EE2S1 Signals and Systems

(3rd ed) Ch. 10.5.1, 10.5.2, 10.5.5
(4th ed) Ch. 8.5.1, 8.5.2; 8.5.5
} **Inverse z-transform**

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Recall this example [Lecture 14]

Compute the z-transform of the two-sided signal:

$$x[n] = \left(\frac{1}{2}\right)^{|n|}$$

- For the causal part ($n \geq 0$) we find:

$$x_c[n] = \left(\frac{1}{2}\right)^{n} u[n] \;\Leftrightarrow\; X_c(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}},$$

ROC: $|z| > \frac{1}{2}$

- For the anti-causal part ($n \leq 0$):

$$x_{ac}[n] = \left(\frac{1}{2}\right)^{-n} u[-n] \;\Leftrightarrow\; X_{ac}(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{n} = \frac{1}{1 - \frac{1}{2}z},$$

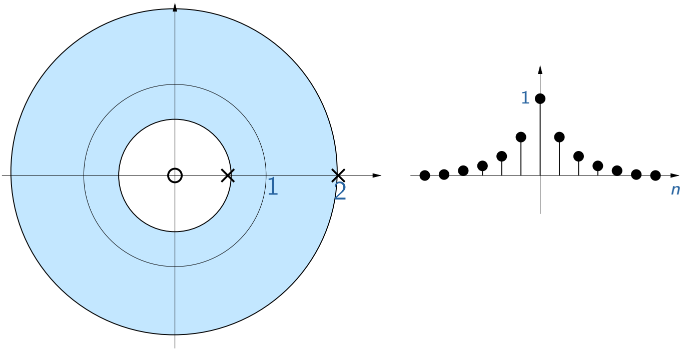
ROC: $|z| < 2$

Example (cont'd)

- For $x[n] = x_c[n] + x_{ac}[n] - 1$ we find

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{1}{1-\frac{1}{2}z} - 1 = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2} = \frac{-1\frac{1}{2}\,z}{(z-\frac{1}{2})(z-2)}$$

with ROC: $\frac{1}{2} < |z| < 2$.



Inverse z-transform

Given $X(z)$ and its ROC. The inverse z-transform is

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^n \frac{\mathrm{d}z}{z}$$

with the contour integral following a counterclockwise closed path in the ROC encircling the origin. This follows from applying the residue theorem to the Laurent series $X(z) = \sum_n x[n] z^{-n}$.

The integral is solved using the residue theorem.

The formula stems from *complex function theory*. It is almost never used except in theoretical derivations. We won't use it in this course.

Inverse z-transform

Given $X(z)$ for a causal signal (ROC: $|z| > R$), how can $x[n]$ be obtained?

- Use the contour integral/residue theorem. General technique but often rather complicated.
- Expansion into a power series of z^n (by long division)

$$\begin{aligned} X(z) &= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \\ \Leftrightarrow \quad x[n] &= x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \end{aligned}$$

  Useful if only a few terms $(x[0], x[1], \cdots)$ are needed.
- Partial fraction expansion, then transforming each term separately (using a table).

Partial fraction expansion

$$X(z) = \frac{B(z)}{A(z)}$$

- Write as function of z^{-1}.
- Ensure that the degree of $B(z)$ is smaller than that of $A(z)$ ("proper rational function"). If necessary, start by splitting off a polynomial, (in z^{-1}), e.g.,

$$X(z) = b_0 + b_1 z^{-1} + \frac{B'(z)}{A(z)}$$

- Determine the poles (i.e. the zeros of $A(z)$). If none of the poles is repeated, then the partial fraction expansion has the form

$$X(z) = b_0 + b_1 z^{-1} + \sum \frac{A_k}{1 - \alpha_k z^{-1}}$$

$$x[n] = b_0 \delta[n] + b_1 \delta[n-1] + \sum A_k \alpha_k^n u[n]$$

Partial fraction expansion

- In the case of double poles, we use

$$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2} \quad \Leftrightarrow \quad x[n] = na^n u[n]$$

Example

$$X(z) = \frac{2 + z^{-2}}{1 + 2z^{-1} + z^{-2}}, \qquad \text{ROC: } |z| > 1$$

- (Write as function of z^{-1}, already the case here.)
- Make "proper"

$$X(z) = 1 + \frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}}$$

- The poles are $z = -1$ (twice)

$$X(z) = 1 + \frac{1 - 2z^{-1}}{(1 + z^{-1})^2} = 1 + \frac{A}{1 + z^{-1}} + \frac{Bz^{-1}}{(1 + z^{-1})^2}$$

with $A = 1$ and $B = -3$. Hence

$$x[n] = \delta[n] + (-1)^n u[n] + 3n(-1)^n u[n]$$

Partial fraction expansion

If $X(z)$ has as ROC the *inside* of a circle, $|z| < R$, then $x[n]$ is anti-causal.

- Write $X(z)$ as function of z.
- Make proper and form partial fraction decomposition as before. Use tables to find the inverse. Example:

$$X(z) = b_0 + b_1 z + \sum \frac{A_k}{1 - \alpha_k z}$$

$$x[n] = b_0 \delta[n] + b_1 \delta[n+1] + \sum A_k {\alpha_k^{-}}^{n} u[-n]$$

Partial fraction expansion

If $X(z)$ has as ROC a ring (donut-shape), then $x[n]$ has mixed causality.

- Determine the poles
- The poles inside the ring correspond to causal terms
  The poles outside the ring correspond to anti-causal terms

[To understand this, you need to understand how the forward z-transform works – see the initial example.]

Exercise (trial exam 2016)

Given

$$X(z) = \frac{1}{1 - 1\frac{1}{2}\,z^{-1} + \frac{1}{2}\,z^{-2}}\,, \quad z \in \mathrm{ROC},$$

determine $x[n]$ using the inverse z-transform if (i) ROC: $|z| > 1$, (ii) ROC: $|z| < \frac{1}{2}$, (iii) ROC: $\frac{1}{2} < |z| < 1$.

Exercise (trial exam 2016)

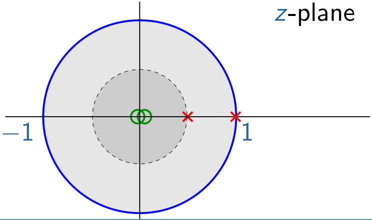
Given

$$X(z) = \frac{1}{1 - 1\frac{1}{2}\, z^{-1} + \frac{1}{2}\, z^{-2}}, \quad z \in \text{ROC},$$

determine $x[n]$ using the inverse z-transform if (i) ROC: $|z| > 1$, (ii) ROC: $|z| < \frac{1}{2}$, (iii) ROC: $\frac{1}{2} < |z| < 1$.

First write this in terms of z^{-1} (already done), make it 'proper' (already done), then split (partial fraction expansion).

$$X(z) = \frac{1}{1 - 1\frac{1}{2}\, z^{-1} + \frac{1}{2}\, z^{-2}} = \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2} z^{-1}}$$



Exercise (cont'd)

$i)$ The region of convergence runs until $z \to \infty$: causal response. Hence

$$x[n] = 2u[n] - \left(\frac{1}{2}\right)^n u[n]$$

$ii)$ The region of convergence includes $z = 0$: anti-causal reponse. Rewrite $X(z)$ as

$$X(z) = -\frac{2z}{1-z} + \frac{2z}{1-2z}\,.$$

The inverse z-transform of $\frac{1}{1-z}$ is $u[-n]$ and of $\frac{1}{1-2z}$ is $2^{-n}u[-n]$, while multiplication with z is equivalent to an 'advance', so that

$$x[n] = -2u[-n-1] + 2 \cdot 2^{-n-1}u[-n-1]$$

Exercise (cont'd)

iii) Rewrite $X(z)$ as

$$X(z) = -\frac{2z}{1-z} - \frac{1}{1-\frac{1}{2}z^{-1}}\,.$$

For this ROC, the first term results in an anti-causal response (pole at the outside of the ROC) , while the second term results in a causal response (pole at the inside of the ROC). Hence,

$$x[n] = -2u[-n-1] - \left(\frac{1}{2}\right)^n u[n]$$