Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Resit exam EE2S11 SIGNALS & SYSTEMS
July 19, 2021
Block 1 (13:30-15:00)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:55–15:10

This block consists of three questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (11 points)

Given a causal time-domain signal $i(t)$. On its ROC, the one-sided Laplace transform of $i(t)$ is given by

$$I(s) = \frac{s-2}{s^2+2s+2}\,.$$

(a) What is its ROC?

(b) Determine $i(0^+)$.

(c) Determine $i(t)$.

(d) Compute $\frac{\mathrm{d}i}{\mathrm{d}t}$.

(e) Determine the inverse Laplace transform of

$$U(s) = \frac{s^2-2s}{s^2+2s+2}\,,$$

which has the same ROC as $I(s)$.

Solution

(a) Poles of $I(s)$ are $s = -1 \pm \mathrm{j}$. Its ROC is $\mathrm{ROC} = \{s \in \mathbb{C}; \mathrm{Re}(s) > -1\}$.

(b) $i(0^+) = \lim_{s\to\infty} sI(s) = 1\,.$

(c)

$$I(s) = \frac{s-2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - 3\frac{1}{(s+1)^2+1}\,.$$

Use table of Laplace transforms: $i(t) = e^{-t}[\cos(t) - 3\sin(t)]u(t)$.

(d)

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \delta(t) - 2e^{-t}[2\cos(t) - \sin(t)]u(t)\,.$$

(e) Directly: $u(t) = \frac{\mathrm{d}i}{\mathrm{d}t}$.

$$U(s) = \frac{s^2 - 2s}{s^2 + 2s + 2} = 1 - 4\frac{s+1}{(s+1)^2 + 1} + 2\frac{1}{(s+1)^2 + 1}\,.$$

Use table of inverse Laplace transforms to arrive at $u(t) = \delta(t) - 2e^{-t}[2\cos(t) - \sin(t)]u(t)$.

Question 2 (5 points)

Given the signal $x(t) = te^{-\alpha t}u(t)$ with $\alpha > 0$.

(a) Determine the convolution $y(t) = x(t) * x(t)$ directly using the convolution integral.

(b) Determine the convolution $y(t) = x(t) * x(t)$ using the Laplace transform.

Solution

(a) $x(t)$ is causal so $y(t)$ is causal as well: $y(t) = 0$ for $t < 0$. For $t > 0$ we have

$$y(t) = \int_{\tau=0}^{t} (t-\tau)e^{-\alpha(t-\tau)}\tau e^{-\alpha\tau}\,\mathrm{d}\tau = e^{-\alpha t}\int_{\tau=0}^{t} (t-\tau)\tau\,\mathrm{d}\tau = \frac{1}{6}e^{-\alpha t}t^3\,.$$

In total: $y(t) = \frac{1}{6}e^{-\alpha t}t^3u(t)$, $t \in \mathbb{R}$.

(b) $X(s) = \dfrac{1}{(s+\alpha)^2}$ with ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > -\alpha\}$. Furthermore, $Y(s) = X(s) \cdot X(s) = \dfrac{1}{(s+\alpha)^4}$ on the above ROC. Table of Laplace transforms gives

$$y(t) = \frac{1}{3!}e^{-\alpha t}t^3u(t)\,.$$

Question 3 (9 points)

A periodic signal $x(t)$ with a fundamental period T_0 has a Fourier series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\alpha}{\beta + (k\pi)^2}e^{\mathrm{j}k\pi t} \quad \text{with } \alpha > 0 \text{ and } \beta > 0.$$

(a) What is the fundamental period T_0?

(b) What is the average value of $x(t)$?

(c) Is $x(t)$ even, odd, or neither? Motivate your answer.

One of the harmonics of $x(t)$ is expressed as $a\cos(4\pi t)$.

(d) What is a?

Solution

(a) $T_0 = 2$

(b) Average value is $X_0 = \alpha/\beta$.

(c) $x(t)$ is even.

(d) Clearly, this harmonic corresponds to the $k = \pm 4$ terms:

$$a\cos(4\pi t) = X_{-4}e^{-\mathrm{j}4\pi t} + X_4 e^{\mathrm{j}4\pi t} = 2X_4\cos(4\pi t)\,.$$

We observe that $a = 2X_4 = \dfrac{2\alpha}{\beta + 16\pi^2}$.

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Resit exam EE2S11 SIGNALS & SYSTEMS
July 19, 2021
Block 2 (15:25-16:55)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 16:50–17:05

This block consists of four questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 4 (9 points)

(a) Given the signals $x[n] = [\cdots, 0, 1, 2, \boxed{3}, 0, 0, \cdots]$ and $h[n] = [\cdots, \boxed{0}, 2, 1, 0, \cdots]$, where the 'box' denotes the value for $n = 0$.

Determine $y[n] = h[n] * x[n]$ using the convolution sum.

(b) Given an input signal $x[n] = \left(\frac{1}{3}\right)^{|n|}$. Determine the z-transform $X(z)$, also specify the ROC.

(c) Let $h[n] = (\frac{1}{2})^n u[n]$ be the impulse response of a filter, let

$$Y(z) = \frac{2z^2}{2z^2 + z - 1}\,, \qquad \text{ROC: } |z| > 1\,,$$

and let $y[n]$ be the corresponding signal.

Compute the input signal $x[n]$ for which this $y[n]$ is the output of the filter.

(d) A filter $H(z)$ is called *allpass* if its magnitude response (amplitude spectrum) is constant over frequency.

Consider

$$H(z) = z^{-1}\frac{1 + 3z^{-2}}{3 + z^{-2}}\,, \qquad \text{ROC: } |z| > \frac{1}{\sqrt{3}}$$

Determine if $H(z)$ is an allpass filter.

(e) Compute all poles and zeros of $H(z)$ in (d) and draw a pole-zero plot.

Solution

(a) 2 pnt $y[n] = \sum_{k=1}^{2} h[k]x[n-k]$,

$$\begin{array}{rr|cccccc}
k=1: & 2x[n-1]: & 2 & \boxed{4} & 6 & 0 & 0 & 0\cdots \\
k=2: & x[n-2]: & 0 & \boxed{1} & 2 & 3 & 0 & 0\cdots \\
\hline
 & y[n]: & 2 & \boxed{5} & 8 & 3 & 0 & 0\cdots
\end{array}$$

(b) 2 pnt Write $x[n] = (\frac{1}{3})^n u[n] + (\frac{1}{3})^{-n} u[-n] - 1$, then

$$X(z) = \frac{1}{1-(\frac{1}{3})z^{-1}} + \frac{1}{1-\frac{1}{3}z} - 1 = \frac{\frac{1}{9}+\frac{1}{3}z^{-1}+\frac{1}{3}z}{\frac{10}{9}-\frac{1}{3}z^{-1}-\frac{1}{3}z}$$

ROC: $\frac{1}{3} < |z| < 3$.

(c) 2 pnt $Y(z) = X(z)H(z)$, with

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}\,.$$

Hence

$$\begin{aligned} X(z) &= [H(z)]^{-1}Y(z) \\ &= (1-\tfrac{1}{2}z^{-1})\,\frac{1}{1+\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}} \\ &= \frac{1-\frac{1}{2}z^{-1}}{(1+z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= \frac{1}{1+z^{-1}}\,, \end{aligned}$$

with ROC: $|z| > 1$. Then $x[n] = (-1)^n u[n]$.

(d) 2 pnt

$$H(e^{j\omega}) = e^{-j\omega}\frac{1+3e^{-2j\omega}}{3+e^{-2j\omega}}$$

$$[H(e^{j\omega})]^* = e^{j\omega}\frac{1+3e^{2j\omega}}{3+e^{2j\omega}}$$

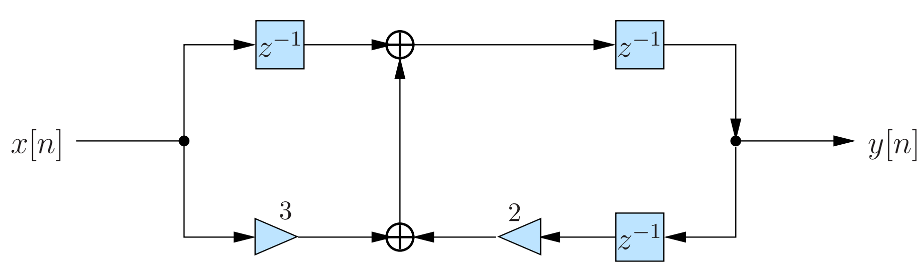
$$\begin{aligned} |H(e^{j\omega})|^2 = H(e^{j\omega})[H(e^{j\omega})]^* &= e^{-j\omega}e^{j\omega}\cdot\frac{(1+3e^{-2j\omega})(1+3e^{2j\omega})}{(3+e^{-2j\omega})(3+e^{2j\omega})} \\ &= 1\cdot\frac{1+9+3e^{-2j\omega}+3e^{2j\omega}}{9+1+3e^{-2j\omega}+3e^{2j\omega}} \\ &= 1 \end{aligned}$$

(e) 1 pnt Poles: $z = 0$, $z = \pm j\frac{1}{\sqrt{3}}$.

Zeros: $\pm j\sqrt{3}$, ∞.

Question 5 (5 points)

Consider the following system realization:



(a) Determine the transfer function $H(z)$.

(b) Is this a minimal realization? (Why?)

(c) Draw the corresponding Direct Form no. 2 realization.

(d) Determine $h[1]$, the impulse response at time $n = 1$.

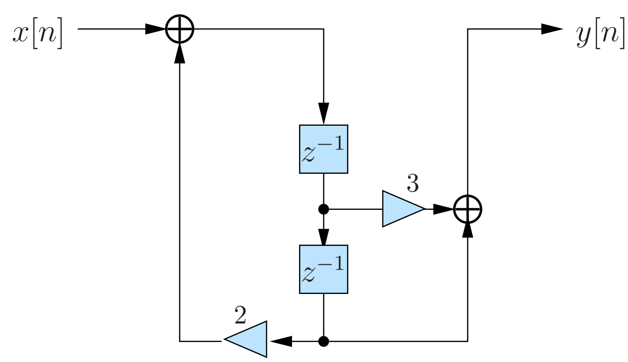
Solution

(a) 2 pnt

$$H(z) = \frac{3z^{-1} + z^{-2}}{1 - 2z^{-2}}$$

(b) 1 pnt Not minimal: 2nd order transfer function, and used 3 delays.

(c) 1 pnt



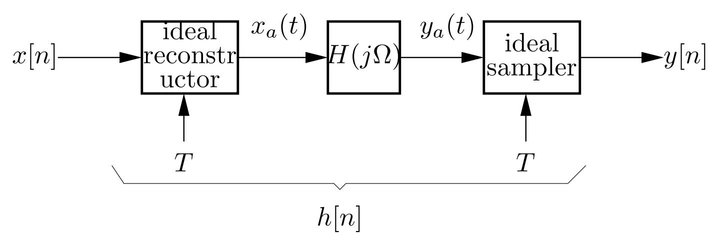
(d) 1 pnt Give an impulse at time $n = 0$, then follow how this propagates through the system. The top-left delay stores "1", the top-right delay stores "3", and the bottom-right delay stores "0". Then at time $n = 1$, the output will be 3. Hence $h[1] = 3$.

Alternatively: Write $H(z) = (3z^{-1}+z^{-2})(1+2z^{-2}+4z^{-4}+\cdots) = 3z^{-1}+\cdots$ (higher order terms), which shows that $h[1] = 3$.

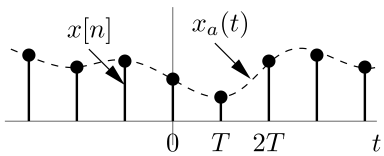
Question 6 (6 points)

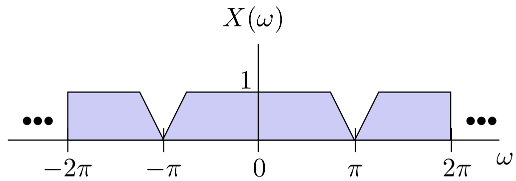
We have a discrete time sequence $x[n]$, and wish to implement a delay: $y[n] = x[n - \Delta]$. If Δ is not an integer, this has no formal meaning as we cannot shift the sequence $x[n]$ by anything but an integer.

To implement the effect of a non-integer delay, we consider the following setup:



The signal is first reconstructed (D/A conversion including an ideal reconstruction filter) assuming a certain sampling period of T, resulting in $x_a(t)$. Next, a suitable continuous-time filter $H(j\Omega)$ is applied, and the resulting signal $y_a(t)$ is sampled again with period T so that we obtain the series $y[n] = y_a(nT)$.





(a) The spectrum $X(\omega)$ corresponding to $x[n]$ is drawn schematically above. Sketch the spectrum $X_a(\Omega)$ after ideal reconstruction. (Also indicate values on the horizontal and vertical axes.)

(b) Express $x_a(t)$ in terms of $x[n]$.

(c) Relate $y_a(t)$ to $x_a(t)$ in an equation.

Based on this, specify $H(j\Omega)$ such that the desired delay is obtained.

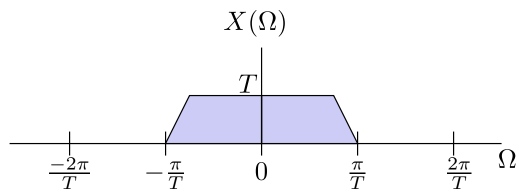
(d) Express $y[n]$ in terms of $x[n]$ in an equation.

Based on this, specify the equivalent discrete-time filter $h[n]$ that implements the non-integer delay.

(e) How should T be selected?

Solution

(a) 1 pnt



(b) 1 pnt

$$x_a(t) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(t-kT)/T]}{\pi(t-kT)/T}$$

(c) 1 pnt $y_a(t) = x_a(t - \Delta T)$, hence $H(j\Omega) = e^{-j\Delta T\Omega}$.

(d) 2 pnt

$$y[n] = y_a(nT) = x_a(nT - \Delta T) = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(nT-\Delta T-kT)/T]}{\pi(nT-\Delta T-kT)/T} = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n-\Delta-k)]}{\pi(n-\Delta-k)}$$

hence $y[n] = x[n] * h[n]$ with

$$h[n] = \frac{\sin[\pi(n-\Delta)]}{\pi(n-\Delta)}$$

(e) 1 pnt $T > 0$ can be selected arbitrarily. (While sampling, the Nyquist condition is always satisfied because $x_a(t)$ is automatically bandlimited due to the ideal reconstruction.)

Question 7 (5 points)

We use the bilinear transform to design a digital lowpass filter $H(z)$ with the following specifications:

– Passband: $0 \le |\omega| \le 0.25\pi$, maximal ripple 0.5 dB

– Stopband: $0.4\pi \le |\omega| \le \pi$, minimal damping 50 dB.

(a) Specify the passband and stopband frequencies for the design of the corresponding analog lowpass filter

(b) What is the required filter order if we use a Butterworth filter?

(c) Suppose $G(z) = H(-z)$. Give a plot of the magnitude response $|G(e^{j\omega})|$. Also specify values on both axes (derived from the specifications of $H(z)$).

Solution

(a) 1 pnt The bilinear transform relates $\Omega = \tan(\frac{\omega}{2})$. Hence

$$\Omega_p = \tan(0.25\pi/2) = 0.4142\,, \quad \Omega_s = \tan(0.4\pi/2) = 0.7265$$

(b) 2 pnt The expression for the magnitude of a Butterworth filter is

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$$

Hence, filling in the passband frequency,

$$\text{Passband ripple:} \quad \frac{1}{1+\epsilon^2} = 10^{-0.5/10} \quad \Rightarrow \quad \epsilon = 0.3493$$

$$\text{Stopband damping:} \quad \frac{1}{1+\epsilon^2(\Omega_s/\Omega_p)^{2N}} \le \delta_s^2 = 10^{-50/10}$$

Define $\delta = \sqrt{\frac{1}{\delta_s^2} - 1} = 316.2$. This gives

$$N \ge \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)} = 12.1$$

We will need to use $N = 13$.

(c) 2 pnt $G(e^{j\omega}) = H(-e^{j\omega}) = H(e^{j(\omega-\pi)})$. The frequency response of $G(z)$ is obtained by that of $H(z)$ by translating by π.

This will translate the lowpass filter into a highpass Butterworth. The passband will be $\pi - \omega_p = 0.75\pi$ (damping 0.5 dB). The stopband will be $\pi - \omega_s = 0.6\pi$ (damping 50 dB).