Signals and Systems

The Laplace Transform 2

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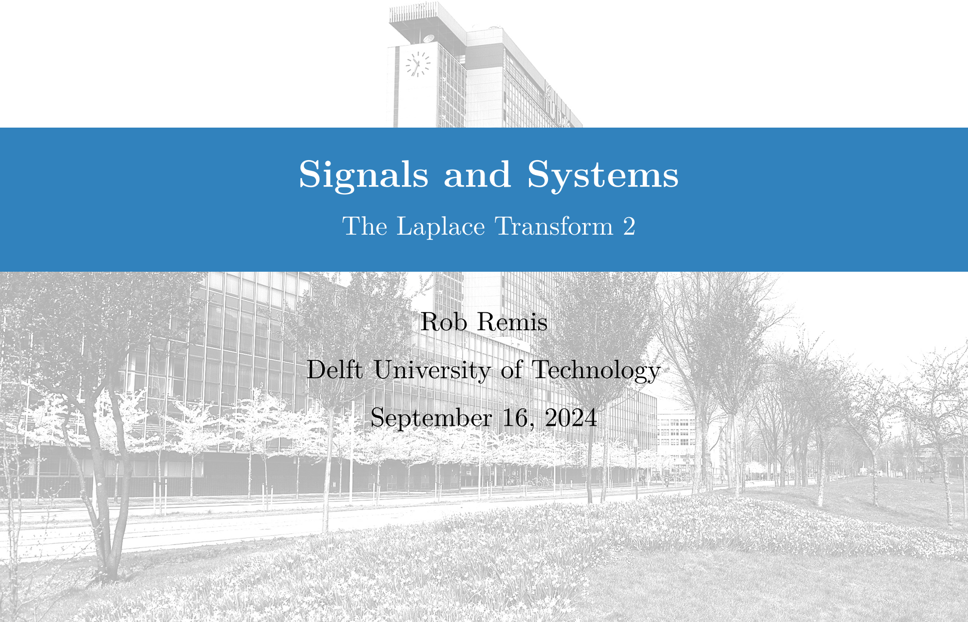

1 Outline

1. Overview
2. The one-sided Laplace transform
3. Properties of the one-sided Laplace transform
4. Circuit Theory Revisited
5. Solving differential equations using the Laplace transform
6. The inverse Laplace transform via inspection

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1 Overview

Contents

- The one-sided Laplace transform
- circuit theory revisited
- Solving differential equations using the Laplace transform
- The inverse Laplace transform via inspection

Book
Sections 3.5 – 3.8

Exercises:
3.1, 3.2, 3.4, 3.5, 3.7, 3.9 – 3.13, 3.18, 3.20, 3.22, 3.25, 3.29, 3.30

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2 The one-sided Laplace transform

- Let $x(t)$ denote a causal signal: $x(t) = x(t)u(t)$
- The two-sided Laplace transform simplifies to

$$X(s) = \int_{t=0}^{\infty} x(t)e^{-st}\,\mathrm{d}t \qquad s \in \mathrm{ROC}_x$$

- This transform is known as the *one-sided Laplace transform*
- A separate study is warranted, since many (most/all) signals and systems encountered in practice/Nature are causal
- Switch-on phenomena (initial-value problems) are conveniently studied using the one-sided Laplace transform

2 The one-sided Laplace transform

- To incorporate the Dirac distribution $\delta(t)$, we define the one-sided Laplace transform of a signal $x(t)$ as

$$X(s) = \int_{t=0^-}^{\infty} x(t)e^{-st}\,\mathrm{d}t = \lim_{\epsilon \downarrow 0} \int_{t=-\epsilon}^{\infty} x(t)e^{-st}\,\mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- Many properties of the two-sided Laplace transform carry over to the one-sided Laplace transform
- We only discuss three properties of the one-sided transform that do not have a two-sided counterpart

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- **Differentiation in the time-domain**
- Let $X(s)$ denote the one-sided Laplace transform of the time-domain signal $x(t)$
- What is the one-sided Laplace transform of

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\,?$$

- By definition

$$Y(s) = \int_{t=0^-}^{\infty} \frac{\mathrm{d}x(t)}{\mathrm{d}t} e^{-st}\,\mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- Integration by parts gives

$$Y(s) = \lim_{T\to\infty,\, \epsilon\downarrow 0} x(t)e^{-st}\Big|_{-\epsilon}^{T} - \int_{t=0^-}^{\infty} x(t)\left[-se^{-st}\right] \mathrm{d}t = -x(0^-) + sX(s)$$

with $x(0^-) = \lim_{\epsilon\downarrow 0} x(-\epsilon)$ and $s \in \mathrm{ROC}_x$

- We have found that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \text{ transforms into } sX(s) - x(0^-)$$

with $s \in \mathrm{ROC}_x$

3 Properties of the one-sided Laplace transform

- Similarly, by repeated integration by parts we find that

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} \text{ transforms into } s^2 X(s) - s x(0^-) - \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=0^-}$$

with $s \in \mathrm{ROC}_x$

- **Abel's initial-value theorem**
- Let $X(s)$ be the one-sided Laplace transform of $x(t)$, $s \in \mathrm{ROC}_x$
- Abel's initial-value theorem states that

$$\lim_{s \to \infty} sX(s) = x(0^+),$$

with $x(0^+) = \lim_{\epsilon \downarrow 0} x(\epsilon)$, provided $x(t)$ is regular at $t = 0$

- Note that
  - Left-hand side: Laplace-domain
  - Right-hand side: time-domain

- **Abel's final-value theorem**
- Let $X(s)$ denote the one-sided Laplace transform of $x(t)$, $s \in \mathrm{ROC}_x$
- Abel's final-value theorem states that

$$\lim_{s \to 0} sX(s) = \lim_{t \to \infty} x(t)$$

  provided $\lim_{t \to \infty} x(t)$ exists
- Note that
  - Left-hand side: Laplace-domain
  - Right-hand side: time-domain

3 Properties of the one-sided Laplace transform

One-Sided Laplace Transforms

Time signal	One-sided Laplace transform	ROC	parameters
$e^{at}u(t)$	$\frac{1}{s-a}$	$\text{Re}(s) > \text{Re}(a)$	$a \in \mathbb{C}$
$-e^{at}u(-t)$	0	$\mathbb{C}$	$a \in \mathbb{C}$
$\frac{t^{k-1}e^{at}}{(k-1)!}u(t)$	$\frac{1}{(s-a)^k}$	$\text{Re}(s) > \text{Re}(a)$	$a \in \mathbb{C}, k \in \mathbb{N}$
$-\frac{t^{k-1}e^{at}}{(k-1)!}u(-t)$	0	$\mathbb{C}$	$a \in \mathbb{C}, k \in \mathbb{N}$
$e^{at}\cos(\Omega_0 t)u(t)$	$\frac{s-a}{(s-a)^2+\Omega_0^2}$	$\text{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$e^{at}\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{(s-a)^2+\Omega_0^2}$	$\text{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$\delta(t)$	1	$\mathbb{C}$	–
$\delta'(t)$	s	$\mathbb{C}$	–

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3 Properties of the one-sided Laplace transform

Properties of the One-Sided Laplace Transform: $x_c(t) = x(t)u(t),\ t \in \mathbb{R}$

Property	Time signal	One-sided Laplace transform	ROC	Parameters
Convolution	$y_c(t) = h_c(t) * x_c(t)$	$Y(s) = H(s)X(s)$	$\mathrm{ROC}_{h_c} \cap \mathrm{ROC}_{x_c}$	–
Diff. s-domain	$-tx(t)$	$\frac{\mathrm{d}X(s)}{\mathrm{d}s}$	ROC_{x_c}	–
Diff. t-domain	$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$sX(s) - x(0^-)$	ROC_{x_c}	–
Int. t-domain	$\int_{\tau=0^-}^{t} x(\tau)\,\mathrm{d}\tau$	$\frac{1}{s}X(s)$	$\{\mathrm{ROC}_{x_c} \mid \mathrm{Re}(s) > 0\}$	–
Shift t-domain	$x_c(t-\tau)$	$e^{-s\tau}X(s)$	ROC_{x_c}	$\tau \in \mathbb{R},\ \tau > 0$
Shift s-domain	$e^{at}x(t)$	$X(s-a)$	$s-a \in \mathrm{ROC}_{x_c}$	$a \in \mathbb{C}$
Scaling	$x(at)$	$\frac{1}{a}X\left(\frac{s}{a}\right)$	$s/a \in \mathrm{ROC}_{x_c}$	$a \in \mathbb{R},\ a > 0$

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4 Circuit Theory Revisited

- **KCL** Kirchhoff's current law: the algebraic sum of all branch currents flowing into any node must be zero
- For a node with N branches

$$\sum_{n=1}^{N} i_n(t) = 0$$

- **KVL** Kirchhoff's voltage law: the algebraic sum of the branch voltages around any closed path in a network must be zero
- For a closed path consisting of N branches

$$\sum_{n=1}^{N} v_n(t) = 0$$

4 Circuit Theory Revisited

- Let

$$I_n(s) \text{ be the one-sided Laplace transform of } i_n(t)$$

$n = 1, 2, ..., N$

$$V_n(s) \text{ be the one-sided Laplace transform of } V_n(t)$$

$n = 1, 2, ..., N$

- Since the Laplace transform is linear, we have

- **KCL** Kirchhoff's current law in the Laplace domain:

$$\sum_{n=1}^{N} I_n(s) = 0$$

- **KVL** Kirchhoff's voltage law in the Laplace domain:

$$\sum_{n=1}^{N} V_n(s) = 0$$

4 Circuit Theory Revisited

- **Constitutive relations**
- *Resistor*

$$v(t) = R\,i(t) \quad \text{transforms into} \quad V(s) = RI(s)$$

- *Capacitor*

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \quad \text{transforms into} \quad I(s) = sC\,V(s) - Cv(0^-)$$

- *Inductor*

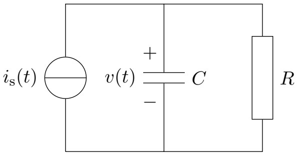
$$v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \quad \text{transforms into} \quad V(s) = sL\,I(s) - Li(0^-)$$

- For circuits with vanishing initial conditions (the circuit is initially at rest), we define the Laplace impedance $Z(s)$ through the relation

$$V(s) = Z(s)\, I(s)$$

- **Resistor**: $Z(s) = R$
- **Capacitor**: $Z(s) = \frac{1}{sC}$
- **Inductor**: $Z(s) = sL$

- **Example.** Consider the circuit sketched below
- Input signal: $i_s(t) = I_0\delta(t)$
- Output signal: $v(t)$
- The circuit is initially at rest



- Kirchhoff's current law in the time-domain:

$$C\frac{\mathrm{d}v}{\mathrm{d}t} + R^{-1}v(t) = I_0\delta(t), \qquad t > 0^-$$

$v(0^-) = 0$

- Kirchhoff's current law in the s-domain:

$$sCV(s) + R^{-1}V(s) = I_0$$

- Divide by C to obtain

$$\left(s + \tau^{-1}\right)V(s) = C^{-1}I_0, \qquad \tau = RC$$

- We find

$$V(s) = C^{-1}I_0\frac{1}{s + \tau^{-1}}, \qquad \tau = RC$$

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- Using the table for one-sided Laplace transforms, the voltage is found as
$$v(t) = C^{-1} I_0 e^{-t/\tau} u(t), \qquad \tau = RC$$
- The current through the capacitor follows as
$$i_{\mathrm{c}}(t) = C \frac{\mathrm{d}v(t)}{\mathrm{d}t} = I_0 \left[\delta(t) - \frac{1}{\tau} e^{-t/\tau} u(t) \right], \qquad \tau = RC$$
- Observe that we can also write
$$I_c(s) = \frac{Y_{\mathrm{cap}}(s)}{Y_{\mathrm{cap}}(s) + Y_{\mathrm{res}}(s)} I_0$$
- $Y_{\mathrm{cap}}(s) = sC$ and $Y_{\mathrm{res}}(s) = R^{-1}$ are the Laplace domain *admittances* of the capacitor and resistor, respectively ($Y(s) = Z^{-1}(s)$)

- Substitution gives

$$I_c(s) = \frac{sC}{sC + R^{-1}} I_0 = \left(1 - \frac{1}{\tau}\frac{1}{s + \frac{1}{\tau}}\right) I_0, \qquad \tau = RC$$

- Using the table for the one-sided Laplace transform, we again arrive at

$$i_c(t) = I_0 \left[\delta(t) - \frac{1}{\tau} e^{-t/\tau} u(t)\right], \qquad \tau = RC$$

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5 Solving differential equations

- We consider the differential equation

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = x(t),$$

for $t > 0^-$ with initial conditions

$$y(0^-) = \alpha \quad \text{and} \quad \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0^-} = \beta$$

- The problem consists of finding $y(t)$ for a given signal $x(t)$ and initial values α and β
- The signal $x(t)$ can be seen as the input signal of a SISO system
- The signal $y(t)$ can be seen as the output signal of the SISO system
- The coefficients a_i are typically real-valued
  Determined by R, L, and/or C in an electric circuit, for example

- Applying a one-sided Laplace transform to the differential equation and taking the initial conditions into account, we obtain

$$(a_2 s^2 + a_1 s + a_0)Y(s) = X(s) + a_1\alpha + a_2(\beta + \alpha s)$$

- and the Laplace transform of the solution is found as

$$Y(s) = Y_{\mathrm{zs}}(s) + Y_{\mathrm{zi}}(s)$$

with

$$Y_{\mathrm{zs}}(s) = \frac{X(s)}{a_2 s^2 + a_1 s + a_0}$$

the Laplace transform of the zero-state response $y_{\mathrm{zs}}(t)$, and

$$Y_{\mathrm{zi}}(s) = \frac{a_1\alpha + a_2(\beta + \alpha s)}{a_2 s^2 + a_1 s + a_0}$$

the Laplace transform of the zero-input response $y_{\mathrm{zi}}(t)$

5 Solving differential equations

- Note that for $x(t) = \delta(t)$ and vanishing initial conditions ($\alpha = 0$ and $\beta = 0$), the time-domain output signal is the impulse response $h(t)$ with the transfer function

$$H(s) = \frac{1}{a_2 s^2 + a_1 s + a_0}$$

as its Laplace transform

- Determining $y(t)$ amounts to finding the time-domain signal that corresponds to $Y(s)$ (inverse Laplace transform)
- We determine $y(t)$ from $Y(s)$ using tables (via inspection)

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6 The inverse Laplace transform via inspection

6 The inverse Laplace transform via inspection

- The Laplace transform signals that we consider are of the form

$$X(s) = \frac{p_M(s)}{q_N(s)}$$

- $p_M(s)$ and $q_N(s)$ are polynomials in s of degree M and N, respectively
- Our Laplace transform signals are rational functions in s

* For $M > N$, $X(s)$ is an *improper* rational function
* For $M \leq N$, $X(s)$ is a *proper* rational function
* For $M < N$, $X(s)$ a *strictly proper* rational function

6 The inverse Laplace transform via inspection

- It can be shown that if $X(s)$ is proper or improper then it can always be written as

$$X(s) = R_{M-N}(s) + \frac{S(s)}{T(s)}$$

- $R_{M-N}(s)$ is a polynomial in s of degree $M - N$
- S and T are polynomials such that the rational function S/T is strictly proper

6 The inverse Laplace transform via inspection

- **Example 1**

$$X(s) = \frac{s}{s+1}$$

is a proper rational function, which can be written as

$$X(s) = 1 - \frac{1}{s+1}$$

- In this example, $R_0(s) = 1$ and $-1/(s+1)$ is strictly proper

- **Example 2**

$$X(s) = \frac{s^3}{s+4}$$

is an improper rational function, which can be written as

$$X(s) = s^2 - 4s + 16 - \frac{64}{s+4}$$

- In this example, $R_2(s) = s^2 - 4s + 16$ and $-64/(s+4)$ is strictly proper

6 The inverse Laplace transform via inspection

- Finding $x(t)$ from $X(s)$ amounts to using partial fraction decompositions of $X(s)$ such that $X(s)$ consists of polynomials and/or strictly proper rational functions that have tabulated time-domain counterparts
- **Example** Suppose

$$X(s) = \frac{1}{(s+1)(s+4)}$$

- A partial fraction decomposition gives

$$X(s) = \frac{1}{3}\left(\frac{1}{s+1} - \frac{1}{s+4}\right)$$

- Using the Laplace transform table, we find

$$x(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)u(t)$$