Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science

EE2S11 SIGNALS AND SYSTEMS

Midterm exam, 13 December 2023, 13:30–15:30

Closed book; two sides of one A4 page with handwritten notes permitted. Graphic calculators not permitted. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

This exam has four questions (16 points).

Question 1 (5 points)

Given the two signals $h(t) = u(t-1)$ and $x(t) = u(t-2)$, where $u(t)$ is the Heaviside unit step function.

(a) Determine the convolution $y(t) = h(t) * x(t)$ of the signals h and x by directly using the convolution integral.

(b) Determine the convolution $y(t) = h(t) * x(t)$ using the Laplace transform.

Suppose that this $x(t)$ is the input signal of a Linear and Time-Invariant (LTI) system and suppose that $h(t)$ is the impulse response of this system.

(c) Is the input signal $x(t)$ causal? Motivate your answer.

(d) Is the LTI system causal? Motivate your answer.

(e) Is the LTI system BIBO stable? Motivate your answer.

Answer

(a)

$$\begin{aligned} y(t) &= \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau \\ &= \int_{\tau=1}^{\infty} u(t-\tau-2)\,\mathrm{d}\tau \\ &\overset{p=t-\tau-2}{=} \int_{p=-\infty}^{t-3} u(p)\,\mathrm{d}p \\ &= (t-3)u(t-3) \end{aligned}$$

(b) $H(s) = e^{-s}/s$, $X(s) = e^{-2s}/s$, and $Y(s) = H(s)X(s) = e^{-3s}/s^2$.
Inverse Laplace transform gives $y(t) = (t-3)u(t-3)$.

(c) Yes, $x(t) = 0$ for $t < 0$.

(d) Yes, $h(t) = 0$ for $t < 0$.

(e) No, $h(t)$ is not absolutely integrable.

Question 2 (4 points)

(a) Given the signal $f(t) = e^{-t}u(t)$. Determine the two-sided Laplace transform of $\frac{\mathrm{d}f}{\mathrm{d}t}$ and give its ROC.

(b) Determine the two-sided Laplace transform of the signal $g(t) = \delta(2t+4)$, where $\delta(t)$ is the Dirac distribution, and give its ROC.

(c) Determine the Laplace transform of the signal $w(t) = (t^2 - 2t + 5)u(t-1)$ and give its ROC.

Answer

(a)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \delta(t) - e^{-t}u(t).$$

The two-sided Laplace transform is $1 - 1/(s+1) = s/(s+1)$, ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > -1\}$.

(b) $g(t) = \delta(2t+4) = \delta[2(t+2)] = \frac{1}{2}\delta(t+2)$. $G(s) = \frac{1}{2}e^{2s}$. ROC $= \mathbb{C}$.

(c) $w(t) = [(t-1)^2 + 4]u(t-1) = 2 \cdot \frac{1}{2}(t-1)^2 u(t-1) + 4u(t-1)$.

$$W(s) = 2\frac{e^{-s}}{s^3} + 4\frac{e^{-s}}{s} = 2\frac{e^{-s}}{s}\left(2 + \frac{1}{s^2}\right).$$

ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$.

Question 3 (3 points)

Determine the inverse Laplace transforms of

(a) $F(s) = \dfrac{s}{s^2 - a^2}, \quad a > 0, \quad \mathrm{Re}(s) > a.$

(b) $G(s) = \dfrac{3s-1}{s(s-1)}, \quad \mathrm{Re}(s) > 1.$

(c) $W(s) = \dfrac{6}{s^2 - 6s + 13}, \quad \mathrm{Re}(s) > 3.$

Answer

(a)

$$\frac{s}{s^2 - a^2} = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}.$$

Using the table, we find

$$f(t) = \frac{1}{2}e^{at}u(t) + \frac{1}{2}e^{-at}u(t) = \cosh(at)u(t).$$

(b)

$$G(s) = \frac{3s-1}{s(s-1)} = \frac{1}{s} + \frac{2}{s-1}$$

Using the table, we find

$$g(t) = (1 + 2e^{t})u(t).$$

(c)

$$W(s) = 3\frac{2}{(s-3)^2 + 2^2}.$$

Using the table, we find

$$w(t) = 3e^{3t}\sin(2t)u(t).$$

Question 4 (4 points)

Given the periodic signal $x(t)$ with fundamental period $T_0 = 2\pi$ and

$$x(t) = e^t, \quad -\pi < t < \pi.$$

(a) Determine the power P_x of this periodic signal.

(b) Determine the Fourier coefficients X_k of this periodic signal.

(c) Show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \frac{\pi}{\tanh(\pi)}.$$

Answer

(a)

$$P_x = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} e^{2t}\,\mathrm{d}t = \frac{1}{4\pi}(e^{2\pi} - e^{-2\pi}) = \frac{1}{4\pi}(e^{\pi} - e^{-\pi})(e^{\pi} + e^{-\pi}).$$

(b)

$$X_k = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} e^t e^{-\mathrm{j}kt}\,\mathrm{d}t = \frac{(-1)^k}{2\pi}\frac{1}{1-\mathrm{j}k}(e^{\pi} - e^{-\pi}).$$

(c) Use Parseval's power relation

$$\sum_{k=-\infty}^{\infty} |X_k|^2 = P_x.$$

Substitution gives

$$\frac{1}{4\pi^2}(e^{\pi} - e^{-\pi})^2 \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \frac{1}{4\pi}(e^{\pi} - e^{-\pi})(e^{\pi} + e^{-\pi})$$

and we obtain

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \pi\frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = \frac{\pi}{\tanh(\pi)}.$$