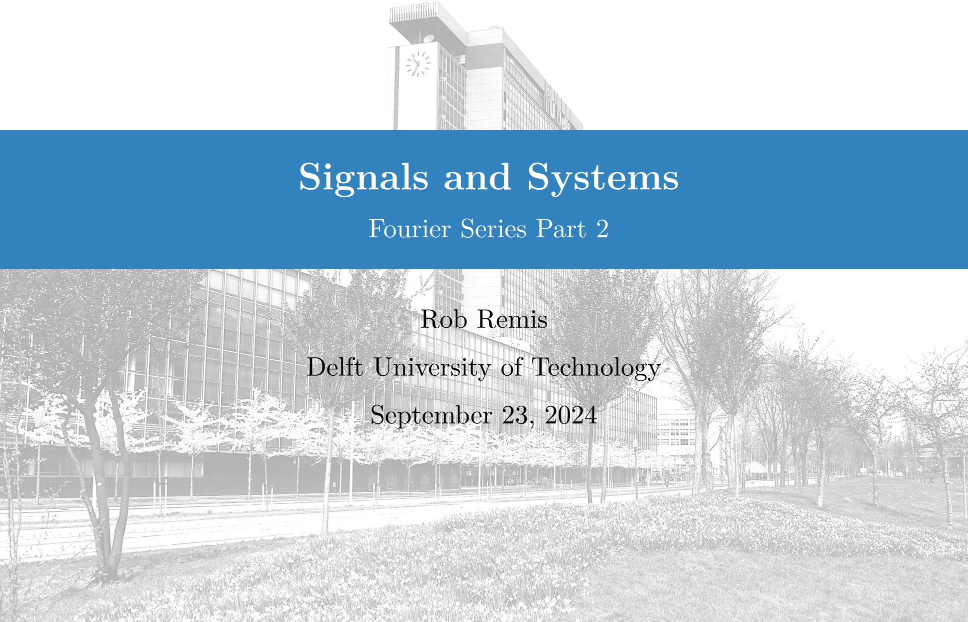

# Signals and Systems

## Fourier Series Part 2

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# 1 Outline

# 1 Overview



- Reflection and even and odd periodic signals
- Real-valued periodic signals
- Time and frequency shifting
- Sum and multiplication of periodic signals
- Derivatives and integrals of periodic signals
- Amplitude and time scaling of periodic signals

- Book: Chapter 4
- Sections/subsections: 4.3.4, 4.3.6, 4.5
- Exercises: 4.2, 4.3, 4.4, 4.5, 4.7, 4.11

# 2 Outline





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## 2 Reflection and even and odd periodic signals

- **Reflection:**
- Let $x(t)$ be a periodic signal with a fundamental period $T_0$ and a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t}$$

- What is the Fourier expansion of $x(-t)$?

$$x(-t) = \sum_{k=-\infty}^{\infty} X_k e^{-\mathrm{j}k\Omega_0 t} = \sum_{k=-\infty}^{\infty} X_{-k} e^{\mathrm{j}k\Omega_0 t}$$

- Conclusion: if the Fourier coefficients of $x(t)$ are given by $X_k$ then the Fourier coefficients of $x(-t)$ are given by $X_{-k}$

# 2 Reflection and even and odd periodic signals



- **Even periodic signals:**
- An even signal $x(t)$ is characterized by

$$x(-t) = x(t) \quad \text{for all } t \in \mathbb{R}$$

- Using the result of the previous slide we find that for an even signal

$$X_{-k} = X_k \quad \text{(even signal)}$$

- For the expansion coeffcients of the trigonometric Fourier series we have

$$c_k = \frac{X_k + X_{-k}}{2} = X_k \quad \text{and} \quad d_k = \mathrm{j}\frac{X_k - X_{-k}}{2} = 0$$

- The trigonometric Fourier series of an even signal has cosine expansion functions (even functions) only

- **Odd periodic signals:**
- An odd signal $x(t)$ is characterized by

$$x(-t) = -x(t) \quad \text{for all } t \in \mathbb{R}$$

- For an odd signal we have

$$X_{-k} = -X_k \quad \text{(odd signal)}$$

- The expansion coeffcients of the trigonometric Fourier series are

$$c_k = \frac{X_k + X_{-k}}{2} = 0 \quad \text{and} \quad d_k = \mathrm{j}\frac{X_k - X_{-k}}{2} = \mathrm{j}X_k$$

- The trigonometric Fourier series of an odd signal has sine expansion functions (odd functions) only

## 3 Real-valued periodic signals



- Let $x(t)$ be a periodic signal with a fundamental period $T_0$ and a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t}$$

- Taking the complex conjugate, we find

$$x^*(t) = \sum_{k=-\infty}^{\infty} X_k^* e^{-\mathrm{j}k\Omega_0 t} = \sum_{k=-\infty}^{\infty} X_{-k}^* e^{\mathrm{j}k\Omega_0 t}$$

- Conclusion: if the Fourier coefficients of $x(t)$ are given by $X_k$ then the Fourier coefficients of $x^*(t)$ are given by $X_{-k}^*$

## 3 Real-valued periodic signals



- If the signal $x(t)$ is real-valued then $x^*(t) = x(t)$. Consequently, for a real-valued signal we have

$$X^*_{-k} = X_k \quad \text{or} \quad X_{-k} = X^*_k \quad \text{(real-valued signal)}$$

- If the signal $x(t)$ is *even and real-valued*, we have

$$X_k = X_{-k} = X^*_k$$

  showing that the Fourier coefficients $X_k$ are real. Note that the coefficients $c_k$ are real as well.

# 3 Real-valued periodic signals



- If the signal $x(t)$ is *odd and real-valued*, we have

$$X_k = -X_{-k} = -X_k^*$$

showing that the Fourier coefficients $X_k$ are imaginary. Note that the coefficients $d_k$ are real in this case.

# 4 Outline

## 4 Time and frequency shifting



- Let $x(t)$ be a periodic signal with a fundamental period $T_0$ and a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t}$$

- What are the Fourier coefficients of $x(t-\tau)$, where $\tau \in \mathbb{R}$ is a time shift?

## 4 Time and frequency shifting



- From the Fourier expansion

$$x(t-\tau) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0(t-\tau)} = \sum_{k=-\infty}^{\infty} X_k e^{-\mathrm{j}k\Omega_0\tau} e^{\mathrm{j}k\Omega_0 t}$$

- Conclusion: if the Fourier coefficients of $x(t)$ are given by $X_k$ then the Fourier coefficients of $x(t-\tau)$ are given by $X_k e^{-\mathrm{j}k\Omega_0\tau}$.

# 4 Time and frequency shifting



- We are given a periodic signal $x(t)$ with fundamental period $T_0$. We consider the *modulated signal*

$$y(t) = x(t)e^{\mathrm{j}\Omega_1 t}$$

- The frequency $\Omega_1$ is called the *modulation frequency*
- For this frequency we take: $\Omega_1 = M\Omega_0$, with $M$ an integer, $M \gg 1$
- The signal $y(t)$ is periodic with a fundamental period $T_0$

# 4 Time and frequency shifting



- For the signals $x(t)$ and $y(t)$ we have the Fourier expansions

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{\mathrm{j}k\Omega_0 t}$$

- How are the Fourier coefficients of $y(t)$ related to the Fourier coeffcients of $x(t)$?

$$\begin{aligned} y(t) = x(t)e^{\mathrm{j}\Omega_1 t} &= \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}(k\Omega_0+\Omega_1)t} \\ &= \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}(k+M)\Omega_0 t} = \sum_{k=-\infty}^{\infty} X_{k-M} e^{\mathrm{j}k\Omega_0 t} \end{aligned}$$

## 4 Time and frequency shifting



- Conclusion: if the Fourier coefficients of $x(t)$ are given by $X_k$ then the Fourier coefficients of $x(t)e^{\mathrm{j}M\Omega_0 t}$ are given by $X_{k-M}$.
- The spectrum of $x(t)$ is shifted in frequency by $\Omega_1 = M\Omega_0$ rad/s

## 5 Outline

## 5 Sum and multiplication of periodic signals



- Let $x(t)$ be a periodic signal with fundamental period $T_1$. Its fundamental frequency is

$$\Omega_1 = \frac{2\pi}{T_1}$$

- Let $y(t)$ be a periodic signal with fundamental period $T_2$. Its fundamental frequency is

$$\Omega_2 = \frac{2\pi}{T_2}$$

- Consider the signal $z(t)$, which is a linear combination of $x(t)$ and $y(t)$:

$$z(t) = \alpha x(t) + \beta y(t)$$

$\alpha$ and $\beta$ are constants.

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# 5 Sum and multiplication of periodic signals



- As we have seen, if
$$\frac{T_2}{T_1} = \frac{N}{M}$$
with $N$ and $M$ integers $\geq 1$ with no common factor then
- $z(t)$ is periodic with fundamental period and frequency
$$T_0 = MT_2 = NT_1, \quad \text{and} \quad \Omega_0 = \frac{2\pi}{T_0}$$
respectively
- Note that
$$\Omega_1 = \frac{2\pi}{T_1} = N\Omega_0 \quad \text{and} \quad \Omega_2 = \frac{2\pi}{T_2} = M\Omega_0$$

## 5 Sum and multiplication of periodic signals



- Remark: when $N$ and $M$ have no common factor, then $N$ and $M$ are said to be *relatively prime* or *coprime*
- Example: $N = 4$ and $M = 6$ are *not* relatively prime. Common factor is 2.
$$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{2}{3}$$
- Example: $N = 2$ and $N = 3$ are relatively prime. These numbers do not have a common factor.

## 5 Sum and multiplication of periodic signals



- Since $x(t)$ is periodic with fundamental frequency $\Omega_1 = N\Omega_0$ it has a Fourier expansion of the form

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_1 t} = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}kN\Omega_0 t}$$

- Since $y(t)$ is periodic with fundamental frequency $\Omega_2 = M\Omega_0$ it has a Fourier expansion of the form

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{\mathrm{j}k\Omega_2 t} = \sum_{k=-\infty}^{\infty} Y_k e^{\mathrm{j}kM\Omega_0 t}$$

# 5 Sum and multiplication of periodic signals



- Since $z(t)$ is periodic with fundamental frequency $\Omega_0$ it has a Fourier expansion of the form

$$z(t) = \sum_{k=-\infty}^{\infty} Z_k e^{\mathrm{j}k\Omega_0 t}$$

- How are the coeffcients $Z_k$ related to the coeffcients $X_k$ and $Y_k$?

- Using the Fourier expansions of $x(t)$ and $y(t)$, we find

$$
\begin{aligned}
z(t) &= \alpha x(t) + \beta y(t) \\
&= \sum_{k=-\infty}^{\infty} \alpha X_k e^{\mathrm{j}kN\Omega_0 t} + \sum_{k=-\infty}^{\infty} \beta Y_k e^{\mathrm{j}kM\Omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} Z_k e^{\mathrm{j}k\Omega_0 t}
\end{aligned}
$$

- The integers: $\mathbb{Z}$
- Given an integer $N \geq 1$
- We say that an integer $k \in \mathbb{Z}$ is an integer multiple of $N$ if there exists an integer $p \in \mathbb{Z}$ such that $k = pN$

# 5 Sum and multiplication of periodic signals



- Example: $N = 3$
  - $k = 0$ is an integer multiple of $N$, since for $p = 0 \in \mathbb{Z}$, we have $k = 0 \cdot N = 0$.
  - $k = 0$ is an integer multiple of any $N$
  - $k = 1$ is not an integer multiple of 3
  - $k = -1$ is not an integer multiple of 3
  - $k = 3$ is an integer multiple of $N = 3$ ($p = 1$)
  - $k = -3$ is an integer multiple of $N = 3$ ($p = -1$)

# 5 Sum and multiplication of periodic signals



- If $k$ is an integer multiple of $N$ **and** $k$ is an integer multiple of $M$:

$$Z_k = \alpha X_{k/N} + \beta Y_{k/M}$$

- If $k$ is *not* an integer multiple of $N$ **and** $k$ is *not* an integer multiple of $M$:

$$Z_k = 0$$

- If $k$ is an integer multiple of $N$ **and** $k$ is *not* an integer multiple of $M$:

$$Z_k = \alpha X_{k/N}$$

- If $k$ is *not* an integer multiple of $N$ **and** $k$ is an integer multiple of $M$:

$$Z_k = \beta Y_{k/M}$$

# 5 Sum and multiplication of periodic signals



- Example: $N = 2$ and $M = 3$, $\alpha = \beta = 1$

$$
\begin{aligned}
Z_0 &= X_0 + Y_0 \\
Z_1 &= 0 \\
Z_2 &= X_1 \\
Z_3 &= Y_1 \\
Z_4 &= X_2 \\
Z_5 &= 0 \\
Z_6 &= X_3 + Y_2 \\
&\dots
\end{aligned}
$$

# 5 Sum and multiplication of periodic signals

- Book:

$$z(t) = \sum_{k=-\infty}^{\infty} (\alpha X_{k/N} + \beta Y_{k/M}) e^{\mathrm{j}k\Omega_0 t}$$

with $k/N$ and $k/M$ integers

## 5 Sum and multiplication of periodic signals



- Let $x(t)$ and $y(t)$ be periodic signals with fundamental period $T_0$
- Fourier expansions of these signals

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t} \quad \text{and} \quad y(t) = \sum_{m=-\infty}^{\infty} Y_m e^{\mathrm{j}m\Omega_0 t}$$

- We multiply the signals $x(t)$ and $y(t)$ to obtain

$$z(t) = x(t)y(t)$$

- The signal $z(t)$ is also periodic with fundamental period $T_0$

## 5 Sum and multiplication of periodic signals



- Fourier expansion of $z(t)$:

$$z(t) = \sum_{n=-\infty}^{\infty} Z_n e^{\mathrm{j}n\Omega_0 t}$$

- How are the Fourier coefficients of $x(t)$ and $y(t)$ related to the Fourier coefficients of $z(t)$?

- We compute

$$
\begin{aligned}
z(t) &= x(t)y(t) \\
&= \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t} \sum_{m=-\infty}^{\infty} Y_m e^{\mathrm{j}m\Omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k Y_m e^{\mathrm{j}(k+m)\Omega_0 t} \\
&\overset{n=k+m}{=} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_k Y_{n-k} e^{\mathrm{j}n\Omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_k Y_{n-k} e^{\mathrm{j}n\Omega_0 t} = \sum_{n=-\infty}^{\infty} Z_n e^{\mathrm{j}n\Omega_0 t}
\end{aligned}
$$

## 5 Sum and multiplication of periodic signals

- We conclude that the Fourier coefficients of $z(t)$ are given by

$$Z_n = \sum_{k=-\infty}^{\infty} X_k Y_{n-k}$$

- $Z_n$ is equal to the convolution of the discrete sequences $X_k$ and $Y_k$
- Compare with

$$z(t) = \int_{\tau=-\infty}^{\infty} x(\tau) y(t-\tau)\, \mathrm{d}\tau$$

- Discrete convolutions will be discussed extensively further on in the course

# 6 Outline

# 6 Derivatives and integrals of periodic signals



- Periodic signal $x(t)$ with a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t}$$

- Let $y(t)$ be the derivative of this signal. We have

$$y(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{k=-\infty}^{\infty} X_k \cdot \mathrm{j}k\Omega_0 \cdot e^{\mathrm{j}k\Omega_0 t} = \sum_{k=-\infty}^{\infty} Y_k e^{\mathrm{j}k\Omega_0 t}$$

- We conclude

$$Y_k = X_k \cdot \mathrm{j}k\Omega_0$$

# 6 Derivatives and integrals of periodic signals



- Periodic signal $y(t)$ with a Fourier expansion

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{\mathrm{j}k\Omega_0 t}$$

- Signal has no dc component: $Y_0 = 0$

$$y(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k e^{\mathrm{j}k\Omega_0 t}$$

# 6 Derivatives and integrals of periodic signals



- Integral of $y(t)$:

$$z(t) = \int_{\tau=-\infty}^{t} y(\tau)\,\mathrm{d}\tau$$

- With $MT_0 \leq t$ and $M$ an integer, we have

$$\begin{aligned}
z(t) &= \int_{\tau=-\infty}^{t} y(\tau)\,\mathrm{d}\tau \\
&= \underbrace{\int_{\tau=-\infty}^{MT_0} y(\tau)\,\mathrm{d}\tau}_{=0} + \int_{\tau=MT_0}^{t} y(\tau)\,\mathrm{d}\tau \\
&= \int_{\tau=MT_0}^{t} y(\tau)\,\mathrm{d}\tau
\end{aligned}$$

# 6 Derivatives and integrals of periodic signals



- Substitute the Fourier series of $y(t)$ to obtain

$$
\begin{aligned}
z(t) &= \int_{\tau=MT_0}^{t} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k e^{\mathrm{j}k\Omega_0\tau}\,\mathrm{d}\tau \\
&= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k \int_{\tau=MT_0}^{t} e^{\mathrm{j}k\Omega_0\tau}\,\mathrm{d}\tau \\
&= -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k \frac{1}{\mathrm{j}k\Omega_0} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k \frac{1}{\mathrm{j}k\Omega_0} e^{\mathrm{j}k\Omega_0 t} \\
&= Z_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Z_k e^{\mathrm{j}k\Omega_0 t}
\end{aligned}
$$

# 6 Derivatives and integrals of periodic signals

- with

$$Z_0 = -\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} Y_k \frac{1}{\mathrm{j}k\Omega_0}$$

and

$$Z_k = Y_k \frac{1}{\mathrm{j}k\Omega_0} \quad k \neq 0$$

# 7 Outline





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# 7 Amplitude and time scaling of periodic signals

- Periodic signal $x(t)$ with a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{\mathrm{j}k\Omega_0 t}$$

- What are the Fourier coefficients of $y(t) = Ax(\alpha t)$, $\alpha > 0$?
- From the Fourier expansion of $x(t)$:

$$y(t) = Ax(\alpha t) = \sum_{k=-\infty}^{\infty} AX_k e^{\mathrm{j}k\alpha\Omega_0 t}$$

- We observe:
  - $y(t)$ is a periodic signal with fundamental frequency $\alpha\Omega_0$
  - and Fourier coefficients $Y_k = AX_k$
  - Note that time scaling with an $\alpha > 0$ does not affect the Fourier coeffcients