Resit exam EE2S11 SIGNALS AND SYSTEMS
8 July 2024, 13:30–16:30

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of seven questions (35 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important.

Question 1 (6 points)

Given the signal $x(t) = e^{-t}[u(t-5) - u(t-6)]$, where $u(t)$ is the Heaviside unit step function.

(a) Is this signal causal? Motivate your answer.

(b) Determine the energy of $x(t)$.

(c) Determine the power of $x(t)$.

(d) What is the support of the signal $z(t) = x(t) * x(t)$? Motivate your answer.

(e) Determine the Laplace transform of $x(t)$, and specify its ROC.

Solution:

0.5p (a) Yes, $x(t) = 0$ for $t < 0$.

1p (b) $E_x = (e^{-10} - e^{-12})/2$

0.5p (c) $P_x = 0$, $x(t)$ is a finite-energy signal.

2p (d) Support of $z(t)$ is the interval $[10, 12]$.

2p (e)

$$X(s) = \frac{1}{s+1} e^{-5(s+1)}[1 - e^{-(s+1)}]$$

No singularity at $s = -1$. ROC $= \mathbb{C}$

Question 2 (6 points)

Determine the inverse Laplace transforms of

(a) $F(s) = \dfrac{s-2}{s^2 - 2s - 3}$, $\quad$ Re$(s) > 3$.

(b) $G(s) = \dfrac{3s+2}{s^2 + 25}$, $\quad$ Re$(s) > 0$.

(c) $W(s) = \dfrac{5}{(s+2)^3}$, $\quad$ Re$(s) > -2$.

Solution:

2p (a)

$$f(t) = \frac{1}{4}(3e^{-t} + e^{3t})u(t)$$

2p (b)

$$g(t) = 3\cos(5t)u(t) + \frac{2}{5}\sin(5t)u(t)$$

2p (c)

$$w(t) = \frac{5}{2}e^{-2t}t^2 u(t)$$

Question 3 (5 points)

Given the periodic signal $x(t)$ with fundamental period $T_0 = 2\pi$ and

$$x(t) = e^t, \quad -\pi < t < \pi.$$

(a) Determine the power P_x of this periodic signal.

(b) Determine the Fourier coefficients X_k of this periodic signal.

(c) Show that
$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \frac{\pi}{\tanh(\pi)}.$$

Solution:

1.5p (a)

$$P_x = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} e^{2t}\,\mathrm{d}t = \frac{1}{4\pi}(e^{2\pi} - e^{-2\pi}) = \frac{1}{4\pi}(e^{\pi} - e^{-\pi})(e^{\pi} + e^{-\pi}).$$

1.5p (b)

$$X_k = \frac{1}{2\pi}\int_{t=-\pi}^{\pi} e^t e^{-\mathrm{j}kt}\,\mathrm{d}t = \frac{(-1)^k}{2\pi}\frac{1}{1-\mathrm{j}k}(e^{\pi} - e^{-\pi}).$$

2p (c) Use Parseval's power relation

$$\sum_{k=-\infty}^{\infty} |X_k|^2 = P_x.$$

Substitution gives

$$\frac{1}{4\pi^2}(e^{\pi} - e^{-\pi})^2 \sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \frac{1}{4\pi}(e^{\pi} - e^{-\pi})(e^{\pi} + e^{-\pi})$$

and we obtain

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \pi\frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = \frac{\pi}{\tanh(\pi)}.$$

Question 4 (7 points)

(a) Given the signal $x[n] = [\cdots, 0, \boxed{1}, 2, 3, 0, 0, \cdots]$, where the 'box' denotes the value for $n = 0$. Determine $r[n] = x[n] * x[-n]$ using the convolution sum.

(b) Determine the z-transform for the following discrete-time signal, also specify the ROC:

$$x[n] = u[n] + 2^n\, u[-n]$$

(c) Determine the signal $x[n]$ corresponding to the z-transform:

$$X(z) = \frac{1.2}{(1 - z^{-1})(1 + 0.2z^{-1})}, \qquad \text{ROC: } 0.2 < |z| < 1\,.$$

(d) Determine the frequency response $H(e^{j\omega})$ for the system defined by the difference equation:

$$y[n] = 0.5y[n-1] + x[n] + x[n-1], \qquad n \geq 0$$

Solution:

2p (a) Let $y[n] = x[-n] = [\cdots, 0, 3, 2, \boxed{1}, 0, 0, \cdots]$, then $r[n] = \sum_{k=1}^{2} x[k]y[n-k]$,

$k=0:$	$1y[n]:$	3	2	$\boxed{1}$	0	0	$0\cdots$
$k=1:$	$2y[n-1]:$	0	6	$\boxed{4}$	2	0	$0\cdots$
$k=2:$	$3y[n-2]:$	0	0	$\boxed{9}$	6	3	$0\cdots$
	$r[n]:$	3	8	$\boxed{14}$	8	3	$0\cdots$

1.5p (b)

$$X(z) = \frac{1}{1-z^{-1}} + \frac{1}{1-(\frac{1}{2})z}, \qquad \text{ROC: } 1 < |z| < 2\,.$$

2p (c) Partial fraction expansion gives

$$X(z) = \frac{1}{1-z^{-1}} + \frac{0.2}{1+0.2z^{-1}}\,.$$

Inspecting the ROC, we see that the first term will give an anti-causal signal and the second term will become causal. We therefore write

$$X(z) = \frac{-z}{1-z} + \frac{0.2}{1+0.2z^{-1}}\,.$$

Applying the inverse transform gives

$$x[n] = -u[-n-1] + 0.2(-0.2)^n u[n]\,.$$

1.5p (d) We first determine $H(z)$ as

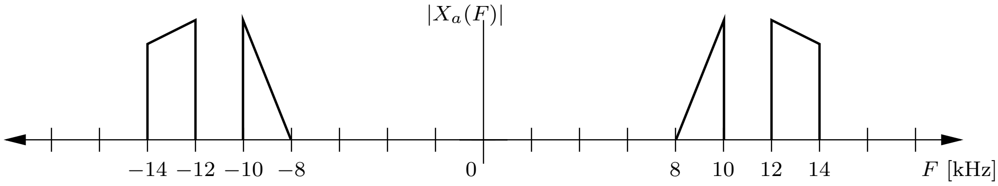
$$H(z) = \frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\,.$$

Therefore, the frequency response is

$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}\,.$$

Question 5 (3 points)

A continuous-time signal $x_a(t)$ has a Fourier transform $X_a(F)$ as shown in the figure:



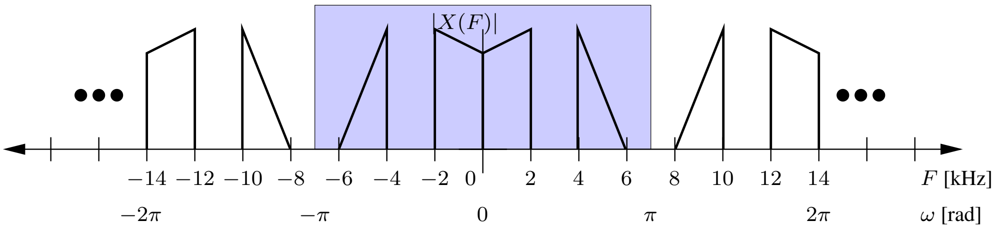
(a) What is the required sample frequency according to the Nyquist sample rate condition?

(b) The signal is sampled at a rate $F_s = 14$ kHz; no filtering is applied. Sketch the amplitude spectrum of the resulting discrete-time signal (carefully indicate the frequencies).

Solution:

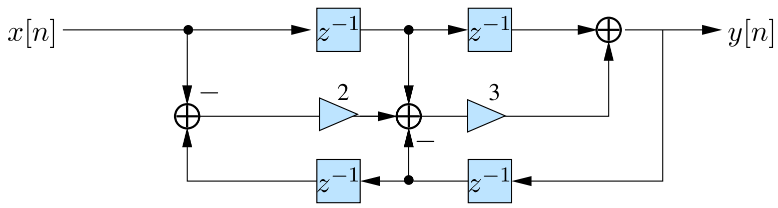
1p (a) $F_s = 28$ kHz.

2p (b) The fundamental interval is -7 kHz till 7 kHz, corresponding to $\omega \in [-\pi, \pi]$ (outside this interval the spectrum is periodic). All components are shifted by multiples of F_s.



Question 6 (3 points)

Consider the following system realization:



(a) Determine the transfer function $H(z)$.

(b) Is this a stable system? (Why?)

(c) Is this a minimal realization? (Why?)

Solution:

2p (a) Introduce additional variables $P(z) = z^{-1}X(z)$ at the output of the first delay, and $Q(z) = z^{-1}Y(z)$ at the output of the delay in the bottom branch. The equation for $Y(z)$ is

$$Y = z^{-1}P + 3(P + 2(z^{-1}Q - X) - Q)$$

Substitute $P(z)$ and $Q(z)$ to obtain

$$Y = z^{-2}X + 3(z^{-1}X + 2(z^{-2}Y - X) - z^{-1}Y)$$

$$(1 + 3z^{-1} - 6z^{-2})Y = (-6 + 3z^{-1} + z^{-2})X$$

The transfer function is

$$H(z) = \frac{-6 + 3z^{-1} + z^{-2}}{1 + 3z^{-1} - 6z^{-2}}$$

(This shows that it is a 2nd order allpass function.)

0.5p (b) Solve $z^2 + 3z - 6 = 0$. The poles are $z = -1\frac{1}{2} \pm j\frac{1}{2}\sqrt{15}$. Clearly the poles are outside the unit circle: unstable.

0.5p (c) Not minimal because 4 delays are used and $H(z)$ is 2nd order.

Question 7 (5 points)

A second-order analog lowpass filter (Butterworth filter) has transfer function

$$G_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1} .$$

The 3-dB cut-off frequency of this filter is $\Omega_c = 1$ rad/s.

Using the bilinear transform and the above filter as a template, we will now design a digital *high*-pass filter $H(z)$ with cut-off frequency $\omega_c' = \frac{2}{3}\pi$.

(a) What should be the corresponding cut-off frequency in the analog frequency domain?

(b) Which frequency transformation should be used?

(c) What is $H_a(s)$?

(d) What is $H(z)$?

(e) Verify that the design meets the specifications.

Solution:

1p (a) The bilinear transform gives $\Omega_c' = \tan(\frac{1}{2}\omega_c') = \tan(\frac{1}{3}\pi) = \sqrt{3}$.

1p (b) $\Omega \to \dfrac{\sqrt{3}}{\Omega}$. This transforms lowpass into highpass, with cutoff at $\sqrt{3}$.

1p (c) Likewise $s \to \dfrac{\sqrt{3}}{s}$. This gives

$$H_a(s) = \frac{1}{\frac{3}{s^2} + \sqrt{6}s + 1} = \frac{s^2}{s^2 + \sqrt{6}s + 3}$$

1p (d) Apply the bilinear transform: $s \to \dfrac{1 - z^{-1}}{1 + z^{-1}}$. This gives

$$H(z) = \frac{(1 - z^{-1})^2}{(1 - z^{-1})^2 + \sqrt{6}(1 - z^{-1})(1 + z^{-1}) + 3(1 + z^{-1})^2} = \cdots$$

1p (e) We evaluate at $\omega = 0$, $\omega = \pi$ and $\omega = \omega_c$:

$$H(z = e^{j0}) = 0$$
$$H(z = e^{j\pi}) = \tfrac{2^2}{2^2+0+0} = 1$$
$$H(z = e^{j2\pi/3}) = H(z = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}) = H_a(s = j\sqrt{3}) = \tfrac{-3}{-3+j\sqrt{6\cdot 3}+3} = j\tfrac{1}{\sqrt{2}}$$

This shows that indeed $H(z)$ is a highpass filter with -3 dB point at ω_c.