) The signal $x[n]$ is given by its DTFT (assume that $X(\omega)$ is real-valued):



Determine and draw the DTFT of $y[n] = x[n]\cos(\frac{\pi}{4}n)$.

Solution

(a) 1.5p Determine the z-transforms:

$$X(z) = \frac{1}{1 - z^{-1}}, \quad \text{and} \quad H(z) = z^{-1} - z^{-2}\,.$$

Then $Y(z) = H(z)X(z) = z^{-1}$, and $y[n] = \delta[n-1]$.

(b) 2p First write

$$x[n] = u[n] + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^{n-2} u[n-2]\,.$$

Then

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{1}{4}\,\frac{z^{-2}}{1 - \frac{1}{2}z^{-1}}\,, \qquad \text{ROC: } \{|z| > 1\}.$$

(c) 3p Poles at $z = -2$ and $z = 0$. Stable implies the unit circle is in the ROC: $\{0 < |z| < 2\}$.

For this ROC, we expect an anti-causal signal. Since $z = 0$ is not in the ROC, $h[n]$ could have a finite number of terms in the "causal" part $n > 0$.

To determine $h[n]$, it is convenient to split $H(z)$ and rewrite as function of z:

$$H(z) = \frac{z^{-1}}{1 + 2z^{-1}} - \frac{z^{-2}}{1 + 2z^{-1}} = \frac{\frac{1}{2}}{1 + \frac{1}{2}z} - z^{-1}\frac{\frac{1}{2}}{1 + \frac{1}{2}z}\,.$$

Then (recalling that multiplication by z^{-1} means $n \to n - 1$)

$$\begin{aligned} h[n] &= \tfrac{1}{2}(-\tfrac{1}{2})^{-n}u[-n] - \tfrac{1}{2}(-\tfrac{1}{2})^{-(n-1)}u[-(n-1)] \\ &= -(-\tfrac{1}{2})^{-n+1}u[-n] + (-\tfrac{1}{2})^{-n+2}u[-n+1]\,. \end{aligned}$$

This result could be derived or written in several other, equivalent, ways:

$$H(z) = -\tfrac{1}{2}z^{-1} + \frac{3}{4}\frac{1}{1 + \frac{1}{2}z} \quad \Rightarrow \quad h[n] = -\tfrac{1}{2}\delta[n-1] + \frac{3}{4}(-\tfrac{1}{2})^{-n}u[-n]$$

$$H(z) = -\tfrac{1}{2}z^{-1}\left(2 + \frac{-3}{1 + \frac{1}{2}z}\right) \quad \Rightarrow \quad h[n] = \delta[n-1] - \frac{3}{2}(-\tfrac{1}{2})^{-(n-1)}u[-(n-1)]$$

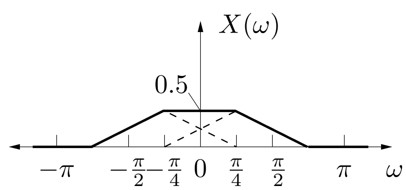
(d) 1p Since it is stable, the unit circle is in the ROC and we can simply substitute $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{e^{-j\omega}(1 - e^{-j\omega})}{1 + 2e^{-j\omega}}\,.$$

This could be simplified a bit by making the denominator real:

$$H(e^{j\omega}) = \frac{2 - e^{-j\omega} - e^{-2j\omega}}{5 + 4\cos(\omega)}$$

(e) 1.5p $Y(\omega) = \frac{1}{2}X(\omega - \frac{\pi}{4}) + \frac{1}{2}X(\omega + \frac{\pi}{4})$.



Question 2 (7 points)

Given is the difference equation of a causal system:

$$y[n] = x[n] + x[n-1] - 0.81\,y[n-2]\,.$$

(a) Determine the corresponding transfer function $H(z)$, also specify the ROC.

(b) Determine the poles and zeros of the transfer function (also those at $z = 0$ and $z = \infty$) and draw the corresponding pole-zero plot.

(c) Based on the pole-zero plot, give a sketch of the amplitude spectrum $|H(e^{j\omega})|$.

(d) Is $H(z)$ a stable transfer function? (Why?)

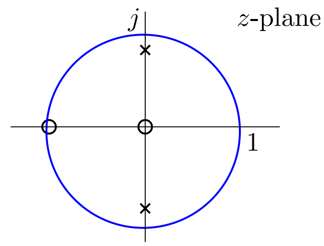
Solution

(a) 2p $Y(z) = X(z) + z^{-1}X(z) - 0.81z^{-2}Y(z)$, so

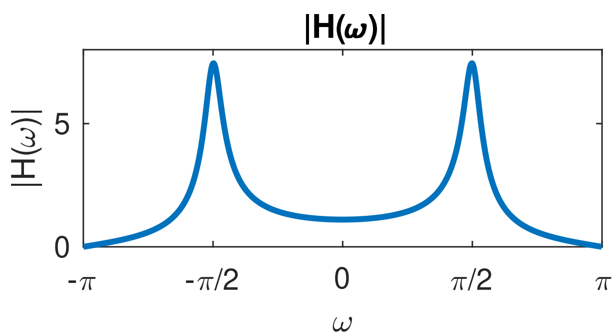
$$H(z) = \frac{1 + z^{-1}}{1 + 0.81z^{-2}}\,.$$

ROC: $\{|z| > 0.9\}$.

(b) 2p Poles at $z = \pm 0.9j$, a zero at $z = -1$ and a zero at $z = 0$.



(c) 2p



Close to poles ($\omega = \pm\pi/2$), the response peaks. At $\omega = \pi$, the response is zero because there is a zero at $z = -1$. At $\omega = 0$ ($z = 1$), the response is not zero, but not very big either. The zero at $z = 0$ only has an effect on the phase. Calculate $H(e^{j0}) = 2/1.81 = 1.11$.

The spectrum is periodic with period 2π; only one period is plotted.

(d) 1p Causal system: the ROC is $\{|z| > 0.9\}$.

Stable, because the unit circle is in the ROC.

Question 3 (6 points)

A continuous-time signal $x_a(t)$ has a Fourier transform $X_a(\Omega) = \delta(\Omega + 1) + \delta(\Omega - 1)$.

(a) Determine $x_a(t)$.

(b) What is the largest value of the sampling period T_s that would not cause aliasing when sampling $x_a(t)$?

(c) We sample the signal at $T_s = \pi$. Draw the sampled signal $x[n]$ (also specify the values on the axes).

(d) Determine and draw the corresponding spectrum $X(\omega)$ (also specify the values on the axes).

Solution:

(a) 1.5p Directly from the inverse FT integral:

$$x_a(t) = \frac{1}{2\pi}(e^{jt} + e^{-jt}) = \frac{1}{\pi}\cos(t)\,.$$

(b) 1.5p The signal is bandlimited with bandwidth $\Omega_{\max} = 1$, or $F_{\max} = 1/(2\pi)$. Therefore, we have to sample at Nyquist rate $F_s = 2F_{\max} = 1/\pi$. The corresponding sampling period is $T_s = \pi$.

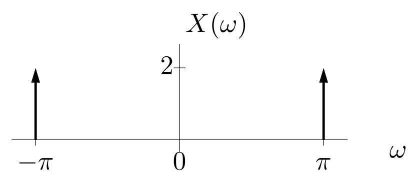
(c) 1p In this case we sample the cos-function exactly at its peaks. Therefore

$$x[n] = \frac{1}{\pi}(-1)^n\,.$$

(d) 2p From $x[n]$ we obtain

$$X(\omega) = 2\sum_{k=-\infty}^{\infty}\delta(\omega - \pi - k2\pi)\,.$$

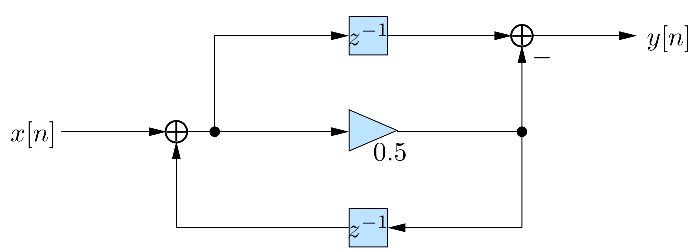
The spectrum is a spike train with spikes at multiples of $\pm\pi, \pm 3\pi, \cdots$.



(Only the fundamental period is shown.)

Question 4 (6 points)

Given the realization of a causal system:



(a) Determine the transfer function $H(z)$ of this realization.

(b) Is this a minimal realization? (Why?)

(c) Draw the "direct form no. 2" realization.

Solution

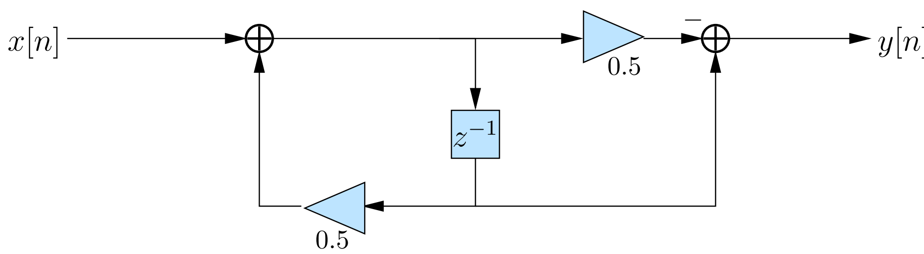
(a) 3p Let $P(z)$ be the signal at the output of the multiplier. Then

$$\begin{cases} P = \frac{1}{2}(X + z^{-1}P) \\ Y = z^{-1}(X + z^{-1}P) - P \end{cases} \Leftrightarrow \begin{cases} P = \frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}}X \\ Y = (2z^{-1} - 1)P \end{cases}$$

$$H(z) = \frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}}\,.$$

(b) 1p Not minimal, because two delays are used but the system is of first order.

(c) 2p



Question 5 (7 points)

A "template" third-order Butterworth filter has the transfer function

$$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}\,.$$

The corresponding amplitude response is $\quad |H(j\Omega)|^2 = \dfrac{1}{1+\Omega^6}\,.$

(a) Which frequency transform should we apply to the template to construct a *high*-pass Butterworth filter with a 3dB cut-off frequency of Ω_c?

(b) What is the corresponding transfer function $G(s)$?

We aim to design an analog 3rd order high-pass Butterworth filter $G(s)$ with a pass-band frequency of 6 rad/s, a stop-band frequency of 3 rad/s and a maximal damping in the pass-band of 0.5 dB.

(c) Give a suitable expression for the amplitude response $|G(j\Omega)|^2$ of this filter and determine its parameters.

(d) For this filter, what is the minimal damping in the stop-band ?

(e) Which transform should be applied to the template $|H(j\Omega)|^2$ to obtain this filter?

Using this, determine the transfer function $G(s)$ of the high-pass filter.

Solution

(a) 1p Substitute $\Omega \to \dfrac{\Omega_c}{\Omega}$.

(b) 1p Substitute $s \to \dfrac{\Omega_c}{s}$ in the expression for $H(s)$, this results in

$$G(s) = \frac{1}{(\frac{\Omega_c}{s})^3 + 2(\frac{\Omega_c}{s})^2 + 2(\frac{\Omega_c}{s}) + 1} = \frac{s^3}{\Omega_c^3 + 2\Omega_c^2 s + 2\Omega_c s^2 + s^3}\,.$$

(c) 2p The general expression is

$$|G(j\Omega)|^2 = \frac{1}{1+\epsilon^2(\frac{\Omega_p}{\Omega})^6}\,.$$

For $\Omega = \Omega_p = 6$ we obtain

$$\frac{1}{1+\epsilon^2} = 10^{-0.5/10} \quad \Rightarrow \quad \epsilon = 0.3493\,.$$

(d) 1p For $\Omega = \Omega_s = 3$ we obtain

$$\frac{1}{1+\epsilon^2(\frac{6}{3})^6} = 0.1135 = -9.45\,\mathrm{dB}\,.$$

(e) 2p First, we determine Ω_c by comparing (a) to (c):

$$\left(\frac{\Omega_c}{\Omega}\right)^6 = \epsilon^2\left(\frac{\Omega_p}{\Omega}\right)^6 \quad \Rightarrow \quad \Omega_c = \Omega_p\,\epsilon^{1/3} = 4.23\,\mathrm{rad/s}\,.$$

The transfer function of the requested 3rd order Butterworth filter is:

$$G(s) = \frac{s^3}{4.23^3 + 2s\,4.23^2 + 2s^2\,4.23 + s^3}\,.$$