Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science

EE2S11 SIGNALS AND SYSTEMS

Resit exam, 10 July 2023, 13:30–16:30

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of seven questions (31 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each answer sheet.

Question 1 (4 points)

Determine the one-sided Laplace transform of the following signals and specify their ROC as well. *Note:* $u(t)$ denotes the Heaviside unit step function.

(a) $f(t) = \delta(2t)$

(b) $g(t) = t\,u(t)$

(c) $m(t) = (t-1)u(t-1)$

(d) $n(t) = (t-1)u(t)$

Solution

1p (a) $F(s) = 1/2$, ROC $= \mathbb{C}$

0.5p (b) $G(s) = 1/s^2$, ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$

1p (c) $M(s) = e^{-s}/s^2$, ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$

1.5p (d) $N(s) = 1/s^2 - 1/s$, ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$

Question 2 (4 points)

Find the inverse Laplace transform of the following signals:

(a) $F(s) = \dfrac{4s+20}{s^2+4s+13}, \quad \mathrm{Re}(s) > -2.$

(b) $G(s) = \dfrac{1}{(s^2+4)(s^2+9)}, \quad \mathrm{Re}(s) > 0.$

Solution

2p (a) $F(s) = \dfrac{4(s+2)+12}{(s+2)^2+3^2} \quad \longleftrightarrow \quad f(t) = 4e^{-2t}(\cos 3t + \sin 3t)u(t)$

2p (b) $G(s) = \dfrac{1}{5}\dfrac{1}{s^2+4} - \dfrac{1}{5}\dfrac{1}{s^2+9} \quad \longleftrightarrow \quad g(t) = \left(\dfrac{1}{10}\sin 2t - \dfrac{1}{15}\sin 3t\right)u(t)$

Question 3 (6 points)

Let $x(t)$ be a periodic signal with period $T_0 = 1$. A single period of $x(t)$ is denoted by $x_1(t)$ and is given by

$$x_1(t) = \begin{cases} \sin(2\pi t) & 0 \leq t \leq 0.5 \\ 0 & 0.5 < t \leq 1. \end{cases}$$

(a) Determine the one-sided Laplace transform of $x_1(t)$ and its ROC.

(b) Determine the DC component of the periodic signal $x(t)$.

(c) Determine the Fourier coefficient X_1 of the periodic signal $x(t)$.

(d) Determine the Fourier coefficients X_k for k even and $k \geq 2$.

(e) Determine the Fourier coefficients X_k for k odd and $k \geq 2$.

Solution

2p (a) $X_1(s) = \dfrac{2\pi}{s^2 + 4\pi^2}(1 + e^{-0.5s}), \quad \text{ROC} = \mathbb{C}$

1p (b) $X_0 = \dfrac{1}{\pi}$

1p (c) $X_1 = \dfrac{1}{4\mathrm{j}}$

1p (d) $X_k = \dfrac{1}{\pi - k^2}$, $k \geq 2$ and k even

1p (e) $X_k = 0$, $k \geq 2$ and k odd

Question 4 (6 points)

(a) Given the signal $x[n] = [\cdots, 0, \boxed{3}, 2, 1, 0, 0, \cdots]$, where the 'box' denotes the value for $n = 0$. Determine $r[n] = x[n] * x[-n]$ using the convolution sum.

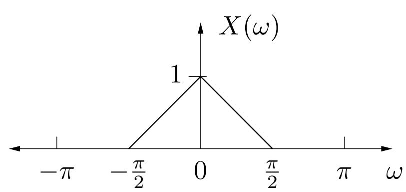
Determine the z-transform for the following discrete-time signal, and specify also the ROC:

(b) $x[n] = (2)^n\, u[-n]$,

Determine the signal $x[n]$ corresponding to the following z-transform:

(c) $X(z) = \dfrac{2z^2}{(z-1)(z-2)}, \qquad \text{ROC} = \{1 < |z| < 2\}.$

The signal $x[n]$ is specified by its DTFT (here we assume that $X(\omega)$ is real-valued):



(d) Determine and draw the DTFT of $x_1[n] = x[n]\cos(\pi n/4)$.

Note: specify the DTFT in terms of $X(\omega)$.

Solution

2p (a) Let $y[n] = x[-n] = [\cdots, 0, 1, 2, \boxed{3}, 0, 0, \cdots]$, then $r[n] = \sum_{k=1}^{2} x[k]y[n-k]$,

$k=0:$	$3y[n]:$	3	6	$\boxed{9}$	0	0	$0\cdots$
$k=1:$	$2y[n-1]:$	0	2	$\boxed{4}$	6	0	$0\cdots$
$k=2:$	$1y[n-2]:$	0	0	$\boxed{1}$	2	3	$0\cdots$
	$r[n]:$	3	8	$\boxed{14}$	8	3	$0\cdots$

1p (b) $X(z) = 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \cdots = \dfrac{1}{1-\frac{1}{2}z}$; ROC $= \{|z| < 2\}$.

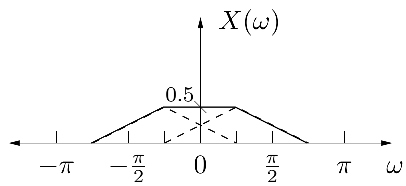
2p (c)

$$X(z) = \frac{-2z}{z-1} + \frac{4z}{z-2} = \underbrace{\frac{2}{1-z^{-1}}}_{\text{ROC:}|z|>1} - \underbrace{\frac{2z}{1-\frac{1}{2}z}}_{\text{ROC:}|z|<2}$$

The first term corresponds to a causal signal and the second to an anticausal signal. Hence

$$x[n] = 2u[n] - 2^{n+2}u[-n-1]$$

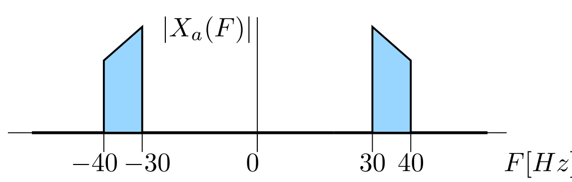
1p (d) $X_1(\omega) = \frac{1}{2}X(\omega - \frac{\pi}{4}) + \frac{1}{2}X(\omega + \frac{\pi}{4})$.



Question 5 (3 points)

A continuous-time signal $x_a(t)$ has frequencies in the range 30 until 40 Hz. The signal is sampled with period T so that we obtain a time series $x[n] = x_a(nT)$.

The amplitude spectrum of $x_a(t)$ appears as follows:



(a) What is the Nyquist frequency at which we would have to sample to avoid any aliasing?

(b) We sample the signal at 40 Hz. Make a drawing of the amplitude-spectrum $|X(\omega)|$ resulting from this sample frequency. Also mark the frequencies.

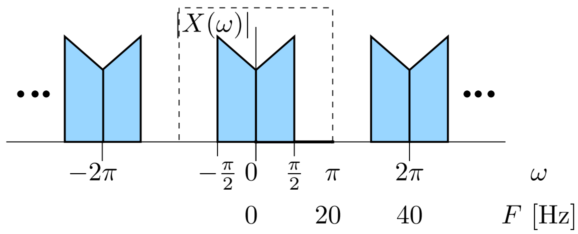
Solution

1p (a) Twice the highest frequency, 80 Hz.

2p (b) The spectrum is periodic with period $F_s = 40$ Hz, take all shifts at multiples of 40 Hz and add these.

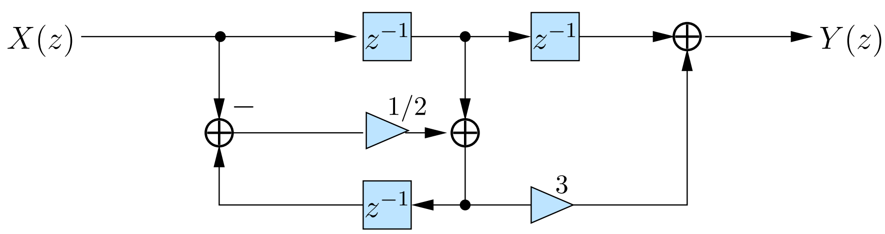
The component between 30 and 40 Hz also appears between -10 and 0 Hz.

The component between -30 and -40 Hz also appears between 0 and 10 Hz. Further, the sample frequency $F = 40$ Hz corresponds to $\omega = 2\pi$.



Question 6 (3 points)

Given the realization



(a) Determine the transfer function $H(z)$ corresponding to this realization.

(b) Is this a stable realization? (Why?)

(c) Is this a minimal realization? (Why?)

Solution

2p (a) Introduce an extra parameter $P(z)$ to the input of the multiplier "3". We obtain

$$\begin{cases} P(z) &= z^{-1}X(z) - \frac{1}{2}X(z) + \frac{1}{2}z^{-1}P(z) \\ Y(z) &= z^{-2}X(z) + 3P(z) \end{cases}$$

An expression for $P(z)$ is

$$P(z)(1 - \frac{1}{2}z^{-1}) = X(z)(z^{-1} - \frac{1}{2})$$

$$P(z) = X(z)\frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}}$$

Eliminate $P(z)$ in the expression for $Y(z)$, this results

$$\begin{aligned} Y(z) &= X(z)\left(z^{-2} + \frac{3(z^{-1}-\frac{1}{2})}{1-\frac{1}{2}z^{-1}}\right) \\ &= X(z)\frac{-\frac{1}{2}z^{-3}+z^{-2}+3z^{-1}-\frac{3}{2}}{1-\frac{1}{2}z^{-1}} \end{aligned}$$

Hence

$$H(z) = \frac{-\frac{1}{2}z^{-3} + z^{-2} + 3z^{-1} - \frac{3}{2}}{1 - \frac{1}{2}z^{-1}}$$

0.5p (b) The pole is located at $z = \frac{1}{2}$, within the unit circle, hence stable.

0.5p (c) The highest degree (filter order) is 3, the number of delays is 3, hence minimal.

Question 7 (5 points)

Design an analog 3rd order low-pass Butterworth filter $H(\Omega)$ with a passband frequency of 2 rad/s, a stopband frequency of 4 rad/s and a maximal damping in the passband of 1 dB.

(a) What is the general expression for the frequency response (squared-amplitude) of a prototype Butterworth low-pass filter.

(b) Determine the unknown filter parameters such that the specifications are met.

(c) For this filter, what is the minimal damping in the stopband ?

(d) We wish to transform this filter to a *high*-pass filter $G(\Omega)$ with a passband frequency $\Omega'_p = 2$ rad/s. What transformation do we use, what is the resulting frequency response $|G(\Omega)|^2$, and what is the resulting stopband frequency?

Solution

1p (a) $|H(\Omega)|^2 = \dfrac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$.

1p (b) At $\Omega_p = 2$ we find $|H(\Omega_p)|^2 = 10^{-1/10}$.

$$\frac{1}{1+\epsilon^2} = 10^{-1/10} \qquad \Rightarrow \qquad \epsilon = \sqrt{10^{1/10}-1} = 0.5088$$

We already know that $N = 3$.

1p (c) At $\Omega_s = 4$ we find

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(4/2)^6} = 0.0569$$

This is -12.5 dB.

2p (d)

$$\begin{aligned} \Omega &\rightarrow \frac{\Omega_p\Omega'_p}{\Omega} = \frac{4}{\Omega} \\ |G(\Omega)|^2 &= \frac{1}{1+\epsilon^2(2/\Omega)^6} \\ \Omega'_s &= \frac{\Omega_p\Omega'_p}{\Omega_s} = \frac{4}{4} = 1 \end{aligned}$$