Signals and Systems

Linear and Time-Invariant Systems

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Contents
- ▶ Linear and time-invariant systems
- ▶ The impulse response and the convolution integral
- ▶ Linear ordinary differential equations and LTI systems
- ▶ Causality
- ▶ BIBO stability

Book
Chapter 2

Exercises:
2.1 – 2.5, 2.8, 2.9, 2.11

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2 Linear and time-invariant systems

We consider a system with a single input and a single output

Input signal $x(t)$, output signal $y(t)$

Such systems are called SISO systems

SISO stands for Single Input Single Output

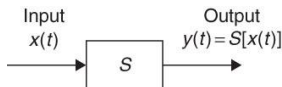If a system has Multiple Inputs and Multiple Outputs it is called a MIMO system

We restrict ourselves to SISO systems

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The action of the system on the input signal $x(t)$ is described by the system operator S. We write

$$y(t) = S\{x(t)\}$$

In this course we are particularly interested in systems that are **L**inear and **T**ime-**I**nvariant

Such systems are called LTI systems

Linearity Suppose we have two input signals $x_1(t)$ and $x_2(t)$. Denote the corresponding output signals by $y_1(t)$ and $y_2(t)$:

$$y_1(t) = S\{x_1(t)\} \quad \text{and} \quad y_2(t) = S\{x_2(t)\}$$

The system is called *linear* if

$$\begin{aligned} y(t) &= S\{\alpha x_1(t) + \beta x_2(t)\} \\ &= \alpha S\{x_1(t)\} + \beta S\{x_2(t)\} \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

for any two constants α and β

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Any linear combination of input signals produces the same linear combination of their corresponding output signals

Taking $\beta = 0$, it follows from the above definition that

$$y(t) = S\{\alpha x_1(t)\} = \alpha S\{x_1(t)\} = \alpha y_1(t)$$

In other words, for a linear system, if you scale the input signal by a factor α, the output signal will scale with the same factor

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Example Consider a SISO system with input signal $x(t)$ and an output signal given by

$$y(t) = \frac{1}{T} \int_{\tau=t-T}^{t} x(\tau) \,\mathrm{d}\tau + B,$$

where B is a constant. Such a system is called a biased averager (can you see why?)

Scaling the input signal by a factor α, we obtain the output signal

$$\frac{\alpha}{T} \int_{\tau=t-T}^{t} x(\tau) \,\mathrm{d}\tau + B,$$

which is not equal to $\alpha y(t)$ unless $B = 0$. The averager is nonlinear for $B \neq 0$ and linear for $B = 0$

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Time-Invariance Let $y(t)$ be the output signal that corresponds to an input signal $x(t)$:

$$y(t) = S\{x(t)\}$$

The system is called *time-invariant* if

$$y(t - \tau) = S\{x(t - \tau)\}$$

for any time shift $\tau \in \mathbb{R}$

In words: shifting your input signal produces an equally time-shifted output signal

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Let the Dirac delta function be the input signal of an LTI system

The corresponding output signal is written as $h(t)$ and is called the *impulse response*:

$$h(t) = S\{\delta(t)\}$$

We claim that if you know the impulse response of an LTI system then you know the response to any other input signal!

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To show this, let $y(t)$ be the output signal that corresponds to an input signal $x(t)$:

$$y(t) = S\{x(t)\}$$

Because of the sifting property of the delta function, we have

$$x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau$$

The right-hand side of the above expression can be seen as a continuous weighted summation of shifted Dirac distributions

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Substitution gives

$$y(t) = S\Big\{\int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau\Big\}$$

Now note that S is a linear system operator and acts on functions that depend on time t

This allows us to write

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)S\big\{\delta(t-\tau)\big\}\,\mathrm{d}\tau$$

Since the system is time-invariant as well, we have

$$h(t-\tau) = S\big\{\delta(t-\tau)\big\}$$

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We arrive at

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

Knowing the impulse response $h(t)$, we can determine the response $y(t)$ to any input signal $x(t)$ by evaluating the above integral

This integral is called the *convolution integral* or *convolution product* of the signals x and h

Short-hand notation:

$$y = x * h \quad \text{or} \quad y(t) = x(t) * h(t)$$

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For two real numbers a and b, we have $ab = ba$

The product of two real numbers commutes

Is this also true for the convolution product? In other words, do we have $x * h = h * x$?

The answer is yes. Let's check it.

$$y(t) = x * h = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau \stackrel{p=t-\tau}{=} \int_{p=-\infty}^{\infty} x(t-p)h(p)\,\mathrm{d}p$$

$$= \int_{p=-\infty}^{\infty} h(p)x(t-p)\,\mathrm{d}p = h * x$$

Conclusion: the convolution product of two signals commutes

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For the product of real numbers, there exists an identity element called "one" and written as 1 for which $a = a \cdot 1 = 1 \cdot a$

What is the identity element for the convolution product?

We already know the answer to this question. It is the Dirac delta function!

$$x = x * \delta = \delta * x$$

The convolution product is also *associative*, that is, for three signals u, v, and w, we have (check this yourself)

$$(u * v) * w = u * (v * w)$$

This property can be exploited to determine the total impulse function of two LTI systems interconnected in cascade

System 1: input signal $x(t)$, impulse function $h_1(t)$, output signal $y_1(t)$

System 2: input signal $y_1(t)$, impulse function $h_2(t)$, output signal $y(t)$

We assume that System 2 does not "load" System 1

Response of the total system:

$$y = y_1 * h_2 = (x * h_1) * h_2 = x * (h_1 * h_2) = x * h$$

where we have introduced the impulse function of the total system as

$$h = h_1 * h_2 = h_2 * h_1$$

Note that since the convolution product of two signals commute, we can interchange the order of the subsystems without affecting the output signal $y(t)$ (provided both systems do not "load" each other)

Finally, if the support of a signal x is (ℓ_x, u_x) and the support of a signal h is (ℓ_h, u_h) then

the support of $y(t) = x(t) * h(t)$ is $(\ell_x + \ell_h, u_x + u_h)$

Verify this statement!

Examples

Exercise 1. Let $p(t)$ denote the rectangular pulse signal. Determine $x(t) = p(t) * p(t)$.

Exercise 2. (*13-12-2023*) Determine the signal $y(t) = h(t) * x(t)$, where $h(t) = u(t-1)$, $x(t) = u(t-2)$, and $u(t)$ is the Heaviside unit step function.

Exercise 3. (*19-07-2021*) Given the signal $x(t) = te^{-\alpha t}u(t)$ with $\alpha > 0$. Determine the convolution $y(t) = x(t) * x(t)$ directly using the convolution integral.

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Up till now we have been looking at fairly general systems whose action on the input signal is described by some operator S

Let us now be more specific and consider systems described by the linear ordinary differential equation

$$\left(a_N \frac{\mathrm{d}^N}{\mathrm{d}t^N} + a_{N-1} \frac{\mathrm{d}^{N-1}}{\mathrm{d}t^{N-1}} + ... + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) y(t) =$$
$$\left(b_M \frac{\mathrm{d}^M}{\mathrm{d}t^M} + b_{M-1} \frac{\mathrm{d}^{M-1}}{\mathrm{d}t^{M-1}} + ... + b_1 \frac{\mathrm{d}}{\mathrm{d}t} + b_0\right) x(t)$$

which holds for $t > 0$ and N and M are nonnegative integers

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In the above differential equation, $x(t)$ is the prescribed (known) input signal and $y(t)$ is the desired output signal

To obtain the output signal $y(t)$, we also need the N initial conditions

$$y(0) \quad \text{and} \quad \left.\frac{d^k y(t)}{dt^k}\right|_{t=0} \quad \text{for } k = 1, 2, ..., N-1$$

RLC circuits, mechanical systems, etc. can all be described by a differential equation of the above form

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Further on we will show how to solve the differential equation using the Laplace transform

Here, we state that the solution (output signal) can be written as

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

▶ $y_{zs}(t)$ is called the *zero-state response*. This is the solution exclusively due to the input signal with initial conditions set to zero

    > **Special case:** The *impulse response* $h(t)$ of the system is a zero-state response of the system in case the input signal is the Dirac impulse function: $x(t) = \delta(t)$

▶ $y_{zi}(t)$ is called the *zero-input response*. This is the solution exclusively due to the initial conditions with the input set to zero

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► For vanishing initial conditions the system is linear and time-invariant.

► For nonvanishing initial conditions, the system is no longer an LTI system

Example. Consider a circuit consisting of a resistor R in series with an inductor L and a voltage source $v(t) = Bu(t)$. The initial current in the inductor is I_0. The input signal of the system is $v(t)$, the current $i(t)$ is the output signal.

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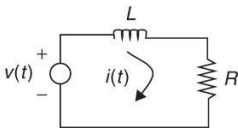
The output signal is given by

$$i(t) = i_{\mathrm{zs}}(t) + i_{\mathrm{zi}}(t) \quad \text{for } t > 0$$

with

$$i_{\mathrm{zs}}(t) = \frac{B}{R}(1 - e^{-t/\tau}), \quad i_{\mathrm{zi}}(t) = I_0 e^{-t/\tau}, \quad \text{and} \qquad \tau = L/R$$

▶ If we double the amplitude of the input signal the output signal becomes $i(t) = 2i_{\mathrm{zs}}(t) + i_{\mathrm{zi}}(t)$ with $i_{\mathrm{zs}}(t)$ and $i_{\mathrm{zi}}(t)$ as above

▶ Clearly, the output is *not* doubled, since $i_{\mathrm{zi}}(t)$ does not vanish: the system is not linear

▶ However, for $I_0 = 0$ (vanishing initial condition) we have $i_{\mathrm{zi}}(t) = 0$ and the system is linear

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Consider a continuous-time system with input signal $x(t)$ and output signal $y(t)$. The system is *causal*

▶ if the output signal $y(t) = 0$ for a vanishing input signal and vanishing initial conditions

▶ if the output $y(t)$ does not depend on future inputs

An LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0$$

Indeed, for an LTI system we have the convolution integral

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau$$

Writing the integral as

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)h(t-\tau)\,\mathrm{d}\tau + \int_{\tau=t}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

we observe that in the second term on the right-hand side integration takes place over future inputs

For a causal system, these inputs cannot contribute to the output signal at time instant t

Consequently, for a causal system we must have $h(t-\tau) = 0$ for $t < \tau < \infty$ or $h(t) = 0$ for $t < 0$

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In case the LTI system is causal, we are left with

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

In addition, if the input signal $x(t)$ also vanishes for $t < 0$ then the convolution integral simplifies even further to

$$y(t) = \int_{\tau=0}^{t} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

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Finally, we discuss the concept of BIBO stability

BIBO stands for Bounded Input Bounded Output

We are given a bounded input signal $x(t)$, that is, a signal that satisfies

$$|x(t)| \leq M$$

for some positive M

We ask: Under what conditions is the output $y(t)$ of an LTI system also bounded?

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Consider

$$\begin{aligned}
\left| y(t) \right| &= \left| \int_{\tau=-\infty}^{\infty} x(t-\tau) h(\tau) \, \mathrm{d}\tau \right| \\
&\leq \int_{\tau=-\infty}^{\infty} \left| x(t-\tau) \right| \left| h(\tau) \right| \mathrm{d}\tau \leq M \int_{\tau=-\infty}^{\infty} \left| h(\tau) \right| \mathrm{d}\tau
\end{aligned}$$

From this last inequality it follows that if

$$\int_{\tau=-\infty}^{\infty} \left| h(\tau) \right| \mathrm{d}\tau < \infty$$

then the output signal $y(t)$ is bounded

Conclusion: If the impulse response $h(t)$ is absolutely integrable (its action is finite) then the LTI system is BIBO stable

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