Signals and Systems

Standard Signals

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September 3, 2024

1 Outline

1 Overview

Contents

- Standard signals
- Even and odd signals
- The energy, action, and power of continuous-time signals
- Periodic signals
- The Dirac delta function

Book
Chapter 1

Exercises:
1.1 – 1.4, 1.6, 1.9, 1.12
Additional exercises at the end of this set of slides

2 Outline

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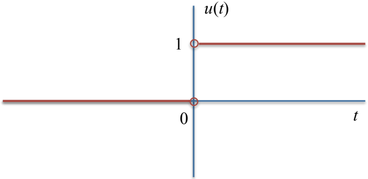
The Heaviside unit step function

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$



- Can be used to model *switch-on* phenomena
- $u(-t)$ can be used to model *switch-off* phenomena

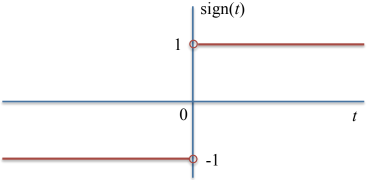
Oliver Heaviside
1850 – 1925

The sign or signum function

$$\text{sign}(t) = \begin{cases} -1 & \text{for } t < 0 \\ +1 & \text{for } t > 0 \end{cases}$$

The sign function in terms of the step function

$$\text{sign}(t) = 2u(t) - 1 \quad \text{or} \quad \text{sign}(t) = u(t) - u(-t)$$

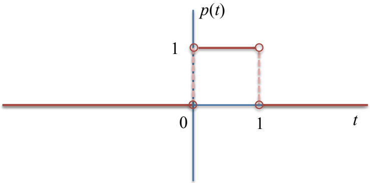


The rectangular pulse function

$$p(t) = \begin{cases} 1 & \text{for } 0 < t < 1 \\ 0 & \text{for } t < 0 \text{ and } t > 1 \end{cases}$$

The pulse function in terms of unit step functions

$$p(t) = u(t) - u(t-1)$$

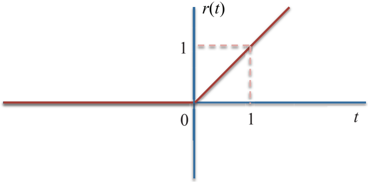


The ramp function

$$r(t) = \begin{cases} t & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

The ramp function in terms of the unit step function

$$r(t) = tu(t) \quad \text{or} \quad r(t) = \int_{\tau=-\infty}^{t} u(\tau)\,\mathrm{d}\tau$$

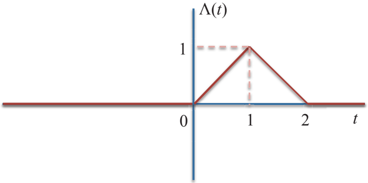


The triangular pulse function

$$\Lambda(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 2-t & \text{for } 1 \leq t \leq 2 \\ 0 & \text{for } t < 0 \text{ and } t > 2 \end{cases}$$

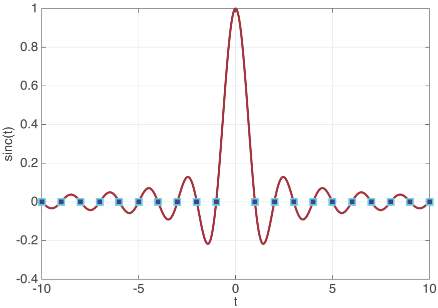
The triangular pulse function in terms of ramp functions

$$\Lambda(t) = r(t) - 2r(t-1) + r(t-2)$$



The sinc function

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{for } t \in \mathbb{R} \setminus \{0\} \\ 1 & \text{for } t = 0 \end{cases}$$



3 Outline

3 Even and odd signals

- A continuous-time signal $x(t)$ is called *even* if

$$x(-t) = x(t) \quad \text{for all } t \in \mathbb{R}$$

- A continuous-time signal $x(t)$ is called *odd* if

$$x(-t) = -x(t) \quad \text{for all } t \in \mathbb{R}$$

- A signal $y(t)$ defined on the entire t-axis can be written as a superposition of an even signal $y_\mathrm{e}(t)$ and an odd signal $y_\mathrm{o}(t)$:

$$y(t) = y_\mathrm{e}(t) + y_\mathrm{o}(t)$$

with

$$y_\mathrm{e}(t) = \frac{y(t) + y(-t)}{2} \quad \text{and} \quad y_\mathrm{o}(t) = \frac{y(t) - y(-t)}{2}$$

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4 The energy and action of a continuous-time signal

- The *energy* of a continuous-time signal $x(t)$ is defined as

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t$$

- A continuous-time signal $x(t)$ is called a *finite-energy signal* or *square integrable* if its energy is finite: $E_x < \infty$

- The integral

$$\int_{t=-\infty}^{\infty} |x(t)| \, \mathrm{d}t$$

  is sometimes called the *action* of the continuous-time signal $x(t)$

- A continuous-time signal $x(t)$ is called a *finite-action signal* or *absolutely integrable* if its action is finite

4 The power of a continuous-time signal

- The *power* of a continuous-time signal $x(t)$ is defined as

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{t=-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

- From this definition it immediately follows that

$P_x = 0$ for a finite-energy signal $x(t)$

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5 Periodic signals

- A continuous-time signal $x(t)$ is called *periodic* if there exists a $T > 0$ called a *period* of $x(t)$ such that

$$x(t+T) = x(t) \quad \text{for every } t \in \mathbb{R}$$

- A period of a periodic signal is not unique
- If T is a period, then $2T$, $3T$, ... are also periods of $x(t)$

- The smallest period T of $x(t)$ is called the *fundamental period* and is denoted as T_0
- The fundamental period T_0 is unique

5 Addition of two periodic signals

Let $x(t)$ and $y(t)$ be two periodic signals

- Signal $x(t)$ has a fundamental period T_0
- Signal $y(t)$ has a fundamental period T_1

Now consider the sum of these two periodic signals

$$z(t) = x(t) + y(t)$$

Under what conditions (if any)
is the signal $z(t)$ periodic?

Answer:

The signal $z(t)$ is periodic if there exists positive integers M and N such that

$$\frac{T_1}{T_0} = \frac{N}{M} = \text{a rational number}$$

Furthermore, if N and M have no common divisor other than one (N and M are relatively prime), then the fundamental period of $z(t)$ is $T_{z0} = NT_0 = MT_1$

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5 Addition of two periodic signals

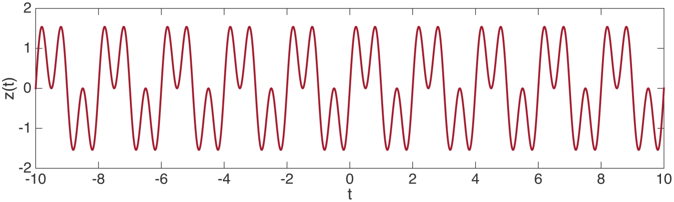
Example 1: Let $x(t) = \sin(\pi t)$ and $y(t) = \sin(3\pi t)$. In this case

$$T_0 = 2, \quad T_1 = 2/3, \quad \text{and} \quad T_1/T_0 = 1/3$$

Clearly, there exist integers M and N such that $T_1/T_0 = 1/3$

Take $M = 3$ and $N = 1$, for example, or $M = 6$ and $N = 2$

The integers M and N have no common divisor other than one for $M = 3$ and $N = 1$ and the period of $z(t) = x(t) + y(t)$ is
$T_{z0} = 1 \cdot T_0 = 3 \cdot T_1 = 2$



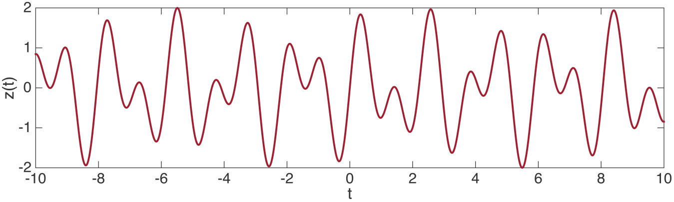
5 Addition of two periodic signals

Example 2: Let $x(t) = \sin(\sqrt{3}\pi t)$ and $y(t) = \sin(\pi t)$. In this case

$$T_0 = 2/\sqrt{3}, \quad T_1 = 2, \quad \text{and} \quad T_1/T_0 = 1/\sqrt{3}$$

In this case, no integers M and N can be found such that $1/\sqrt{3} = M/N$

The signal $z(t) = x(t) + y(t)$ is not periodic



5 Energy and power of periodic signals

Recall that the energy of a signal $x(t)$ is defined as

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \lim_{T \to \infty} \int_{t=-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

and its power as

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{t=-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

Now let $x(t)$ denote a continuous-time periodic signal with fundamental period T_0

5 Energy and power of periodic signals

We introduce the integral

$$E_x^{(N)} = \int_{t=t_0-NT_0}^{t_0+NT_0} |x(t)|^2 \, \mathrm{d}t,$$

where t_0 is an arbitrary fixed time instant and N a positive integer

Observe that

$$E_x = \lim_{N\to\infty} E_x^{(N)} \quad \text{and} \quad P_x = \lim_{N\to\infty} \frac{1}{2NT_0} E_x^{(N)}$$

Using the periodicity of $x(t)$ we find that

$$E_x^{(N)} = 2N \int_{t=t_0}^{t_0+T_0} |x(t)|^2 \, \mathrm{d}t$$

5 Energy and power of periodic signals

We observe that $E_x^{(N)}$ grows linearly in N as N increases

Consequently, a periodic signal is an infinite energy signal

The power of a periodic signal exists and is given by

$$P_x = \frac{1}{T_0} \int_{t=t_0}^{t_0+T_0} |x(t)|^2 \, \mathrm{d}t$$

6 Outline

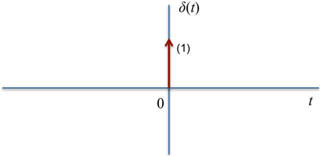
6 The Dirac delta function

The Dirac delta function (also known as the *impulse function*) is actually not a function in the standard sense.

It is a so-called *distribution*. It generalizes the idea of a regular function.

The delta function vanishes everywhere except at a single time instant. Let this time instant be $t = t_0$.

The delta function that acts at $t = t_0$ is denoted by $\delta(t - t_0)$ and is represented graphically by an arrow as illustrated below for a delta function that acts at $t = t_0 = 0$.



6 The Dirac delta function

Now let $\varphi(t)$ denote a regular function/signal defined on the entire real time axis and let φ be continuous at $t = t_0$. The action of the delta function is described by the integral

$$\int_{t=-\infty}^{\infty} \varphi(t)\delta(t - t_0)\,\mathrm{d}t = \varphi(t_0)$$

With the help of the delta function, we select or sample the signal value $\varphi(t_0)$ from $\varphi(t)$. Only the delta function has this property. The above formula can be considered as the definition of the delta function. It is sometimes called the *selective property*, the *sampling property*, or the *sifting property* (Dutch: zeefeigenschap) of the delta function.

Another interpretation: the area of the function $\varphi(t)\delta(t - t_0)$ is equal to $\varphi(t_0)$.

We use the selection/sifting property to derive some additional properties of the delta function.

Integration properties

First, take $\varphi(t) = 1$ for all $t \in \mathbb{R}$ in the sifting formula. We get

$$\int_{t=-\infty}^{\infty} \delta(t - t_0)\, \mathrm{d}t = 1.$$

The delta function is a signal of unit area.

Second, let $a < b$ and take

$$\varphi(t) = \begin{cases} 1 & \text{if } t \in (a, b) \\ 0 & \text{if } t < a \text{ or } t > b \end{cases}$$

This gives

$$\int_{t=a}^{b} \delta(t - t_0)\,\mathrm{d}t = \begin{cases} 1 & \text{if } t_0 \in (a, b) \\ 0 & \text{if } t_0 < a \text{ or } t_0 > b \end{cases}$$

In other words, if t_0 belongs to the integration interval, the integral evaluates to one. If t_0 does not belong to the integration interval, the integral evaluates to zero.

Scaling property The sampling property can also be used to show that

$$\delta(at) = \frac{1}{|a|}\delta(t), \quad a \in \mathbb{R} \setminus \{0\}.$$

Exercise: Verify the above scaling formula.

Special case: for $a = -1$ we get

$$\delta(-t) = \delta(t).$$

The Dirac delta function is even.

Derivative of the Heaviside unit step function

Let us start again with the sifting property of the delta function:

$$\int_{t=-\infty}^{\infty} \varphi(t)\delta(t-t_0)\,\mathrm{d}t = \varphi(t_0)$$

For later convenience, we rewrite this expression in a different form. Specifically, we use τ as an integration variable and use t instead of t_0 in the sifting formula. We obtain

$$\int_{\tau=-\infty}^{\infty} \varphi(\tau)\delta(\tau-t)\,\mathrm{d}\tau = \varphi(t) \qquad (*)$$

Since the delta function is even, this can also be written as

$$\int_{\tau=-\infty}^{\infty} \varphi(\tau)\delta(t-\tau)\,\mathrm{d}\tau = \varphi(t)$$

Now let $u(t)$ denote the Heaviside unit step function and consider the integral

$$\int_{\tau=-\infty}^{\infty} \varphi(\tau)u(t-\tau)\,\mathrm{d}\tau = \int_{\tau=-\infty}^{t} \varphi(\tau)\,\mathrm{d}\tau$$

Take the derivative with respect to t to obtain

$$\int_{\tau=-\infty}^{\infty} \varphi(\tau)\frac{\mathrm{d}u(t-\tau)}{\mathrm{d}t}\,\mathrm{d}\tau = \varphi(t)$$

This is the rewritten sifting property of the delta function! Since the delta function is the only function having this property, we conclude that

$$\delta(t-\tau) = \frac{\mathrm{d}}{\mathrm{d}t}u(t-\tau)$$

The Dirac delta function is equal to the derivative of the Heaviside unit step function

The derivative of the delta function

Consider the sifting property of equation (∗) again:
See Slide 31

$$\int_{\tau=-\infty}^{\infty} \varphi(\tau)\delta(\tau - t)\,\mathrm{d}\tau = \varphi(t) \qquad (*)$$

Let φ be continuously differentiable at $t = t_0$

Differentiate the above sifting property with respect to t to obtain

$$\int_{\tau=-\infty}^{\infty} \varphi(\tau)\delta'(\tau - t)\,\cdot -1 \cdot\,\mathrm{d}\tau = \varphi'(t)$$

or

$$\int_{t=-\infty}^{\infty} \varphi(t)\delta'(t - t_0)\,\mathrm{d}t = -\varphi'(t_0)$$

This is the sifting property of the derivative of the delta function

Approximating the delta function

Consider the Gaussian function

$$f_\epsilon(t) = \frac{1}{\sqrt{\pi\epsilon}} e^{-t^2/\epsilon} \qquad \text{with } \epsilon > 0$$

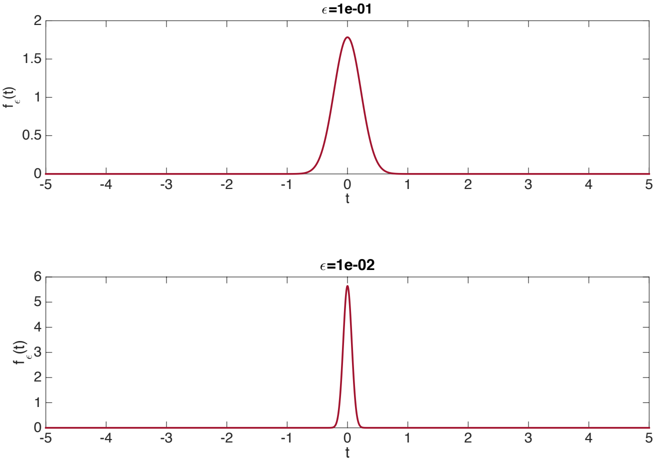
Note that this function is even and is normalized in the sense that

$$\int_{t=-\infty}^{\infty} f_\epsilon(t)\,\mathrm{d}t = 1 \qquad \text{for any } \epsilon > 0$$

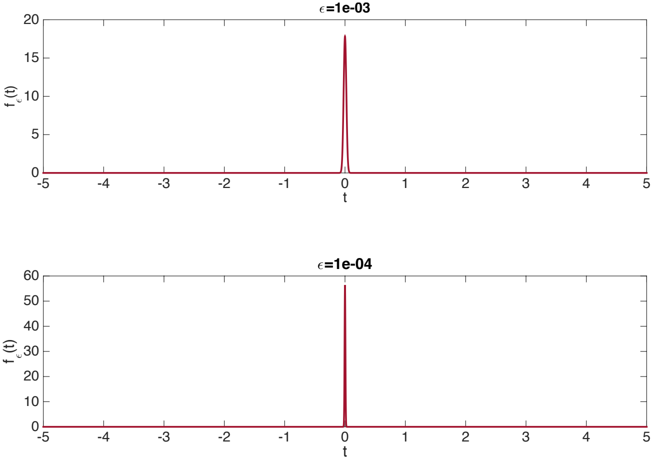
With $\varphi(t)$ a function that is continuous at the origin, consider the integral

$$\int_{t=-\infty}^{\infty} \varphi(t) f_\epsilon(t)\,\mathrm{d}t \quad \text{and let } \epsilon \downarrow 0$$

6 The Dirac delta function



6 The Dirac delta function



For "very small" values of ϵ, we have

$$\int_{t=-\infty}^{\infty} \varphi(t) f_\epsilon(t)\,\mathrm{d}t \approx \varphi(0) \int_{t=-\infty}^{\infty} f_\epsilon(t)\,\mathrm{d}t = \varphi(0)$$

which is approximately the sifting property at $t = 0$

We write

$$\delta(t) = \lim_{\epsilon \downarrow 0} f_\varepsilon(t) = \lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-t^2/\epsilon}$$

Equivalent distributions

Let $f(t)$ be a regular function continuous at $t = t_0$. We claim that

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$$

Sifting property for delta distribution on the left-hand side

$$\int_{t=-\infty}^{\infty} \varphi(t)[f(t)\delta(t - t_0)]\,\mathrm{d}t = \int_{t=-\infty}^{\infty} \varphi(t)f(t)\delta(t - t_0)\,\mathrm{d}t = \varphi(t_0)f(t_0)$$

Sifting property for delta distribution on right-hand side

$$\int_{t=-\infty}^{\infty} \varphi(t)[f(t_0)\delta(t - t_0)]\,\mathrm{d}t = f(t_0)\int_{t=-\infty}^{\infty} \varphi(t)\delta(t - t_0)\,\mathrm{d}t = \varphi(t_0)f(t_0)$$

Equivalent distributions

Let $f(t)$ be a regular function continuously differentiable at $t = t_0$. We claim that

$$f(t)\delta'(t-t_0) = -f'(t_0)\delta(t-t_0) + f(t_0)\delta'(t-t_0)$$

Sifting property for derivative of delta distribution on the left-hand side

$$\begin{aligned}\int_{t=-\infty}^{\infty} \varphi(t)[f(t)\delta'(t-t_0)]\,\mathrm{d}t &= \int_{t=-\infty}^{\infty} \varphi(t)f(t)\delta'(t-t_0)\,\mathrm{d}t \\ &= -(\varphi f)'\Big|_{t=t_0} \\ &= -\varphi'(t_0)f(t_0) - \varphi(t_0)f'(t_0)\end{aligned}$$

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Equivalent distributions

Sifting properties for delta and derivative of delta distribution on right-hand side

$$
\begin{aligned}
&\int_{t=-\infty}^{\infty} \varphi(t)[-f'(t_0)\delta(t-t_0) + f(t_0)\delta'(t-t_0)]\,\mathrm{d}t \\
&\quad = -f'(t_0)\int_{t=-\infty}^{\infty} \varphi(t)\delta(t-t_0)\,\mathrm{d}t + f(t_0)\int_{t=-\infty}^{\infty} \varphi(t)\delta'(t-t_0)\,\mathrm{d}t \\
&\qquad = -f'(t_0)\varphi(t_0) - f(t_0)\varphi'(t_0)
\end{aligned}
$$

Summary:

- Sifting property:

$$\int_{t=-\infty}^{\infty} f(t)\delta(t-t_0)\,\mathrm{d}t = f(t_0) \qquad f \text{ continuous at } t = t_0$$

- Scaling property:

$$\delta(at) = \frac{1}{|a|}\delta(t) \qquad a \in \mathbb{R} \setminus \{0\}$$

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Summary:

- Derivative of unit step:

$$\delta(t) = \frac{\mathrm{d}u}{\mathrm{d}t}$$

- Sifting property of the derivative of the delta function

$$\int_{t=-\infty}^{\infty} f(t)\delta'(t-t_0)\,\mathrm{d}t = -f'(t_0) \quad \text{with } f \text{ cont. diff. at } t = t_0$$

Summary:

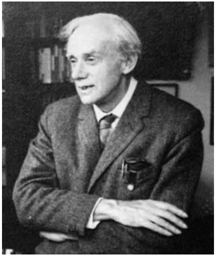
- Multiplication property Dirac delta function:
  f continuous at $t = t_0$

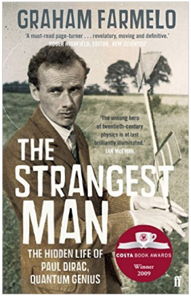
$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

- Multiplication property derivative Dirac delta function:
  f continuously differentiable at $t = t_0$

$$f(t)\delta'(t-t_0) = -f'(t_0)\delta(t-t_0) + f(t_0)\delta'(t-t_0)$$

6 Paul Dirac

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THE HIDDEN LIFE OF PAUL DIRAC, QUANTUM GENIUS
COSTA BOOK AWARDS
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6 Exercises

Exercise 1. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{sign}(t) = 2\delta(t)$$

Exercise 2. Show that

$$\delta(at+b) = \frac{1}{|a|}\delta(t+b/a) \qquad a \neq 0$$

Exercise 3. Determine

$$\int_{\tau=-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau$$

Exercise 4. Determine

$$\int_{\tau=-\infty}^{t} \delta(\tau) f(\tau)\,\mathrm{d}\tau$$

Exercise 5. Compute

$$\int_{t=-\infty}^{\infty} \delta(t) f(t-t_0)\,\mathrm{d}t$$

6 Exercises

Exercise 6. Compute

$$\int_{t=-\infty}^{\infty} \delta(t) t \, \mathrm{d}t$$

Exercise 7. Sketch the signal

$$f(t) = \sin(\pi t) u(t)$$

and determine $f'(t)$.

Exercise 8. Sketch the signal

$$g(t) = \cos(\pi t) u(t)$$

and determine $g'(t)$.

Exercise 9. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} |t| = \operatorname{sign}(t)$$

Exercise 10. Explain why

$$\int_{t=-\infty}^{\infty} \delta'(t) \, \mathrm{d}t = 0$$

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