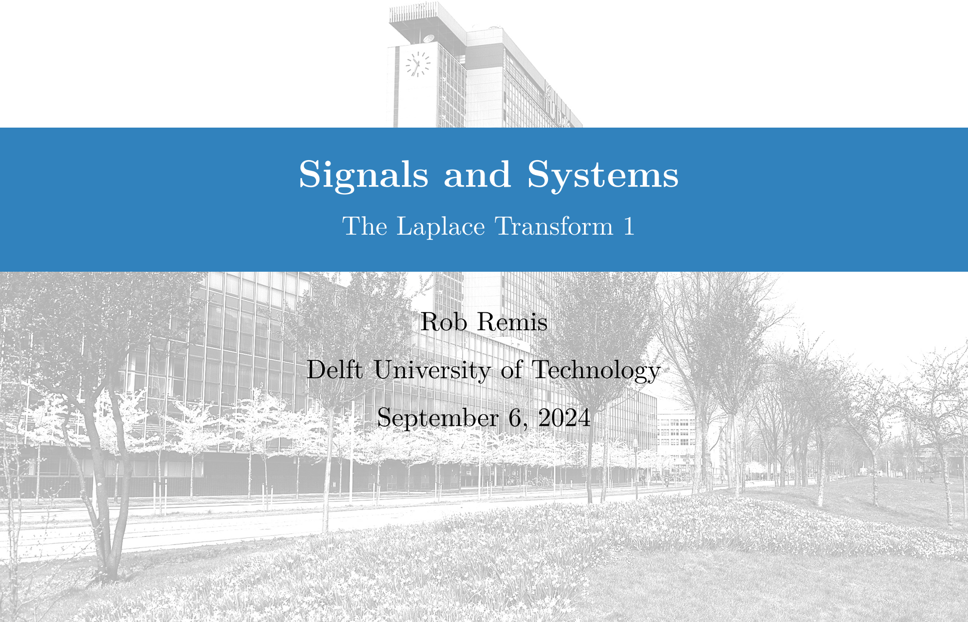
Signals and Systems

The Laplace Transform 1



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1 Outline

1 Overview

Contents

- The two-sided Laplace transform
- Properties of the two-sided Laplace transform

Book
Sections 3.1 – 3.4

Exercises:
3.1, 3.3, 3.4, 3.6, 3.7, 3.8, 3.13, 3.15, 3.17, 3.20, 3.21

2 The two-sided Laplace transform

- Given an LTI system with a single input and a single output
- Input signal: $x(t)$
- Output signal: $y(t)$
- We have seen that the output signal is given by the convolution of the input signal $x(t)$ and the impulse response $h(t)$:

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \int_{\tau=-\infty}^{\infty} x(t-\tau)h(\tau)\,\mathrm{d}\tau$$

2 The two-sided Laplace transform

- Let the input signal be given by

$$x(t) = e^{st} \quad \text{with} \quad s \in \mathbb{C}$$

- The corresponding output signal is

$$y(t) = \int_{\tau=-\infty}^{\infty} e^{s(t-\tau)} h(\tau)\,\mathrm{d}\tau = \int_{\tau=-\infty}^{\infty} h(\tau) e^{-s\tau}\,\mathrm{d}\tau\, e^{st} = H(s)x(t)$$

with

$$H(s) = \int_{\tau=-\infty}^{\infty} h(\tau) e^{-s\tau}\,\mathrm{d}\tau$$

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- We observe that the output signal is a multiple of the input signal (provided the integral converges, of course): $y(t) = H(s)e^{st}$

$$\underbrace{S}_{A}\{\underbrace{e^{st}}_{u}\} = \underbrace{H(s)}_{\lambda}\underbrace{e^{st}}_{u}$$

Compare this with a standard eigenvalue problem in linear algebra:

$$Au = \lambda u$$

- $H(s)$ is an *eigenvalue* of the LTI system corresponding to the *eigenfunction* $x(t) = e^{st}$

2 The two-sided Laplace transform

- The expression for $H(s)$ is precisely the definition of the two-sided Laplace transform of $h(t)$
- Two-sided Laplace transform of a signal $x(t)$:

$$X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t$$

  defined, of course, for those $s \in \mathbb{C}$ for which the integral converges

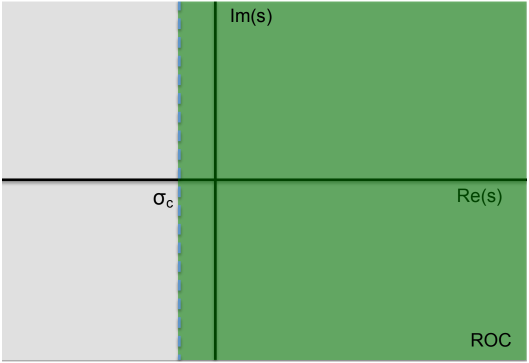
- To investigate under what condition(s) convergence takes place, we consider
  - the Laplace transform of causal signals: $x(t) = 0$ for $t < 0$
  - the Laplace transform of anti-causal signals: $x(t) = 0$ for $t > 0$
  - the Laplace transform of noncausal signals

- A causal signal $x(t)$ is said to be of *exponential order* if there exists constants A and α such that

$$\big|x(t)\big| \leq Ae^{\alpha t} \quad \text{for } t \geq 0$$

- It can be shown that for causal signals of exponential order there exists a unique number $-\infty \leq \sigma_\mathrm{c} < \infty$ such that the Laplace integral converges for $\mathrm{Re}(s) > \sigma_\mathrm{c}$
- The number σ_c is called the *abscissa of convergence*

2 The two-sided Laplace transform



Region of Convergence (ROC) for causal signals

- The set $\{s \in \mathbb{C}; \mathrm{Re}(s) > \sigma_\mathrm{c}\}$ is called the *Region of Convergence* (ROC)
- To avoid confusion, we sometimes write ROC_x to indicate the ROC of the Laplace transform of a signal $x(t)$
- Note that the ROC of a causal signal (of exponential order) is a *right-half plane* (unless $\sigma_\mathrm{c} = -\infty$, of course)
- It can also be shown that $X(s)$ is differentiable w.r.t. s on its ROC

2 The two-sided Laplace transform

- **Example 1:** The two-sided Laplace transform of the Heaviside unit step function $u(t)$

$$U(s) = \int_{t=-\infty}^{\infty} u(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} e^{-st}\,\mathrm{d}t = \frac{1}{s} \quad \text{for } \mathrm{Re}(s) > 0$$

- In this case ROC $= \left\{s \in \mathbb{C}; \mathrm{Re}(s) > 0\right\}$

- **Example 2:** The two-sided Laplace transform of a scaled rectangular pulse function $x(t) = p\left(\frac{t}{T}\right)$, $T > 0$

$$X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{T} e^{-st}\,\mathrm{d}t = \frac{1}{s}\left(1 - e^{-sT}\right), \qquad s \in \mathbb{C}$$

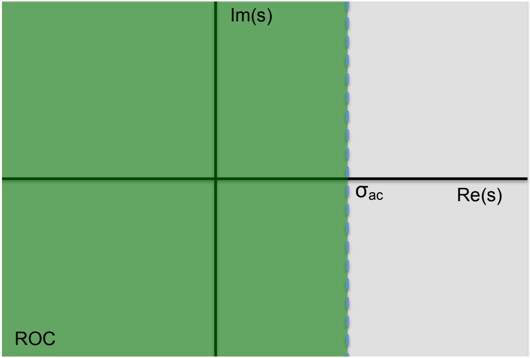
- Note that there is no singularity at $s = 0$
- In this case ROC $= \mathbb{C}$

- An anti-causal signal $x(t)$ is said to be of exponential order if there exists constants B and β such that

$$\left|x(t)\right| \leq Be^{\beta t} \quad \text{for } t \leq 0$$

- For anti-causal signals of exponential order it can be shown that there exists a number $-\infty < \sigma_{\mathrm{ac}} \leq \infty$ such that the Laplace integral converges for $\mathrm{Re}(s) < \sigma_{\mathrm{ac}}$
- The number σ_{ac} is again called the abscissa of convergence and the ROC is a *left-half plane* (unless $\sigma_{\mathrm{ac}} = \infty$)
- The Laplace transform is differentiable w.r.t. s on the ROC

2 The two-sided Laplace transform



Region of Convergence (ROC) for anti-causal signals

- **Example.** The two-sided Laplace transform of $x(t) = u(-t)$

$$X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t = \int_{t=-\infty}^{0} e^{-st}\,\mathrm{d}t = -\frac{1}{s} \quad \text{for } \mathrm{Re}(s) < 0$$

- For this signal the ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) < 0\}$

- *Specifying the ROC is important!*
- For example, $X(s) = 1/s$ can be
  - the Laplace transform of the causal signal $x(t) = u(t)$, or
  - the Laplace transform of the anti-causal signal $x(t) = -u(-t)$
- Which one is intended becomes clear by specifying the ROC

- Finally, what about noncausal signals?
- For such a signal we write

$$x(t) = x(t) \cdot 1 = x(t)\big[u(t) + u(-t)\big] = x_{\mathrm{c}}(t) + x_{\mathrm{ac}}(t)$$

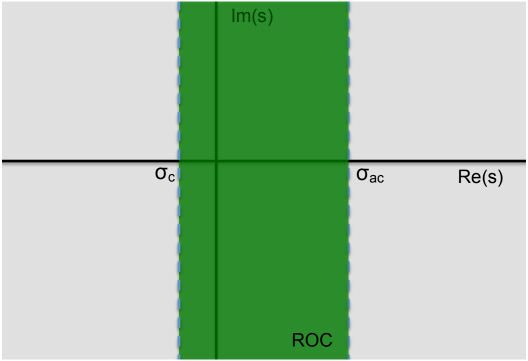
with

$$x_{\mathrm{c}}(t) = x(t)u(t) \quad \text{and} \quad x_{\mathrm{ac}}(t) = x(t)u(-t)$$

2 The two-sided Laplace transform

- The Laplace transform of $x_{\mathrm{c}}(t)$ is $X_{\mathrm{c}}(s)$ with a region of convergence given by $\mathrm{ROC}_{x_{\mathrm{c}}}$
- The Laplace transform of $x_{\mathrm{ac}}(t)$ is $X_{\mathrm{ac}}(s)$ with a region of convergence given by $\mathrm{ROC}_{x_{\mathrm{ac}}}$
- The Laplace transform of the noncausal signal $x(t)$ is given by $X(s) = X_{\mathrm{c}}(s) + X_{\mathrm{ac}}(s)$ with a region of convergence given by $\mathrm{ROC}_x = \mathrm{ROC}_{x_{\mathrm{c}}} \cap \mathrm{ROC}_{x_{\mathrm{ac}}}$
- The two-sided Laplace transform of $x(t)$ does not exist if $\mathrm{ROC}_x = \emptyset$
- If the two-sided Laplace transform exists then in general its ROC is a *strip* in the complex s-plane

2 The two-sided Laplace transform



Region of Convergence (ROC) for noncausal signals

- **Example 1:** The two-sided Laplace transform of the noncausal signal $x(t) = e^{-|t|}$ is

$$X(s) = -\frac{2}{s^2 - 1} \quad \text{with} \quad -1 < \text{Re}(s) < 1$$

- **Example 2:** The two-sided Laplace transform of the noncausal signal $x(t) = e^{at}$ with $a \in \mathbb{C}$ does not exist

Two-sided Laplace transform of standard signals

- The two-sided Laplace transform of the Dirac distribution is

$$\int_{t=-\infty}^{\infty} \delta(t)e^{-st}\,\mathrm{d}t = 1, \qquad s \in \mathbb{C}$$

- The two-sided Laplace transform of the derivative of the Dirac distribution is

$$\int_{t=-\infty}^{\infty} \delta'(t)e^{-st}\,\mathrm{d}t$$

- Recall that

$$f(t)\delta'(t) = -f'(0)\delta(t) + f(0)\delta'(t)$$

- Using the above equation with $f(t) = e^{-st}$ gives

$$e^{-st}\delta'(t) = s\delta(t) + \delta'(t)$$

- and substitution results in

$$\int_{t=-\infty}^{\infty} \delta'(t)e^{-st}\,\mathrm{d}t = s, \qquad s \in \mathbb{C}$$

since

$$\int_{t=-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \quad \text{and} \quad \int_{t=-\infty}^{\infty} \delta'(t)\,\mathrm{d}t = 0$$

Two-Sided Laplace Transforms

Time signal	**Two-sided Laplace transform**	**ROC**	**parameters**
$e^{at}u(t)$	$\frac{1}{s-a}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$	$a \in \mathbb{C}$
$-e^{at}u(-t)$	$\frac{1}{s-a}$	$\mathrm{Re}(s) < \mathrm{Re}(a)$	$a \in \mathbb{C}$
$\frac{t^{k-1}e^{at}}{(k-1)!}u(t)$	$\frac{1}{(s-a)^k}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$	$a \in \mathbb{C},\ k \in \mathbb{N}$
$-\frac{t^{k-1}e^{at}}{(k-1)!}u(-t)$	$\frac{1}{(s-a)^k}$	$\mathrm{Re}(s) < \mathrm{Re}(a)$	$a \in \mathbb{C},\ k \in \mathbb{N}$
$e^{at}\cos(\Omega_0 t)u(t)$	$\frac{s-a}{(s-a)^2+\Omega_0^2}$	$\mathrm{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$e^{at}\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{(s-a)^2+\Omega_0^2}$	$\mathrm{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$\delta(t)$	1	$\mathbb{C}$	–
$\delta'(t)$	s	$\mathbb{C}$	–

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- **Convolution**
  - Let $y(t) = x(t) * h(t)$
  - $X(s)$ is the two-sided Laplace transform of $x(t)$ with a region of convergence ROC_x
  - $H(s)$ is the two-sided Laplace transform of $h(t)$ with a region of convergence ROC_h
  - The Laplace transform of $y(t)$ is

$$Y(s) = X(s)H(s) \quad \text{with } \text{ROC}_y = \text{ROC}_x \cap \text{ROC}_h$$

3 Properties of the two-sided Laplace transform

- A convolution product in the time-domain is transformed into an ordinary product in the s-domain!

- Let's verify this statement
- We start with the definition of the two-sided Laplace transform

$$Y(s) = \int_{t=-\infty}^{\infty} y(t) e^{-st}\,\mathrm{d}t$$

- Substitution of the convolution integral gives

$$Y(s) = \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau)\,\mathrm{d}\tau\, e^{-st}\,\mathrm{d}t$$

- Interchanging the order of integration, we get

$$\begin{aligned}
Y(s) &= \int_{\tau=-\infty}^{\infty} x(\tau) \int_{t=-\infty}^{\infty} h(t-\tau)e^{-st}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\overset{p=t-\tau}{=} \int_{\tau=-\infty}^{\infty} x(\tau) \int_{p=-\infty}^{\infty} h(p)e^{-s(p+\tau)}\,\mathrm{d}p\,\mathrm{d}\tau \\
&= \int_{\tau=-\infty}^{\infty} x(\tau)e^{-s\tau}\,\mathrm{d}\tau \int_{p=-\infty}^{\infty} h(p)e^{-sp}\,\mathrm{d}p \\
&= X(s)H(s)
\end{aligned}$$

with $s \in \mathrm{ROC}_x \cap \mathrm{ROC}_h$

- Given an LTI system with input signal $x(t)$ and output signal $y(t)$
- Let $h(t)$ denote the impulse response of this system
- Output signal:

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

- In the Laplace- or s-domain:

$$Y(s) = H(s)X(s) \qquad \text{with } s \in \mathrm{ROC}_x \cap \mathrm{ROC}_h$$

- $H(s)$ is called the *transfer function* of the system

- **Example** Let $x(t) = u(t)$ and $h(t) = e^{-t}u(t)$. We are interested in the convolution product $y(t) = x(t) * h(t)$.
- Computing this product directly, we find

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } t > 0 \end{cases}$$

- Let us now compute the convolution of the two signals using the Laplace transform
- The two-sided Laplace transform of $x(t)$ is given by

$$X(s) = \frac{1}{s}, \qquad \text{Re}(s) > 0$$

- The two-sided Laplace transform of $h(t)$ is given by

$$H(s) = \frac{1}{s+1}, \qquad \mathrm{Re}(s) > -1$$

- The two-sided Laplace transform of $y(t)$ is given by

$$Y(s) = X(s)H(s) = \frac{1}{s(s+1)}, \qquad \mathrm{Re}(s) > 0$$

- What time signal corresponds this Laplace domain function?
- Observe that

$$Y(s) = \frac{1}{s} - \frac{1}{s+1}, \qquad \mathrm{Re}(s) > 0$$

- Use that the correspondence between a time-domain signal and its Laplace transform is essentially unique
- From the table of Laplace transforms we obtain

$$y(t) = (1 - e^{-t})u(t) \quad \text{or} \quad y(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } t > 0 \end{cases}$$

- **Differentiation in the Laplace-domain**
- Let $X(s)$ be the two-sided Laplace transform of a signal $x(t)$ with a region of convergence given by ROC_x
- We have stated that $X(s)$ can be differentiated w.r.t. s on ROC_x

- We have

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- Interchanging the order of differentiation and integration, we find

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = \int_{t=-\infty}^{\infty} \big[-tx(t)\big]e^{-st}\,\mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- The expression on the right is the Laplace transform of $-tx(t)$

- We conclude that

$$-tx(t) \text{ transforms into } \frac{\mathrm{d}X(s)}{\mathrm{d}s} \qquad s \in \mathrm{ROC}_x$$

- Differentiation in the s-domain corresponds to multiplication by $-t$ in the time-domain

- **Differentiation in the time-domain**
- Suppose we are given a time-domain signal $x(t)$ with a two-sided Laplace transform $X(s)$, $s \in \mathrm{ROC}_x$
- What is the Laplace transform of

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\,?$$

- By definition, we have

$$\begin{aligned} Y(s) &= \int_{t=-\infty}^{\infty} \frac{\mathrm{d}x(t)}{\mathrm{d}t} e^{-st}\,\mathrm{d}t \\ &= \lim_{T\to\infty} x(t)e^{-st}\Big|_{t=-T}^{T} - \int_{t=-\infty}^{\infty} x(t)\big(-se^{-st}\big)\,\mathrm{d}t \\ &= s\int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t \\ &= sX(s), \qquad \text{with } s \in \mathrm{ROC}_x \end{aligned}$$

- The first term on the right-hand side on the second line (blue formula) vanishes because we assumed that $X(s)$ exists

- We have found that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad \text{transforms into } sX(s) \qquad s \in \mathrm{ROC}_x$$

- Differentiation in the time-domain transforms into multiplication by s in the Laplace domain!

- **Integration in the time-domain**
- Suppose that we are again given a time-domain signal $x(t)$ with a two-sided Laplace transform $X(s)$, $s \in \mathrm{ROC}_x$
- What is the Laplace transform of

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)\,\mathrm{d}\tau\,?$$

- Observe that y is the convolution of x and the Heaviside unit step function u:

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)u(t-\tau)\,\mathrm{d}\tau$$

- Using the convolution property, we find

$$Y(s) = X(s) \cdot \frac{1}{s}, \qquad s \in \mathrm{ROC}_x \cap \mathrm{ROC}_u$$

- If $\mathrm{ROC}_x \cap \mathrm{ROC}_u = \emptyset$ then the Laplace transform of $y(t)$ does not exist
- Since $\mathrm{ROC}_u = \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$, we can also write

$$Y(s) = \frac{1}{s}X(s), \qquad s \in \left\{\mathrm{ROC}_x \middle| \mathrm{Re}(s) > 0\right\}$$

- We have found that

$$\int_{\tau=-\infty}^{t} x(\tau)\,\mathrm{d}\tau \quad \text{transforms into } \frac{1}{s}X(s)$$

with $s \in \left\{\mathrm{ROC}_x \middle| \mathrm{Re}(s) > 0\right\}$

- Integration in the time-domain transforms into division by s in the Laplace domain!

- **Shift in the time-domain**
- Again we have a signal $x(t)$ with a Laplace transform $X(s)$, $s \in \mathrm{ROC}_x$
- Let $y(t)$ be a shifted version of $x(t)$ with time shift $\tau \in \mathbb{R}$:

$$y(t) = x(t+\tau), \qquad \tau \in \mathbb{R}$$

- What is the Laplace transform of $y(t)$? We compute

$$\begin{aligned} Y(s) &= \int_{t=-\infty}^{\infty} x(t+\tau)e^{-st}\,\mathrm{d}t \\ &\overset{p=t+\tau}{=} \int_{p=-\infty}^{\infty} x(p)e^{-s(p-\tau)}\,\mathrm{d}p \\ &= e^{s\tau}\int_{p=-\infty}^{\infty} x(p)e^{-sp}\,\mathrm{d}p \\ &= e^{s\tau}X(s), \qquad s \in \mathrm{ROC}_x \end{aligned}$$

- We have found that

$$x(t+\tau) \text{ transforms into } e^{s\tau}X(s), \qquad s \in \text{ROC}_x$$

- **Example.** Suppose the two-sided Laplace transform of a signal $h(t)$ is given by

$$H(s) = \frac{1}{1-e^{-sT}} \qquad \text{with } T>0 \quad \text{and} \quad \text{ROC}_h = \{s \in \mathbb{C}; \text{Re}(s) > 0\}$$

  What is $h(t)$?

- **Solution:** Set $z = e^{-sT}$. We then have

$$\frac{1}{1-e^{-sT}} = \frac{1}{1-z}$$

- Now recall the power series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots \qquad \text{for } |z| < 1 \text{ and } z \in \mathbb{C}$$

- In our case, we have with $s = \sigma + \mathrm{j}\Omega$:

$$|z| = \left|e^{-sT}\right| = \left|e^{-\sigma T - \mathrm{j}\Omega T}\right| = \left|e^{-\sigma T}\right| \cdot \left|e^{-\mathrm{j}\Omega T}\right| = e^{-\sigma T},$$

  since $\left|e^{-\mathrm{j}\Omega T}\right| = 1$
- We also have $\mathrm{Re}(s) = \sigma > 0$ and $T > 0$
- Consequently, $|z| = e^{-\sigma T} < 1$ and

$$\frac{1}{1-e^{-sT}} = 1 + e^{-sT} + e^{-2sT} + \ldots$$

- Using the Laplace transform of the Dirac distribution and the shifting property, we find

$$h(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + ...$$

- Suppose $x(t)$ is a causal signal with support $(0, T_x)$
- For example, $x(t) = \Lambda(t)$, support $(0, 2)$
- The Laplace transform of $x(t)$ is $X(s)$, $s \in \mathbb{C}$
- Given now an LTI system with a transfer function

$$H(s) = \frac{1}{1-e^{-sT}} \qquad \text{with } T > T_x \text{ and } \mathrm{Re}(s) > 0$$

- The Laplace transform of the output signal is

$$Y(s) = \frac{X(s)}{1 - e^{-sT}}, \qquad \mathrm{Re}(s) > 0$$

- The corresponding output signal is given by the convolution integral

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

with

$$h(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + \ldots = \sum_{k=0}^{\infty} \delta(t-kT)$$

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- Substitution gives

$$\begin{aligned} y(t) &= \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau = \int_{\tau=-\infty}^{\infty} \sum_{k=0}^{\infty} \delta(\tau-kT)\,x(t-\tau)\,\mathrm{d}\tau \\ &= \sum_{k=0}^{\infty} \int_{\tau=-\infty}^{\infty} \delta(\tau-kT)x(t-\tau)\,\mathrm{d}\tau = \sum_{k=0}^{\infty} x(t-kT) \\ &= x(t) + x(t-T) + x(t-2T) + \dots \end{aligned}$$

- We have constructed a periodic extension of $x(t)$ for $t > 0$

- In the Laplace-domain

$$Y(s) = \frac{X(s)}{1 - e^{-sT}} = X(s) \sum_{k=0}^{\infty} e^{-skT}$$
$$= X(s) + X(s)e^{-sT} + X(s)e^{-s2T} + \dots$$

- Apply an inverse Laplace transform and use the shift property to each term on the right-hand side to obtain

$$y(t) = x(t) + x(t - T) + x(t - 2T) + \dots$$

- **Shift in the Laplace domain**
- Let $X(s)$ be the two-sided Laplace transform of $x(t)$ with $s \in \mathrm{ROC}_x$
- Is there a time-domain signal that corresponds to $X(s-a)$ with $s-a \in \mathrm{ROC}_x$?
- We use the definition of the Laplace transform

$$X(s-a) = \int_{t=-\infty}^{\infty} x(t)e^{-(s-a)t}\,\mathrm{d}t = \int_{t=-\infty}^{\infty} e^{at}x(t)\,e^{-st}\,\mathrm{d}t$$

- The answer is yes

$$e^{at}x(t) \text{ transforms into } X(s-a), \qquad s-a \in \mathrm{ROC}_x$$

- **Scaling**
- Let $x(t)$ have a two-sided Laplace transform $X(s)$ with $s \in \mathrm{ROC}_x$
- Given a nonzero real number a, what is the Laplace transform of

$$y(t) = x(at)\,?$$

- We use the definition of the Laplace transform
- For $a > 0$, we find

$$\begin{aligned} Y(s) &= \int_{t=-\infty}^{\infty} y(t)e^{-st}\,\mathrm{d}t = \int_{t=-\infty}^{\infty} x(at)e^{-st}\,\mathrm{d}t \\ &\overset{\tau=at}{=} \frac{1}{a}\int_{\tau=-\infty}^{\infty} x(\tau)e^{-(s/a)\tau}\,\mathrm{d}\tau \\ &= \frac{1}{a}X\left(\frac{s}{a}\right), \qquad s/a \in \mathrm{ROC}_x \end{aligned}$$

- Similarly, for $a < 0$ we obtain

$$Y(s) = -\frac{1}{a}X\left(\frac{s}{a}\right) \qquad s/a \in \mathrm{ROC}_x$$

- Combining both results, we have

$$x(at) \text{ transforms into } \frac{1}{|a|}X(\frac{s}{a}) \qquad \text{for } a \in \mathbb{R} \setminus \{0\}$$

and with $s/a \in \mathrm{ROC}_x$

- **Example**
- *Switch on.* We have seen that the two-sided Laplace transform of the Heaviside unit step function $u(t)$ is given by

$$U(s) = \frac{1}{s} \quad \text{with} \quad s \in \mathrm{ROC}_u = \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$$

- *Switch off.* We have also seen that the two-sided Laplace transform of the anti-causal switch-off signal $f(t) = u(-t)$ is

$$F(s) = -\frac{1}{s} \quad \text{with} \quad s \in \mathrm{ROC}_f = \{s \in \mathbb{C}; \mathrm{Re}(s) < 0\}$$

- Clearly,

$$F(s) = U(-s) \quad \text{with } s \in \mathrm{ROC}_f \text{ or } -s \in \mathrm{ROC}_u$$

3 Properties of the two-sided Laplace transform

Properties of the Two-Sided Laplace Transform

Property	Time signal	Two-sided Laplace transform	ROC	Parameters
Convolution	$y(t) = h(t) * x(t)$	$Y(s) = H(s)X(s)$	$\text{ROC}_h \cap \text{ROC}_x$	–
Diff. s-domain	$-tx(t)$	$\frac{\mathrm{d}X(s)}{\mathrm{d}s}$	ROC_x	–
Diff. t-domain	$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$sX(s)$	ROC_x	–
Int. t-domain	$\int_{\tau=-\infty}^{t} x(\tau)\,\mathrm{d}\tau$	$\frac{1}{s}X(s)$	$\{\text{ROC}_x \mid \text{Re}(s) > 0\}$	–
Shift t-domain	$x(t+\tau)$	$e^{s\tau}X(s)$	ROC_x	$\tau \in \mathbb{R}$
Shift s-domain	$e^{at}x(t)$	$X(s-a)$	$s - a \in \text{ROC}_x$	$a \in \mathbb{C}$
Scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{s}{a}\right)$	$s/a \in \text{ROC}_x$	$a \in \mathbb{R} \setminus \{0\}$

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