Resit exam EE2S1 (or EE2S11) SIGNALS AND SYSTEMS 17 December 2024, 09:00–12:00

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of seven questions (35 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important.

Question 1 (7 points)

The behavior of a SISO system is governed by the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = x(t) + \frac{\mathrm{d}x}{\mathrm{d}t},$$

where $x(t)$ is the input signal and $y(t)$ the output signal.

(a) Determine the impulse response $h(t)$ of the system.

(b) Determine the output signal $y(t)$ in case the input signal is given by

$$x(t) = (1-t)e^{-t}u(t)$$

and the system is initially at rest.

(c) Determine $\lim_{t\to\infty} y(t)$ in case the input signal is given by

$$x(t) = 2\,u(t)$$

and the system is initially at rest.

Solution:

2.5p (a) Take $x(t) = \delta(t)$, apply a one-sided Laplace transform to the differential equation, and take the vanishing initial conditions into account. This gives

$$(s^2 + 2s + s)H(s) = s + 1$$

from which we obtain

$$H(s) = \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1}, \quad \mathrm{Re}(s) > -1.$$

Using the table, we find $h(t) = e^{-t}\cos(t)u(t)$.

2.5p (b) The one-sided Laplace transform of $x(t)$ is

$$X(s) = \frac{s}{(s+1)^2}, \quad \mathrm{Re}(s) > -1$$

and therefore

$$Y(s) = H(s)X(s) = \frac{s}{[(s+1)^2+1](s+1)}$$
$$= \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} - \frac{1}{s+1}, \quad \mathrm{Re}(s) > -1.$$

Using the table, we find $y(t) = e^{-t}[\cos(t) + \sin(t) - 1]u(t)$.

2p (c) $\lim_{t\to\infty} y(t) = 1$ follows directly from the differential equation or from Abel's final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 1.$$

Question 2 (5 points)

Determine the inverse Laplace transforms of

(a) $X(s) = \dfrac{s^2+2}{s^3-s}, \quad \text{Re}(s) > 1.$

(b) $Y(s) = \dfrac{2}{s^2(s-1)}, \qquad \text{Re}(s) > 1.$

Solution:

2.5p (a)

$$X(s) = -\frac{2}{s} + \frac{3}{2}\frac{1}{s-1} + \frac{3}{2}\frac{1}{s+1}, \quad \text{Re}(s) > 1.$$

Using the table, we find $x(t) = [-2 + \frac{3}{2}(e^t + e^{-t})]u(t)$.

2.5p (b)

$$Y(s) = -\frac{2}{s^2} - \frac{2}{s} + 2\frac{1}{s-1}, \quad \text{Re}(s) > 1.$$

Using the table, we find $y(t) = 2(e^t - t - 1)u(t)$.

Question 3 (4 points)

A periodic signal $x(t)$ with a fundamental period T_0 has a Fourier series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\alpha}{\beta + (k\pi)^2} e^{\mathrm{j}k\pi t} \quad \text{with } \alpha > 0 \text{ and } \beta > 0.$$

(a) What is the fundamental period T_0?

(b) What is the average value of $x(t)$?

(c) Is $x(t)$ even, odd, or neither? Motivate your answer.

One of the harmonics of $x(t)$ is expressed as $a\cos(4\pi t)$.

(d) What is a?

Solution:

1p (a) $T_0 = 2$.

1p (b) Average value is $X_0 = \alpha/\beta$.

1p (c) $x(t)$ is even.

1p (d) Clearly, this harmonic corresponds to the $k = \pm 4$ terms:

$$a\cos(4\pi t) = X_{-4}e^{-\mathrm{j}4\pi t} + X_4 e^{\mathrm{j}4\pi t} = 2X_4\cos(4\pi t).$$

We observe that $a = 2X_4 = \frac{2\alpha}{\beta + 16\pi^2}$.

Question 4 (4 points)

(a) Determine the z-transform for the following discrete-time signal, also specify the ROC:

$$x[n] = \frac{1}{2}\left(1 + (-1)^n\right) u[n]\,.$$

(b) Assuming $x[n]$ is causal, determine the signal $x[n]$ corresponding to the z-transform:

$$X(z) = \frac{z^{-2} + 10z^{-1}}{z^{-2} + 6z^{-1} + 8}\,.$$

(c) Determine the signal $x[n]$ corresponding to the DTFT

$$X(e^{j\omega}) = e^{j\pi/4}\delta(\omega - 1) + e^{-j\pi/4}\delta(\omega + 1)\,.$$

Answer:

1p (a) $X(z) = \frac{1}{2}\frac{1}{1 - z^{-1}} + \frac{1}{2}\frac{1}{1 + z^{-1}} = \frac{1}{1 - z^{-2}}\,, \quad |z| > 1.$

2p (b)

$$X(z) = \frac{z^{-2} + 10z^{-1}}{z^{-2} + 6z^{-1} + 8} = 1 + \frac{4z^{-1} - 8}{z^{-2} + 6z^{-1} + 8} = 1 - \frac{8}{z^{-1} + 2} + \frac{12}{z^{-1} + 4} = 1 - \frac{4}{1 + \frac{1}{2}z^{-1}} + \frac{3}{1 + \frac{1}{4}z^{-1}}$$

$$x[n] = \delta[n] - 4\left(-\frac{1}{2}\right)^n u[n] + 3\left(-\frac{1}{4}\right)^n u[n]\,.$$

Since $x[0] = 0$, we can simplify to

$$x[n] = -4\left(-\frac{1}{2}\right)^n u[n-1] + 3\left(-\frac{1}{4}\right)^n u[n-1]\,.$$

Alternatively, in the first step write

$$X(z) = z^{-1} \cdot \frac{z^{-1} + 10}{z^{-2} + 6z^{-1} + 8} = z^{-1}\left(\frac{4}{z^{-1} + 2} - \frac{3}{z^{-1} + 4}\right) = z^{-1}\left(\frac{2}{1 + \frac{1}{2}z^{-1}} - \frac{3/4}{1 + \frac{1}{4}z^{-1}}\right)$$

resulting in

$$x[n] = 2\left(-\frac{1}{2}\right)^{n-1} u[n-1] - \frac{3}{4}\left(-\frac{1}{4}\right)^{n-1} u[n-1]$$

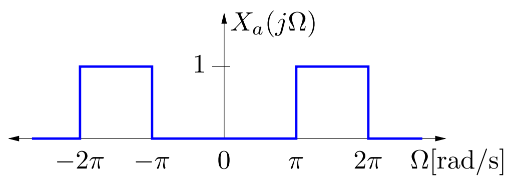
This is the same as the previous answer.

1p (c) $\frac{1}{\pi}\cos(n + \frac{\pi}{4})\,.$

Obtained by using $e^{j\omega_0 n} \leftrightarrow 2\pi\delta(\omega - \omega_0)$, hence $e^{j\pi/4}\delta(\omega - 1) \leftrightarrow \frac{1}{2\pi}e^{j(n+\pi/4)}$.

Question 5 (5 points)

The continuous-time signal $x_a(t)$ has a (real-valued) Fourier transform $X_a(j\Omega)$ as shown here:



(a) Is this a bandlimited signal? What is its maximal frequency (in Hz)?

(b) At what sampling period T_s should we sample this signal, avoiding aliasing?

We sample the signal at $T_s = \frac{1}{2}$ sec, and obtain the discrete-time signal $x[n]$. The DTFT spectrum of $x[n]$ is $X(e^{j\omega})$.

(c) How is ω related to Ω?

(d) Draw $X(e^{j\omega})$; also indicate the values on both axes.

Consider now the signal $y_a(t) = (x_a(t))^2$.

(e) Make a plot of $Y_a(j\Omega)$; also indicate the values on the horizontal axis.

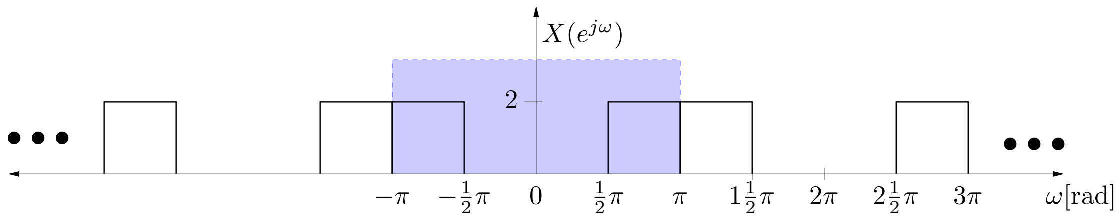
Answer:

1p (a) Yes; since $\Omega = 2\pi F$, the maximal frequency is $F = 1$ Hz.

1p (b) The sample rate should be at least $F_s = 2$ Hz, so $T_s \leq \frac{1}{2}$ s.

1p (c) $\omega = \Omega T_s$.

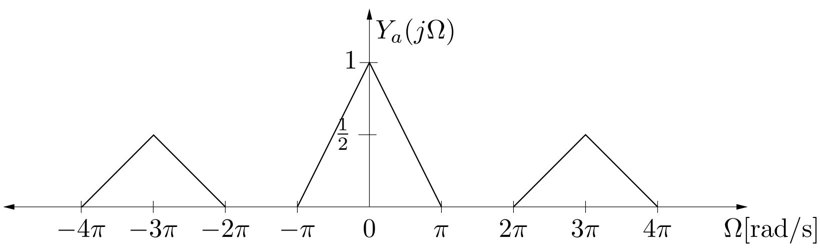
1p (d) $X_s(j\Omega) = \frac{1}{T_s} \sum X_a(j(\Omega - k\Omega_s))$ with $T_s = 1/2, \Omega_s = 2\pi/T_s = 4\pi$.



1p (e) $y_a(t) = x_a^2(t) = x_a(t) \cdot x_a(t) \quad \leftrightarrow \quad Y(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * X_a(j\Omega)$.

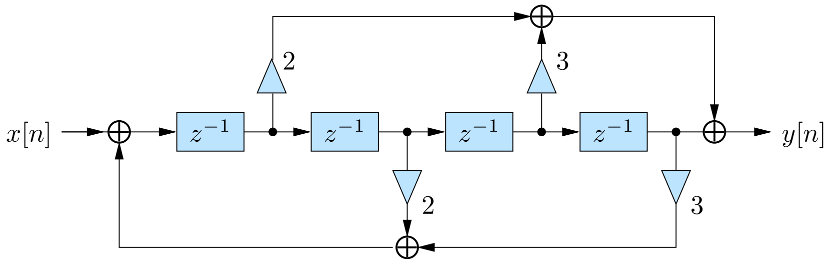
As $X_a(j\Omega)$ consists of two "bricks", the convolution produces 4 terms. Each has a triangular shape (cf. the convolution of a brick with itself). Two of the terms will center at $\Omega = 0$ hence the amplitude will be double at that place (as we assumed a real-valued spectrum here). At $\Omega = 0$, the convolution $X_a(j\Omega) * X_a(j\Omega)$ will be equal to the energy in the brick, which is π. Thus, the resulting amplitude of the triangle is $2\pi/2\pi = 1$.

[Note that the width of each triangle is twice the width of the brick. Thus, we would have to sample the squared signal at twice the rate needed for $x_a(t)$.]



Question 6 (4 points)

Consider the following system realization:



(a) Determine the transfer function $H(z)$.

(b) Determine all poles and zeros of $H(z)$, and draw a pole-zero diagram.

(c) Is this a stable system? (Why?)

(d) Is this a minimal realization? (Why?)

Answer:

2p (a) $H(z) = \dfrac{2z^{-1} + 3z^{-3} + z^{-4}}{1 - 2z^{-2} - 3z^{-4}} = \dfrac{2z^3 + 3z + 1}{z^4 - 2z^2 - 3}$

1p (b) To determine the poles and zeros, work on the version of $H(z)$ using "z" (not "z^{-1}").

Poles: $(z^2 - 3)(z^2 + 1) = 0$ so that $p_{1,2} = \pm\sqrt{3}$, $p_{3,4} = \pm j$.

Zeros: unintentionally, this realization was in a form that makes this too hard to compute without a computer. Nonetheless, you can note that there is a zero at $z = \infty$.

0.5p (c) Not stable, poles outside the unit circle.

0.5p (d) Minimal: 4th order and used 4 delays (in fact it is in Direct form no. II).

Question 7 (6 points)

We would like to design a digital *high*pass filter with the following specifications:

- Passband: starting at 3.0 kHz; ripple in the passband: ≤ 1 dB
- Stopband: below 2.0 kHz; stopband damping: ≥ 40 dB
- Sample rate: 12 kHz

The digital filter will be designed by applying the bilinear transform on an analog transfer function. We will use a Chebyshev filter. The expression for the frequency response of a prototype n-th order low-pass Chebyshev filter is

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)} .$$

(a) What are the passband and stopband frequencies (in rad) in the digital time domain?

(b) What are the filter specifications in the analog time domain?

(c) Which transformation will you use to transform a lowpass into a highpass filter? What are the specifications of the analog lowpass filter?

(d) Compute the required filter order n for a Chebyshev filter.

(Remark: $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$.)

(e) Suppose that the resulting Chebyshev lowpass filter has this form:

$$H(s) = \frac{B(s)}{A(s)} = \frac{1}{a_0 + a_1 s + \cdots + a_n s^n} \,.$$

How can you find the filter coefficients of the digital *high*pass filter?

Answer:

1p (a)

$$\omega_p = \frac{3}{12} 2\pi = \frac{1}{2}\pi \,, \qquad \omega_s = \frac{2}{12} 2\pi = \frac{1}{3}\pi$$

1p (b)

$$\text{passband: } \Omega_p = \tan(\frac{\omega_p}{2}) = 1 \,, \qquad \text{stopband: } \Omega_s = \tan(\frac{\omega_p}{2}) = .5774$$

with the same damping specs as before.

1p (c)

$$s \to \frac{1}{s} \,, \qquad \Omega \to \frac{1}{\Omega}$$

(There are other options, more generally $s \to \frac{\Omega_0}{s}$, you will have to take this into account later in item e.)

Specs for the lowpass filter:

- $\Omega'_p = 1$, ripple in passband smaller than 1 dB
- $\Omega'_s = 1/0.5774 = 1.7321$, damping in stopband larger than 40 dB

2p (d) Passband (using $T_n^2(1) = 1$ for any n):

$$\frac{1}{1+\epsilon^2} = \delta_p^2 = (10^{-1/20})^2 \quad \Rightarrow \quad \epsilon = 0.5087$$

Stopband:

$$\frac{1}{1+\epsilon^2 T_n^2(\Omega'_s/\Omega'_p)} = \delta_s^2 = (10^{-40/20})^2 = 10^{-4} \quad \Rightarrow \quad T_n^2(\Omega'_s/\Omega'_p) = \frac{10^4 - 1}{\epsilon^2} = 38640$$

Next use (for $\Omega > 1$) that $T_n(\Omega) = \cosh(n \cosh^{-1}(\Omega))$. This results in

$$n = \frac{\cosh^{-1}(\sqrt{38640})}{\cosh^{-1}(1.7321)} = 5.2126$$

So we should take $n = 6$.

1p (e) First apply the lowpass to highpass transform: $s \to \frac{1}{s}$, this results in

$$H_{HP}(s) = \frac{1}{A(\frac{1}{s})} = \frac{s^n}{a_0 s^n + \cdots + a_n}$$

Then insert the bilinear transform $s = \frac{1-z^{-1}}{1+z^{-1}}$, this results in

$$H_{HP}(z) = \frac{(1-z^{-1})^n}{a_0(1-z^{-1})^n + a_1(1-z^{-1})^{n-1}(1+z^{-1}) + \cdots + a_n(1+z^{-1})^n}$$

The filter coefficients are found by writing out the numerator and the denominator as a polynomial in z^{-1}.