18. Solutions to the trial exam part 2 of 14 January 2016

Question 1

We are given the signals

$$x[n] = \begin{cases} 1, & 0 \le n \le 5, \\ 0, & \text{elsewhere} \end{cases} \qquad h[n] = [\cdots, 0, \boxed{1}, -2, 1, 0, \cdots]$$

a) Determine $y[n] = x[n] * h[n]$ using the convolution sum in time domain.

Answer

Compute $y[n] = x[n] * h[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$:

$h[0]x[n]$:	$\boxed{1}$	1	1	1	1	1	0	0	$0 \cdots$
$h[1]x[n-1]$:	$\boxed{0}$	-2	-2	-2	-2	-2	-2	0	$0 \cdots$
$h[2]x[n-2]$:	$\boxed{0}$	0	1	1	1	1	1	1	$0 \cdots$
$y[n]$:	$\boxed{1}$	-1	0	0	0	0	-1	1	$0 \cdots$

Question 1

b) Determine the z-transformaties $X(z)$ and $H(z)$, also specify the regions of convergence (ROCs).

c) Determine $y[n] = x[n] * h[n]$ using the (inverse) z-transform.

Answer

$$
\begin{aligned}
X(z) &= 1 + z^{-1} + \cdots + z^{-5} \\
&= \tfrac{1-z^{-6}}{1-z^{-1}} \qquad \text{ROC: } z \neq 0 \\
H(z) &= 1 - 2z^{-1} + z^{-2} \\
&= (1 - z^{-1})^2 \qquad \text{ROC: } z \neq 0
\end{aligned}
$$

$$
\begin{aligned}
Y(z) = H(z)X(z) &= (1 - z^{-1})^2 \tfrac{1-z^{-6}}{1-z^{-1}} \\
&= (1 - z^{-1})(1 - z^{-6}) \\
&= 1 - z^{-1} - z^{-6} + z^{-7}
\end{aligned}
$$

$y[n] = \delta[n] - \delta[n-1] - \delta[n-6] + \delta[n-7]$, same result as for item a).

Question 1

d) Given

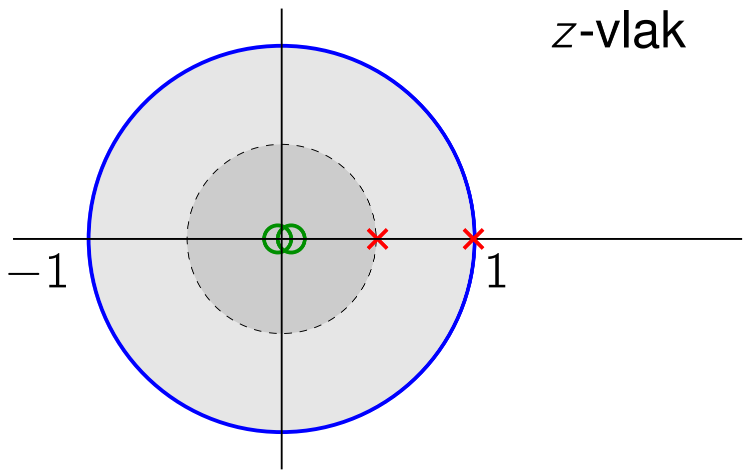
$$X(z) = \frac{1}{1 - 1\frac{1}{2}\,z^{-1} + \frac{1}{2}\,z^{-2}}\,, \qquad z \in \text{ROC},$$

determine $x[n]$ using the inverse z-transform if (d1) ROC: $|z| > 1$, (d2) ROC: $|z| < \frac{1}{2}$, (d3) ROC: $\frac{1}{2} < |z| < 1$.

Answer

First write this in terms of z^{-1} (already done), make it 'proper' (already done), then split (partial fraction expansion).

$$X(z) = \frac{1}{1 - 1\frac{1}{2}\,z^{-1} + \frac{1}{2}\,z^{-2}} = \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2}z^{-1}}$$



Question 1

d1) The region of convergence runs until $z \to \infty$: causal response. Hence

$$x[n] = 2u[n] - \left(\frac{1}{2}\right)^n u[n]$$

d2) The region of convergence includes $z = 0$: anti-causal reponse. Rewrite $X(z)$ as

$$X(z) = -\frac{2z}{1-z} + \frac{2z}{1-2z} \,.$$

The inverse z-transform of $\frac{1}{1-z}$ is $u[-n]$ and of $\frac{1}{1-2z}$ is $2^{-n}u[-n]$, while multiplication with z is equivalent to an 'advance', so that

$$x[n] = -2u[-n-1] + 2 \cdot 2^{-n-1} u[-n-1]$$

d3) Rewrite $X(z)$ as

$$X(z) = -\frac{2z}{1-z} - \frac{1}{1-\frac{1}{2}z^{-1}} .$$

For this ROC, the first term results in an anti-causal response (pole at the outside of the ROC) , while the second term results in a causal response (pole at the inside of the ROC). Hence,

$$x[n] = -2u[-n-1] - \left(\frac{1}{2}\right)^n u[n]$$

Question 1

e) Given $x[n] = (-1)^n u[n]$, determine the DTFT $X(\omega)$.

Answer

The z-transform is

$$X(z) = \frac{1}{1 + z^{-1}}, \qquad \text{ROC: } |z| > 1$$

The resulting DTFT is (continuation of $X(z)$ until the unit circle except for $z = -1$)

$$X(e^{j\omega}) = \frac{1}{1 + e^{-j\omega}}, \qquad \omega \neq \cdots, -\pi, \pi, 3\pi \cdots$$

At the frequencies $\pm\pi + 2\pi k$ we obtain delta spikes. To determine these, use the following derivation:

Question 1

For $y[n] = u[n]$ we have seen that $Y(\omega) = \frac{1}{1-e^{-j\omega}} + \pi \sum_k \delta(\omega - 2\pi k)$.

We also saw that for a modulation:

$$(-1)^n y[n] \quad \leftrightarrow \quad \frac{1}{2}\left[Y(\omega-\pi) + Y(\omega+\pi)\right] = Y(\omega - \pi)$$

(due to periodicity of the spectrum with period 2π, both shifts exactly coincide).

Together, we obtain

$$X(\omega) = \frac{1}{1-e^{-j(\omega-\pi)}} + \pi \sum_k \delta(\omega - \pi - 2\pi k) = \frac{1}{1+e^{-j\omega}} + \pi \sum_k \delta(\omega - \pi - 2\pi k)$$

Question 1

f) Given $X(\omega) = \cos(\omega)$, determine $x[n]$.

Answer

$$X(\omega) = \tfrac{1}{2}e^{j\omega} + \tfrac{1}{2}e^{-j\omega} \quad \rightarrow \quad x[n] = \tfrac{1}{2}\delta[n+1] + \tfrac{1}{2}\delta[n-1]$$

Question 2

The transfer function of a causal LTI system is given by

$$H(z) = \frac{z+1}{(z+1)^2+1}, \quad z \in \text{ROC}$$

a) Determine all poles and zeros of the system, and make a drawing of the complex z-plane.

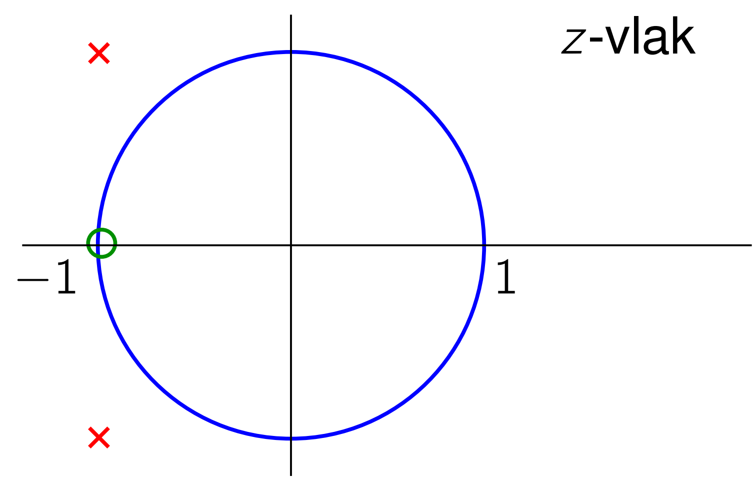
b) Specify the ROC.

Answer

Zeros for $z = -1$ and $z = \infty$.

Poles for $z = -1 \pm j$.

Because the system was specified as being causal, the ROC is the outside of a circle, resulting in $|z| > \sqrt{2}$.



Question 2

c) Is the system BIBO stable? (Why?)

d) Determine the frequency response of the system.

e) Determine the impulse response $h[n]$.

Answer

Not BIBO stable because $|z| = 1$ is not in the ROC.

The frequency response does not exist because $|z| = 1$ is not in the ROC.

Otherwise, we would obtain

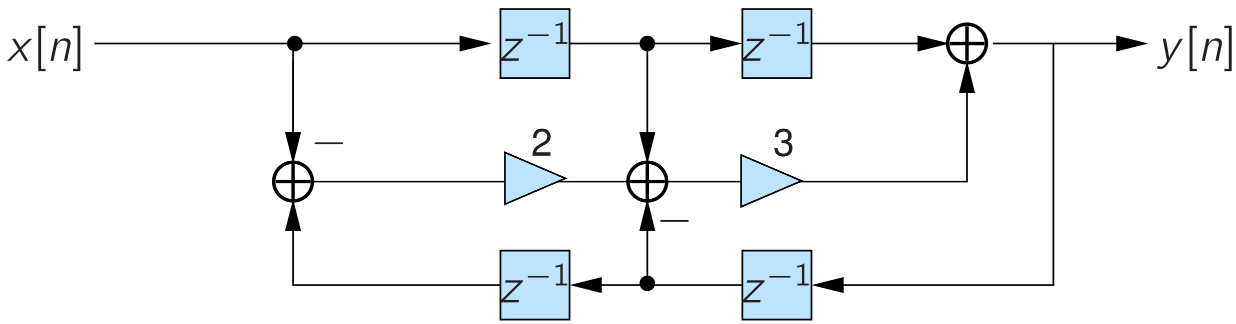
$$H(e^{j\omega}) = \frac{e^{j\omega} + 1}{(e^{j\omega} + 1)^2 + 1}$$

$$H(z) = \frac{1/2}{z+1+j} + \frac{1/2}{z+1-j} = \frac{1/2\, z^{-1}}{1+(1+j)z^{-1}} + \frac{1/2\, z^{-1}}{1+(1-j)z^{-1}}$$

$$\begin{aligned} h[n] &= \tfrac{1}{2}(-1-j)^{n-1}u[n-1] + \tfrac{1}{2}(-1+j)^{n-1}u[n-1] \\ &= \tfrac{1}{2}(\sqrt{2})^{n-1}(e^{j(3\pi/4)(n-1)} + e^{-j(3\pi/4)(n-1)})u[n-1] \\ &= (\sqrt{2})^{n-1}\cos(\tfrac{3\pi}{4}(n-1))u[n-1] \qquad \text{unstable...} \end{aligned}$$

Question 3

We are given the following system:



a) Determine the system transfer function $H(z)$.

b) Is this a stable system? (Why?)

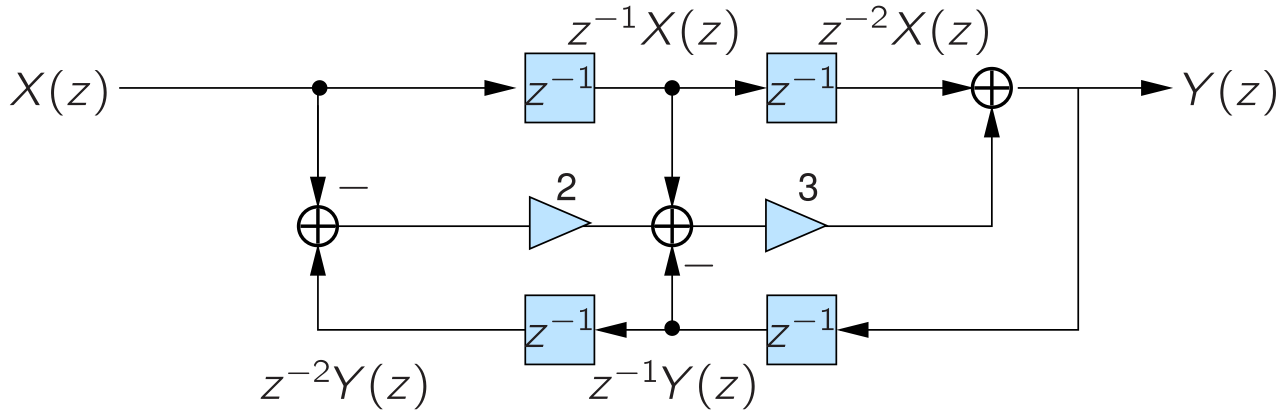
c) Is this a minimal realization? (Why?)

d) Make a drawing of the ”direct form no. II” realization, also specify the coefficients.

Question 3

Answer

a) Introduce additional parameters: here not really necessary, because the output of each delay is a simple function of x or y:



Hence

$$
\begin{aligned}
Y(z) &= z^{-2}X(z) + 3\left(z^{-1}X(z) - z^{-1}Y(z) + 2\left(z^{-2}Y(z) - X(z)\right)\right) \\
&= \left(z^{-2}X(z) + 3z^{-1}X(z) - 6X(z)\right) - 3z^{-1}Y(z) + 6z^{-2}Y(z) \\
Y(z)(1 + 3z^{-1} - 6z^{-2}) &= X(z)(z^{-2} + 3z^{-1} - 6) \\
H(z) = \frac{Y(z)}{X(z)} &= \frac{z^{-2} + 3z^{-1} - 6}{1 + 3z^{-1} - 6z^{-2}}
\end{aligned}
$$

Question 3

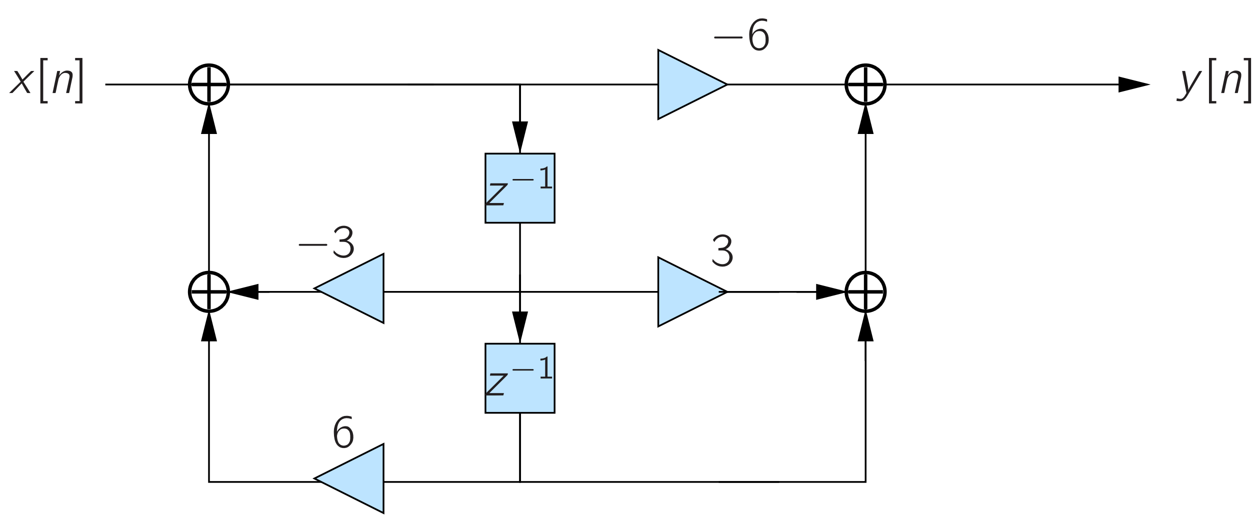
b) Determine the poles:

$$z^2 + 3z - 6 = 0 \quad \rightarrow \quad z_{1,2} = -\frac{3}{2} \pm \frac{1}{2}\sqrt{9+24} = \{-4.37, 1, 37\}$$

The poles are outside the unit circle (and the realization is causal): unstable

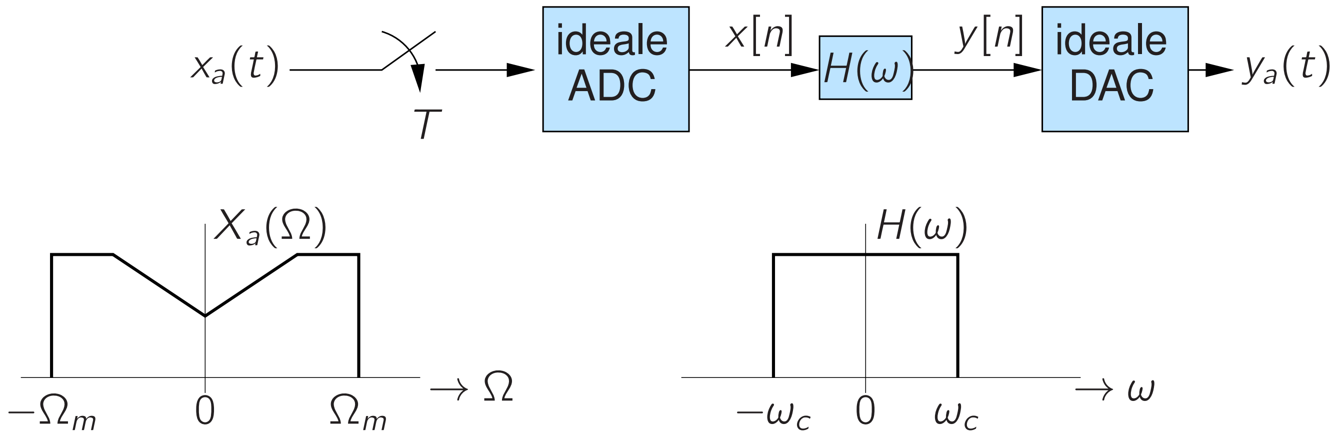
c) Not minimal: 4 delays used for a 2nd order system

d)



Question 4

A continuous-time signal $x_a(t)$ has a spectrum as indicated below. $x_a(t)$ is sampled at a frequency $1/T$ equal to the Nyquist rate, filtered by an ideal low-pass filter $H(\omega)$, and converted back into an analog signal $y_a(t)$. The cut-off frequency of $H(\omega)$ is $\omega_c = \Omega_m T/3$.



a) Give an expression for T.

Answer

a) $T = \dfrac{1}{2F_m} = \dfrac{2\pi}{2\Omega_m} = \dfrac{\pi}{\Omega_m}$.

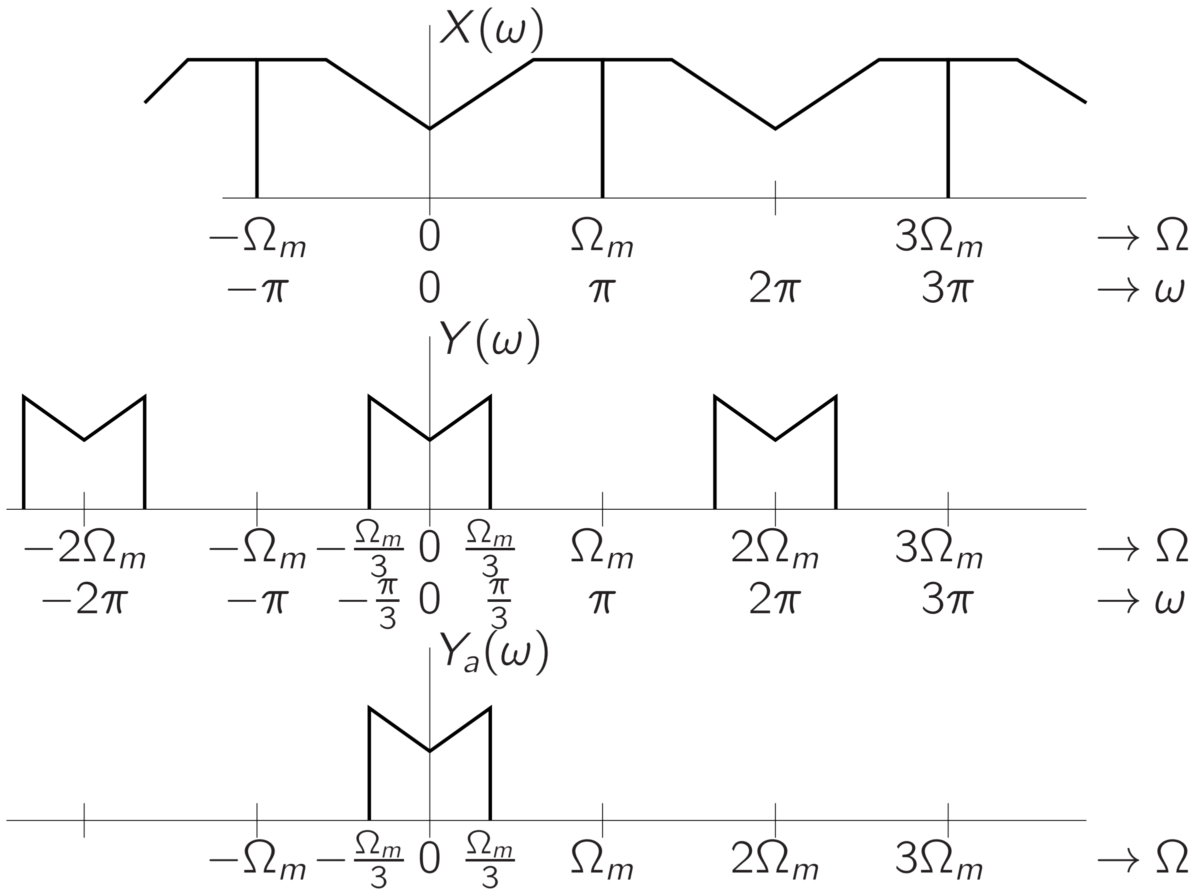
Question 4

b) Draw the spectrum corresponding to $x[n]$. Also indicate the freqencies.

c) Draw the spectrum corresponding to $y[n]$. Also indicate the freqencies.

d) Draw the spectrum corresponding to $y_a(t)$. Also indicate the freqencies.

Answer



Question 5

We would like to design a digital low-pass filter with the following specifications:

– Pass-band ripple: ≤ 1 dB

– Pass-band: 4 kHz

– Stop-band damping: ≥ 40 dB

– Stop-band: from 6.0 kHz

– Sample rate: 24 kHz

The digital filter is designed by applying a bilinear transform to an analog transfer function.

a) What are the pass-band and stop-band frequencies in the digital time-domain?

b) What are the filter specifications in the analog time-domain?

Question 5

Answer

a)

$$f_p = \frac{4}{24} = \frac{1}{6} \Rightarrow \omega_p = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$f_s = \frac{6}{24} = \frac{1}{4} \Rightarrow \omega_s = \frac{2\pi}{4} = \frac{\pi}{2}$$

b) Apply the bilinear transform: $\omega = 2\arctan(\Omega)$, $\Omega = \tan(\frac{\omega}{2})$:

$$\Omega_p = \tan(\frac{\omega_p}{2}) = 0.5774$$

$$\Omega_s = \tan(\frac{\omega_s}{2}) = 1$$

For the ripples: $\delta_p = 10^{-1/20} = 0.8913$, $\delta_s = 10^{-40/20} = 0.01$.

Question 5

c) Compute the required filter order for a Butterworth filter

d) Compute the required filter order for a Chebyshev filter

(Remark: $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$.)

Answer

c) Use the derivation shown in the book or on the slides.

Butterworth: $|H(\omega)|^2 = \dfrac{1}{1 + \epsilon^2(\Omega/\Omega_p)^{2N}}$.

For the pass-band, we find $\dfrac{1}{1+\epsilon^2} = \delta_p^2 \quad \Rightarrow \quad \epsilon = \sqrt{\dfrac{1}{0.7943} - 1} = 0.5089$

For the stop-band, we have: $\delta_s = 10^{-40/20} = 0.01$.

For the filter order:

$$|H(\Omega_s)|^2 = \frac{1}{1 + \epsilon^2(\Omega_s/\Omega_p)^{2N}} = \delta_s^2 \quad \Rightarrow \quad \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{\frac{1}{\delta_s^2} - 1}{\epsilon^2} =: \frac{\delta^2}{\epsilon^2} \quad \Rightarrow \quad N \geq \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)}$$

Substitution gives $\delta = 99.995$ and $N \geq 9.618$, i.e., the filter order is $N \geq 10$.

Question 5

d) Similar derivation for Chebyshev results in

$$N \geq \frac{\cosh^{-1}(\delta/\epsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = 5.212$$

Hence a 6th order filter.

Question 5

e) Make a drawing of the transfer function of the resulting two digital filters after the bilinear transform. Also mark the filter specificaties in the figure.

Answer