Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE2S11 SIGNALS AND SYSTEMS

Midterm exam, 7 December 2020, 13:30–15:50
Block 1 (13:30-14:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:25–14:40

This block consists of three questions (20 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (8 points)

Let $\Lambda(t)$ denote the triangular pulse signal. Furthermore, let $v(t) = \Lambda(2t)$ and

$$w(t) = \sum_{k=-\infty}^{\infty} v(t-k).$$

(a) Sketch the signals $v(t)$ and $w(t)$.

(b) Compute $\frac{\mathrm{d}w}{\mathrm{d}t}$ and express this derivative in terms of (time-shifted) step functions.

(c) Is $v(t)$ a finite-energy signal? Motivative your answer.

(d) Is $w(t)$ a finite-energy signal? Motivative your answer.

(e) Is $\frac{\mathrm{d}w}{\mathrm{d}t}(t)$ a finite-energy signal? Motivative your answer.

Solution

(a) See book, Figure 1.23, page 110.

(b) We have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}v(t-k)}{\mathrm{d}t} = \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\Lambda[2(t-k)] \overset{p=2(t-k)}{=} \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p}\Lambda(p)\,\frac{\mathrm{d}p}{\mathrm{d}t} = 2\sum_{k=-\infty}^{\infty} \frac{\mathrm{d}\Lambda(p)}{\mathrm{d}p}.$$

Since

$$\frac{\mathrm{d}\Lambda(p)}{\mathrm{d}p} = u(p) - 2u(p-1) + u(p-2),$$

with u the unit step function, we obtain

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= 2\sum_{k=-\infty}^{\infty} u[2(t-k)] - 2u[2(t-k)-1] + u[2(t-k)-2] \\ &= 2\sum_{k=-\infty}^{\infty} u(t-k) - 2u(t-k-1/2) + u(t-k-1).\end{aligned}$$

(c) Yes, $v(t)$ is clearly square integrable.

(d) No, $w(t)$ is a periodic signal.

(e) No, $\frac{\mathrm{d}w}{\mathrm{d}t}$ is a periodic signal.

Question 2 (6 points)

Let $r(t) = t\,u(t)$ denote the ramp signal and let $w(t) = \cos(t)u(t)$.

(a) Determine the convolution $y(t) = r(t) * w(t)$ directly using the convolution integral.

(b) Determine the convolution $y(t) = r(t) * w(t)$ using the Laplace transform.

Solution

(a) $r(t)$ and $w(t)$ are both causal. Consequently, $y(t)$ is causal as well and we have $y(t) = 0$ for $t < 0$. For $t > 0$, $y(t)$ is given by

$$y(t) = \int_{\tau=0}^{t} (t - \tau)\cos(\tau)\,\mathrm{d}\tau = 1 - \cos(t).$$

In total: $y(t) = [1 - \cos(t)]u(t)$.

(b) The Laplace transforms of $r(t)$ and $w(t)$ are given by

$$R(s) = \frac{1}{s^2} \quad \text{and} \quad W(s) = \frac{s}{s^2+1},$$

respectively. Both have the right-half of the complex s-plane ($\mathrm{Re}(s) > 0$) as ROC. The Laplace transform of $y(t)$ is

$$Y(s) = \frac{1}{s^2} \cdot \frac{s}{s^2+1} = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}, \qquad \mathrm{Re}(s) > 0.$$

Using the table of Laplace transforms, we find $y(t) = [1 - \cos(t)]u(t)$.

Question 3 (6 points)

(a) The one-sided Laplace transform of a causal signal $f(t)$ is given by

$$F(s) = \frac{2(2s+7)}{(s+4)(s+2)}, \qquad \mathrm{Re}(s) > -2.$$

Determine $f(t)$.

(b) The one-sided Laplace transform of a causal signal $g(t)$ is given by

$$G(s) = e^{-2s}\frac{1}{s^2+s-2}, \qquad \mathrm{Re}(s) > 1.$$

Determine $g(t)$.

Solution

(a)

$$F(s) = \frac{2(2s+7)}{(s+4)(s+2)} = \frac{1}{s+4} + \frac{3}{s+2}, \qquad \mathrm{Re}(s) > -2.$$

Using the table of Laplace transforms, we find $f(t) = (e^{-4t} + 3e^{-2t})u(t)$.

(b)

$$G(s) = e^{-2s}\frac{1}{s^2+s-2} = e^{-2s}\frac{1}{3}\left[\frac{1}{s-1} - \frac{1}{s+2}\right], \quad \mathrm{Re}(s) > 1.$$

Using the table of Laplace transforms, we find $g(t) = \frac{1}{3}\left[e^{t-2} - e^{-2(t-2)}\right]u(t-2)$.

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Block 2 (14:50-15:50)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 15:45–16:00

This block consists of two questions (19 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 4 (7 points)

(a) The one-sided Laplace transform of a causal signal $w(t)$ is given by

$$W(s) = \frac{s^2 + 2k^2}{s(s^2 + 4k^2)}, \quad k > 0, \quad \mathrm{Re}(s) > 0.$$

Determine $w(t)$.

The Laplace transform of the signal

$$f(t) = \sin(\omega t + \varphi)u(t), \quad \omega > 0,$$

can be written as

$$F(s) = \frac{\alpha \sin(\varphi) + \beta \cos(\varphi)}{s^2 + \omega^2}.$$

(b) Determine the ROC of $F(s)$.

(c) Determine α and β.

Solution

(a)

$$W(s) = \frac{s^2 + 2k^2}{s(s^2 + 4k^2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2 + 4k^2}\right), \quad \mathrm{Re}(s) > 0.$$

Using the table of Laplace transforms, we find $w(t) = \frac{1}{2}\left[1 + \cos(2kt)\right]u(t) = \cos^2(kt)u(t)$.

(b) ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$.

(c) For $\varphi = 0$, we have $f(t) = \sin(\omega t)u(t)$. We know that

$$F(s) = \frac{\omega}{s^2 + \omega^2} = \frac{\beta}{s^2 + \omega^2}$$

and we obtain $\beta = \omega$. For $\varphi = \pi/2$, we have $f(t) = \cos(\omega t)u(t)$ and in this case

$$F(s) = \frac{s}{s^2 + \omega^2} = \frac{\alpha}{s^2 + \omega^2}$$

from which we obtain $\alpha = s$.

Question 5 (12 points)

On the interval $-\pi \leq t \leq \pi$, a periodic signal $f(t)$ with a fundamental period $T_0 = 2\pi$ is given by $f(t) = \cos(at)$, where a is **not** an integer. Recall that the trigoniometric Fourier expansion of a periodic signal is given by

$$f(t) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t).$$

(a) Determine the dc-component c_0.

(b) Show that the Fourier coefficients c_k for $k \geq 1$ are given by

$$c_k = \frac{\sin(a\pi)}{\pi}(-1)^k \frac{a}{a^2 - k^2}, \qquad k \geq 1.$$

Hints:

$$\cos(\alpha t)\cos(\beta t) = \frac{1}{2}\Big\{ \cos[(\alpha+\beta)t] + \cos[(\alpha-\beta)t] \Big\} \quad \text{and} \quad \sin[(k+a)\pi] = (-1)^k \sin(a\pi).$$

(c) Determine the Fourier coefficients d_k.

(d) Use the Fourier expansion of $f(t)$ to show that

$$\frac{1}{\sin(z)} = \frac{1}{z} + \sum_{k=1}^{\infty}(-1)^k \left(\frac{1}{z - k\pi} + \frac{1}{z + k\pi} \right),$$

where z is not an integer multiple of π.

Solution

$T_0 = 2\pi$, $\Omega_0 = 1$.

(a)

$$c_0 = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} \cos(at)\,\mathrm{d}t = \frac{1}{\pi}\int_{t=0}^{\pi} \cos(at)\,\mathrm{d}t = \frac{\sin(a\pi)}{a\pi}.$$

(b)

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{t=-\pi}^{\pi} \cos(at)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{t=0}^{\pi} \cos(at)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{t=0}^{\pi} \cos[(k+a)t] + \cos[(k-a)t]\,\mathrm{d}t \\
&= \frac{1}{2\pi}\left[\frac{\sin[(k+a)\pi]}{k+a} + \frac{\sin[(k-a)\pi]}{k-a}\right] \\
&= \frac{1}{2\pi}(-1)^k\sin(a\pi)\left(\frac{1}{k+a} - \frac{1}{k-a}\right) \\
&= \frac{\sin(a\pi)}{\pi}(-1)^k\frac{a}{a^2-k^2}
\end{aligned}$$

(c) The Fourier coefficients $d_k = 0$, since $f(t)$ is even.

(d) The expansion is

$$\cos(at) = \frac{2}{\pi}\sin(a\pi)\left[\frac{1}{2a} + \sum_{k=1}^{\infty}(-1)^k\frac{a\cos(kt)}{a^2-k^2}\right], \quad -\pi \le t \le \pi$$

For $t = 0$ we obtain

$$1 = \frac{2}{\pi}\sin(a\pi)\left[\frac{1}{2a} + \sum_{k=1}^{\infty}(-1)^k\frac{a}{a^2-k^2}\right],$$

which can be written as

$$\frac{1}{\sin(a\pi)} = \frac{1}{a\pi} + \sum_{k=1}^{\infty}(-1)^k\frac{2a\pi}{(a\pi)^2-(k\pi)^2} = \frac{1}{a\pi} + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{a\pi-k\pi} + \frac{1}{a\pi+k\pi}\right).$$

Set $z = a\pi$ and the result follows.