EE2S1 Signals and Systems

Exercises: Exam Part 2 of January 2024

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Question 1a

a) Given the signals
$x[n] = u[n+2] - u[n-2]\,, \quad h[n] = [\cdots, 0, \boxed{1}, -2, 0, 0, \cdots]\,.$
Determine $y[n] = x[n] * h[n]$ using the convolution sum (in time-domain).

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Determine $y[n] = x[n] * h[n]$ using the convolution sum (in time-domain).

$$y[n] = \sum_k h[k]x[n-k]$$

$h[0]x[n]$:	$\cdots$ 0	1	1	$\boxed{1}$	1	0	0	0 $\cdots$
$h[1]x[n-1]$:	$\cdots$ 0	0	-2	$\boxed{-2}$	-2	-2	0	0 $\cdots$
$y[n]$:	$\cdots$ 0	1	-1	$\boxed{-1}$	-1	-2	0	0 $\cdots$

Question 1b

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$$x[n] = u[n] - (\frac{1}{2})^4 (\frac{1}{2})^{n-4} u[n-4]$$

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{(\frac{1}{2})^4 z^{-4}}{1 - \frac{1}{2} z^{-1}}$$

ROC: $\{|z| > 1\}$

Question 1c

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$$x[n] = a^n u[n] \quad \rightarrow \quad X(z) = \frac{1}{1 - az^{-1}} \qquad \text{ROC: } \{|z| > a\}$$

$$x[-n] = a^{-n} u[-n] \quad \rightarrow \quad X(z^{-1}) = \frac{1}{1 - az} \qquad \text{ROC: } \{|z| < 1/a\}$$

Result of the convolution in z-domain is a product:

$$Y(z) = \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - az} \qquad \text{ROC: } \{a < |z| < 1/a\}$$

Question 1c (continued)

To recover $y[n]$, we need to apply a partial fraction expansion, therefore we first write the function using polynomials in z or z^{-1} (but not both). Hence:

$$Y(z) = \frac{z}{(z-a)(1-az)} = \frac{A}{z-a} + \frac{B}{1-az} = \frac{1}{1-a^2}\left(\frac{a}{z-a} + \frac{1}{1-az}\right)$$

Check the ROC to determine which part is causal and which part is anti-causal. The first term (with ROC: $\{|z| > a\}$) is causal and therefore we write it in terms of z^{-1}. The second term (with ROC: $\{|z| < 1/a\}$) is anti-causal and we keep it in terms of z. This results in:

$$Y(z) = \frac{1}{1-a^2}\left(\frac{az^{-1}}{1-az^{-1}} + \frac{1}{1-az}\right) = \frac{1}{1-a^2}\left(\frac{1}{1-az^{-1}} - 1 + \frac{1}{1-az}\right)$$

$$y[n] = \frac{1}{1-a^2}\left(a^n u[n] - \delta[n] + a^{-n} u[-n]\right) = \frac{a^{|n|}}{1-a^2}$$

Question 1

d) Determine, if it exists, the frequency response $H(e^{j\omega})$ for the system defined by the difference equation

$$y[n] = 1.6y[n-1] - 0.64y[n-2] + x[n] - x[n-2]$$

Question 1

d) Determine, if it exists, the frequency response $H(e^{j\omega})$ for the system defined by the difference equation

$$y[n] = 1.6y[n-1] - 0.64y[n-2] + x[n] - x[n-2]$$

First apply a z-transform:

$$Y(z)(1 - 1.6z^{-1} + 0.64z^{-2}) = X(z)(1 - z^{-2})$$

$$H(z) = \frac{1 - z^{-2}}{1 - 1.6z^{-1} + 0.64z^{-2}} = \frac{1 - z^{-2}}{(1 - 0.8z^{-1})^2} \qquad \text{ROC: } |z| > 0.8$$

The poles are $z_{1,2} = 0.8$ (double), the unit circle is in the ROC and the Fourier transform exists. This results in

$$H(e^{j\omega}) = \frac{1 - e^{-2j\omega}}{(1 - 0.8e^{-j\omega})^2}$$

Question 1

e) Given an LTI system with transfer function $H(z) = 1 - 2z^{-1}$. Determine a (bounded) input signal $x[n]$ for which the output signal is equal to $y[n] = \delta[n] + \frac{1}{2}\delta[n-1]$.

Question 1

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$$Y(z) = 1 + \frac{1}{2}z^{-1}$$

$$X(z) = \frac{Y(z)}{H(z)} = \frac{1 + \frac{1}{2}z^{-1}}{1 - 2z^{-1}}$$

Because we require a bounded $x[n]$, the ROC is $\{|z| < 2\}$ which gives an anti-causal sequence. Therefore we rewrite $X(z)$ as

$$X(z) = \frac{z(1 + \frac{1}{2}z^{-1})}{z - 2} = -\frac{1}{2}\frac{\frac{1}{2} + z}{1 - \frac{1}{2}z} = -\frac{1}{2}\left(\frac{1}{2} + \frac{5}{4}\frac{z}{1 - \frac{1}{2}z}\right)$$

$$x[n] = -\frac{1}{2}\left(\frac{1}{2}\delta[n] + \frac{5}{4}(\frac{1}{2})^{-n-1}u[-n-1]\right) = -\frac{1}{4}\delta[n] + \frac{5}{8}(\frac{1}{2})^{-n-1}u[-n-1]$$

Question 2

The transfer function of a causal LTI system is given by
$H(z) = \dfrac{z-1}{z(z+0.9)}$

a) Determine all poles and zeros of the system and make a drawing in the complex z-plane.

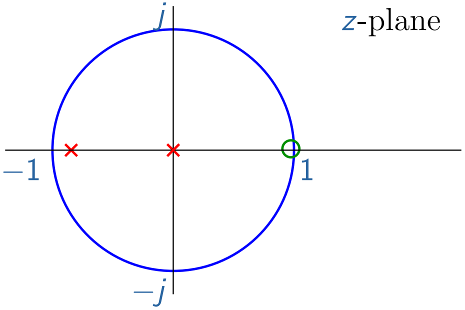
b) Specify the ROC.

c) Is the system BIBO stable? (Why?)

d) Draw, based on the poles and zeros of $H(z)$, the amplitude response. Is this a low-pass, high-pass or other kind of filter?

Question 2: Solutions

a) Poles: $z = 0$, $z = -0.9$. Zeros: $z = 1$, $z = \infty$.

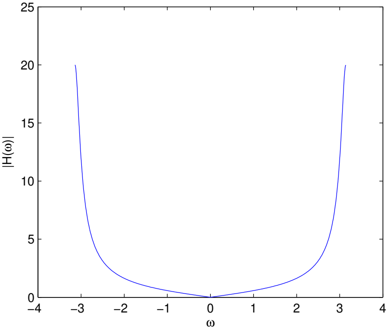


b) Causal results in ROC: $|z| > 0.9$.

c) Unit circle in ROC: BIBO stable.

Question 2: Solutions (continued)

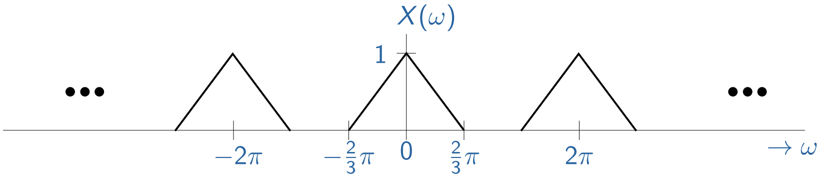
d)



High-pass. Zero at $z = 1$ results in $H(e^{j\omega}) = 0$ for $\omega = 0$. The pole at $z = -0.9$ results in a peak for $\omega = \pm\pi$. The pole at $z = 0$ only has an effect on the phase. Compute: $H(e^{j\pi}) = H(1) = 20$. The shown plot should be symmetric and either plot from $-\pi$ to π, or show periodicity outside this interval.

Question 3

An analog signal $x_a(t)$ with Fourier transform $X_a(\Omega)$ is band-limited at 10 kHz. The signal is sampled without aliasing at a sampling frequency F_s, resulting in the discrete-time signal $x[n]$. The spectrum $X(\omega)$ of $x[n]$ is shown below:



a) What is the relation between Ω and ω?
b) Which sampling frequency was used?
c) What is the smallest frequency at which we can sample $x_a(t)$ without aliasing?
For this case, draw the resulting spectrum.

Question 3: solutions

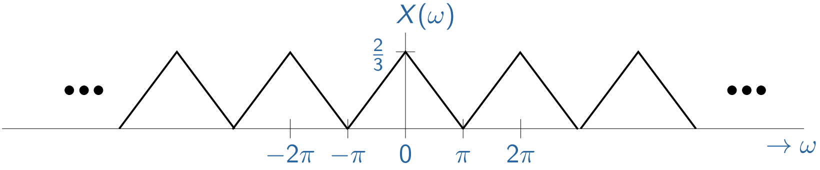
a) $\omega = \Omega T_s$.
This standard result could be rederived if you recall that
$\omega = 2\pi \quad \leftrightarrow \quad \Omega = 2\pi F_s$. This results in $\Omega = \omega F_s$ i.e., $\omega = \Omega T_s$.

b) $\omega = \frac{2}{3}\pi$ results in $\Omega = \frac{2}{3}\pi F_s$. At $F = 10$ kHz we find

$$F = \frac{\Omega}{2\pi} = \frac{\frac{2}{3}\pi F_s}{2\pi} = \frac{1}{3} F_s = 10\text{kHz}$$

hence $F_s = 30$ kHz.

c) $F_s = 20$ kHz.



(Regarding the peak amplitude: use that sampling at T_s will scale the amplitude by $1/T_s = F_s$.)

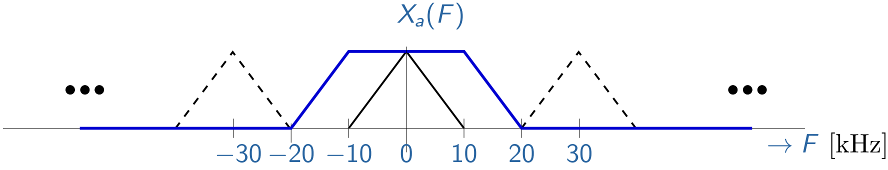
Question 3 (continued)

d) Consider the initial sampling rate. After sampling, $x_a(t)$ is reconstructed from $x[n]$ by means of an ideal D/A converter and a low-pass filter. Specify the pass-band and stop-band frequencies of the filter.

Question 3 (continued)

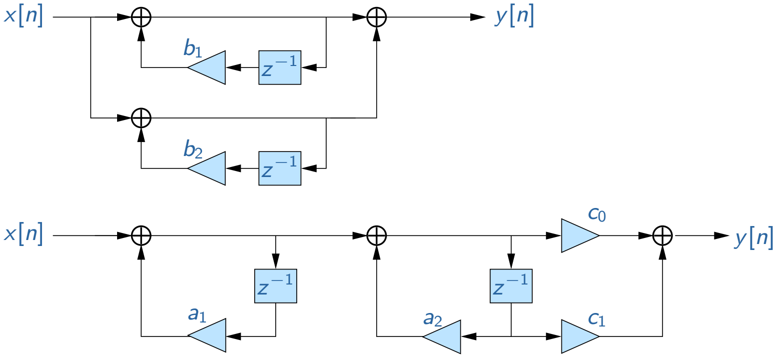
d) Consider the initial sampling rate. After sampling, $x_a(t)$ is reconstructed from $x[n]$ by means of an ideal D/A converter and a low-pass filter. Specify the pass-band and stop-band frequencies of the filter.

After D/A conversion, the signal is analog. In the frequency spectrum, the frequency $\omega_p = \frac{2}{3}\pi$ corresponds to $F_p = 10$ kHz, and the frequency $\omega_s = \frac{4}{3}\pi$ corresponds to $F_{stop} = 20$ kHz. The low-pass filter (in the analog domain! no periodicity) thus has a pass-band running until 10 kHz and a stop-band starting at 20 kHz.



Question 4

Given the realizations:



a) Determine a_1, a_2 and c_0, c_1 in terms of b_1, b_2 such that both systems are equivalent.

b) Are these minimal realizations?

Question 4: solutions

a) First realization:

$$H(z) = \frac{1}{1-b_1z^{-1}} + \frac{1}{1-b_2z^{-1}} = \frac{2-(b_1+b_2)z^{-1}}{(1-b_1z^{-1})(1-b_2z^{-1})}$$

Second realization:

$$H(z) = \frac{1}{1-a_1z^{-1}} \cdot \frac{c_0+c_1z^{-1}}{1-a_2z^{-1}} = \frac{c_0+c_1z^{-1}}{(1-a_1z^{-1})(1-a_2z^{-1})}$$

From this it follows that
$a_1 = b_1,\ a_2 = b_2,\ c_0 = 2,\ c_1 = -(b_0+b_1)$.

b) Both are minimal because the number of delays in the realization is equal to the filter order of $H(z)$.

Question 4: solutions (continued)

$$H(z) = \frac{2 - (b_1 + b_2)z^{-1}}{(1 - b_1 z^{-1})(1 - b_2 z^{-1})}$$

c) Draw the "direct form no. II" realization and also specify the coefficients.

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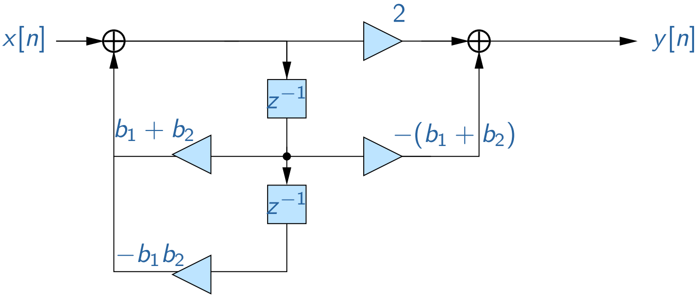

Question 5

A "template" third-order Butterworth filter has the transfer function

$$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

The corresponding frequency response is $|H(j\Omega)|^2 = \frac{1}{1+\Omega^6}$.

a) Which frequency transform should we apply to the template to construct a low-pass Butterworth filter with a 3dB cut-off frequency of Ω_c?

b) What is the corresponding transfer function $G(s)$?

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a) Substitute $\Omega \to \frac{\Omega}{\Omega_c}$.
b) Substitute $s \to \frac{s}{\Omega_c}$, this results in

$$G(s) = \frac{1}{(\frac{s}{\Omega_c})^3 + 2(\frac{s}{\Omega_c})^2 + 2(\frac{s}{\Omega_c}) + 1}$$

Question 5: (continued)

We now design an analog 3rd order low-pass Butterworth filter with a pass-band frequency of 3 rad/s, a stop-band frequency of 6 rad/s and a maximal damping in the pass-band of 0.5 dB.

c) Give a suitable expression for the frequency response (squared-amplitude) of this filter and determine its parameters.
d) For this filter, what is the minimal damping in the stop-band ?
e) Which transform should be applied to $|H(j\Omega)|^2$ to obtain this filter? Determine the corresponding transfer function.

Question 5: solutions

c) The general expression is

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2(\frac{\Omega}{\Omega_p})^6}$$

For $\Omega = \Omega_p = 3$ we obtain

$$\frac{1}{1+\epsilon^2} = 10^{-0.5/10} \quad \Rightarrow \quad \epsilon = 0.3493$$

d) For $\Omega = \Omega_s = 6$ we obtain

$$\frac{1}{1+\epsilon^2(\frac{6}{3})^6} = 0.1135 \doteq -9.45\,\mathrm{dB}$$

The stopband damping is 9.45 dB.

Question 5: solutions (continued)

e) First, we determine Ω_c:

$$(\frac{\Omega}{\Omega_c})^6 = \epsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^6 \quad \Rightarrow \quad \Omega_c = \frac{\Omega_p}{\epsilon^{1/3}} = 4.26\,\text{rad/s}$$

The transformation is $\Omega \to \frac{\Omega}{4.26} = 0.235\Omega$.
The transfer function of the requested Butterworth filter is:

$$H(s) = \frac{1}{(\frac{s}{4.26})^3 + 2(\frac{s}{4.26})^2 + 2(\frac{s}{4.26}) + 1}$$