ee2s11 Signals and Systems

Exercises analog/digital filter design

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1. Question

We use the bilinear transform to design a digital low-pass filter $H(z)$, specified as follows:

Pass-band: $0 \le |\omega| \le 0.3\pi$, maximal ripple 1 dB
Stop-band: $0.35\pi \le |\omega| \le \pi$, minimal damping 60 dB

a Translate these specifications to the analog frequency domain

b What is the required filter order if we use a Butterworth filter?

Answer

a The bilinear transform leidt tot $\Omega = \tan(\frac{\omega}{2})$. Here

$$\Omega_p = \tan(0.3\pi/2) = 0.5095, \qquad \Omega_s = \tan(0.35\pi/2) = 0.6128$$

b The expression for a Butterworth filter is

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$$

Hence

$$\text{Pass-band ripple:} \quad \frac{1}{1+\epsilon^2} = 10^{-1/10} \quad \Rightarrow \quad \epsilon = 0.5088$$

$$\text{Stop-band damping:} \quad \frac{1}{1+\epsilon^2(\Omega_s/\Omega_p)^{2N}} \le \delta_s^2 = 10^{-60/10} = 10^{-6}$$

Define $\delta := \sqrt{\frac{1}{\delta_s^2} - 1} = 999.9$ and simplify:

$$N \ge \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)} = 41.07$$

This gives $N = 42$.

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2. Question

Design a first-order digital low-pass filter $H(z)$ with the following specifications:

Pass-band frequency: $\omega_p = 0.3\pi$,
Damping outside the pass-band: at least 10 dB

Use the bilinear transform en base your design on an analog Butterworth filter.

a What is the pass-band frequency in the analog frequency domain?

b What is the generic expression for a first-order analog Butterworth filter?

c What is the analog filter $H_a(s)$ meeting the specifications?

d What is $H(z)$

e Demonstrate (verify) that the design meets the specifications.

Answer

a $\Omega_p = \tan(\omega_p/2) = 0.5095$. You could also use (like in Chaparro) $\Omega_p = \frac{2}{T}\tan(\omega_p/2)$, giving the same result.

b $|H(\omega)|^2 = \dfrac{1}{1+\epsilon^2(\Omega/\Omega_p)^2}$

Hence $H(s)H(-s) = \dfrac{1}{1-\epsilon^2(s/\Omega_p)^2}$ and $H(s) = \dfrac{1}{1+\epsilon s/\Omega_p}$

c First determine ϵ from the damping requirement at $\omega_p = 0.3\pi$. From $|H(\omega = 0.3\pi)|^2 = 10^{-10/10} = 0.1$ (damping 10 dB) we obtain

$$\frac{1}{1+\epsilon^2} = 0.1 \qquad \Rightarrow \qquad \epsilon = 3$$

Hence $H(s) = \dfrac{1}{1+\epsilon/\Omega_p \cdot s} = \dfrac{1}{1+5.88\cdot s}$

d Use the bilinear transform:

$$s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$$

This results in

$$H(z) = \frac{1}{1+5.88\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{6.88}\cdot\frac{1+z^{-1}}{1-0.7096\cdot z^{-1}}$$

e Verify:

$$|H(\omega=0)| = |H(z^{-1}=1)| = 1$$
$$|H(\omega=\pi)| = |H(z^{-1}=-1)| = 0$$
$$|H(\omega=0.3\pi)|^2 = |H(z^{-1}=0.59-j0.81)|^2 = \frac{1}{(6.88)^2}\frac{(1+0.59)^2+(0.81)^2}{(1-0.7096\cdot 0.59)^2+(0.7096\cdot 0.81)^2} = \cdots = 0.1$$

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3. Question

A generic second-order analog low-pass filter (Butterworth filter) is given by

$$H_a(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

The 3-dB cutoff frequency is $\Omega_c = 1$.

Design using the bilinear transform a digital *high-pass* filter $H(z)$ with cut-off frequency $\omega_c' = \frac{3}{4}\pi$:

a What is the cut-off frequency in the analog frequency domain?

b Which frequency transform do you apply?

c What is $H_a(s)$

d What is $H(z)$

e Verify that the design meets the specififations.

Answer

Cut-off frequency: $\Omega_c' = \tan(\omega_c'/2) = 2.4142$

Frequency transform:

$$s \rightarrow \frac{\Omega_c \Omega_c'}{s}$$

Here: $s \rightarrow \frac{2.4142}{s}$.

The resulting analog high-pass filter is

$$H_a(s) = \frac{1}{(\frac{2.4142}{s})^2 + \sqrt{2}\frac{2.4142}{s} + 1} = \frac{s^2}{s^2 + 3.4142s + 5.8284}$$

The corresponding digital transfer function follows from the bilinear transform:

$$s \rightarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

This results in

$$H(z) = \frac{(\frac{1-z^{-1}}{1+z^{-1}})^2}{(\frac{1-z^{-1}}{1+z^{-1}})^2 + 3.4142(\frac{1-z^{-1}}{1+z^{-1}}) + 5.8284} = \cdots = \frac{z^2 - 2z + 1}{10.2436z^2 + 9.6568z + 3.4142}$$

Verify:

$$\begin{aligned} H(\omega = 0) &= H(z = 1) = 0 \\ H(\omega = \pi) &= H(z = -1) = 1 \\ H(\omega = \omega_c) &= H(z = e^{j\omega_c}) = 0.0133 + j0.7068 \quad \Rightarrow \quad |H(\omega_c)| = 0.7071 = \tfrac{1}{\sqrt{2}} \end{aligned}$$

Indeed, a high-ass filter with 3 dB damping at ω_c.

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4. Question

Design a 2nd order high-pass Chebychev filter with a stop-band frequency of 1 rad/s, a pass-band frequency of 2 rad/s and a maximal damping in the pass-band of 1 dB.

a Which frequency transform do you use?

b What is the frequency response (squared-amplitude) of the corresponding low-pass filter with a pass-band frequency of 1 rad/s.

c What is the stop-band frequency and the maximal damping in the stop-band of the low-pass filter?

d Now apply the frequency transform to convert the low-pass filter into the requested high-pass filter, en give the frequency response (squared-amplitude) van the resulting high-pass filter.

e Make a drawing of this frequency response (squqred-amplitude) en indicate the stop-band and pass-band frequencies, and the corresponding damping.

Answer

a We use the low-pass to high-pass transform:

$$\Omega = \frac{2}{\Omega'}$$

Hence, we need to design a low-pass filter with $\Omega_p = 1$ and then apply the transform.

b The frequency response for a 2nd order Chebychev with $\Omega_p = 1$ is given by:

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2(2\Omega^2 - 1)^2}.$$

We determine ϵ based on the maximal damping in the pass-band of 1 dB:

$$\begin{aligned} \alpha_p &= 10\log(1+\epsilon^2) = 1 \text{ dB} \\ \Rightarrow \quad \epsilon &= \sqrt{10^{\alpha_p/10} - 1} = 0.5 \end{aligned}$$

Hence

$$|H(j\Omega)|^2 = \frac{1}{1 + 0.25(2\Omega^2 - 1)^2}.$$

c The stop-band frequency is:

$$\Omega_s' = 1 \text{ rad/s (for high-pass)} \Rightarrow \Omega_s = \frac{2}{\Omega_s'} = 2 \text{ rad/s (for low-pass)}$$

The minimal damping in the stop-band is

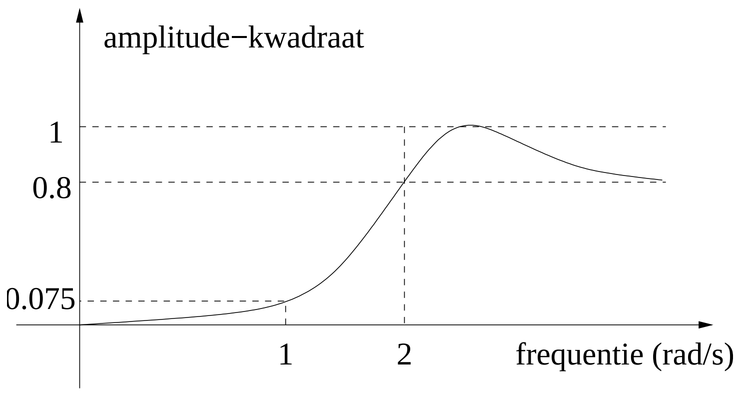
$$\alpha_s = 10\log(1 + \epsilon^2(2\Omega_s^2 - 1)^2) \approx 11\text{dB}$$

The maximal damping in the stop-band is ∞.

d The frequency response of the high-pass filter is then given by

$$|H(j\Omega')|^2 = \frac{1}{1 + 0.25(2(2/\Omega')^2 - 1)^2}.$$

e See figure below:



5. Question

Design an analog 2nd order high-pass Butterworth filter with stop-band frequency 40 Hz, pass-band frequency 60 Hz, and maximal damping in the pass-band 3 dB.

a Which frequency transform do you use?

b Give an expression for the frequency response (squared-amplitude) of the corresponding low-pass filter with a pass-band frequency of 1 rad/s.

c What is the stop-band frequency and the minimal damping in the stop-band of this low-pass filter?

d Now use the frequency transform to convert the low-pass filter into the requested high-pass filter, and give the expression for the frequency response (squared-amplitude) of the final high-pass filter.

e Make a drawing of this frequency response (squared-amplitude); indicate the stop-band and pass-band frequencies, and the corresponding damping.

Answer

a $\Omega = \dfrac{60 \cdot 2\pi}{\Omega'} = \dfrac{377}{\Omega'}$.

This is a mapping of $\Omega' = 60 \cdot 2\pi$ rad to $\Omega = 1$ rad, and of $\Omega' = 40 \cdot 2\pi$ rad to $\Omega = 3/2$ rad. We have to design a low-pass filter with a pass-band frequency of $\Omega_p = 1$ rad and a stop-band frequency of $\Omega_s = 3/2$ rad.

b Second order Butterworth is given by

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 \Omega^4}$$

For $\Omega = 1$ the response should be -3 dB (factor 1/2 in power):

$$|H(1)|^2 = \frac{1}{1+\epsilon^2} = \frac{1}{2} \quad \Rightarrow \quad \epsilon = 1$$

Hence

$$|H(\Omega)|^2 = \frac{1}{1+\Omega^4}$$

c The stop-band frequency is $\Omega_s = 3/2$. The corresponding damping is

$$|H(\Omega_s)|^2 = \frac{1}{1+\Omega_s^4} = 0.165 = -7.8 \text{ dB}$$

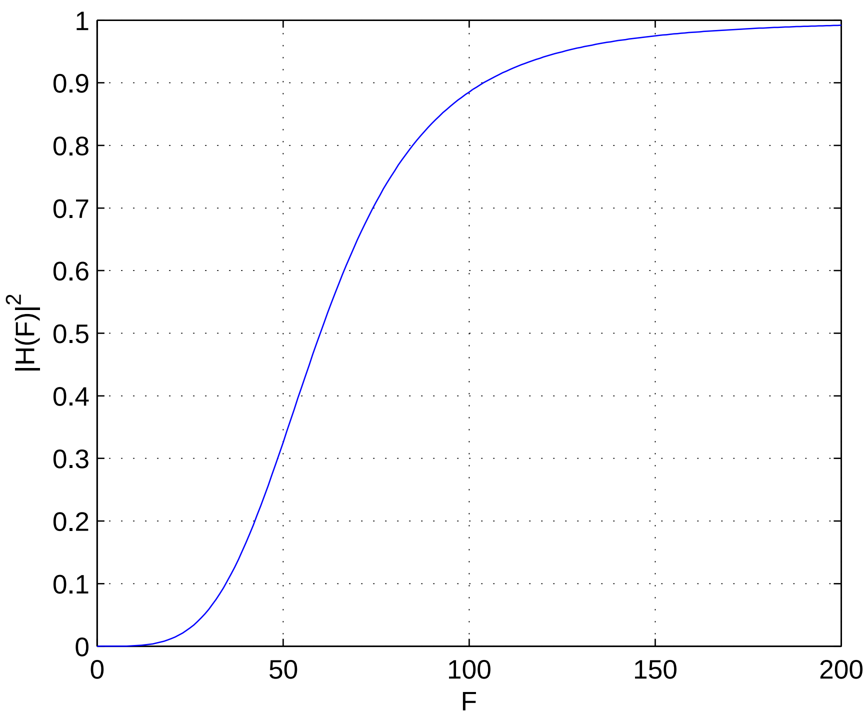
d

$$|H(\Omega)|^2 = \frac{1}{1 + (60 \cdot 2\pi/\Omega)^4}$$

or, with F in Hz:

$$|H(F)|^2 = \frac{1}{1+(60/F)^4} = \frac{F^4}{F^4 + 60^4}$$

e



Verify that the resulting filter satisfies the specs!

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6. Question

We design an analog 2nd order *high-pass* Chebychev filter with a stop-band frequency of 3 rad/s, a pass-band frequency of 6 rad/s, and a maximal damping in the pass-band of 3 dB.

a First, we design a low-pass filter and then apply a frequency transform. Which frequency transform will you use? What are the resulting specifications for the low-pass filter?

b Give an explicit expression for the second-order Chebychev polynomial $T_2(\Omega)$. Make a drawing of this function.

c Give the expression for the frequency response (squared-amplitude) of the corresponding 2nd order Chebychev low-pass filter with a pass-band frequency of 1 rad/s and a maximal damping in the pass-band of 3 dB. Determine all unknown parameters.

d What is the minimal damping in the stop-band of the resulting low-pass filter?

e Now apply the frequency transform to convert the low-pass filter into the requested high-pass filter, and give the expression for the frequency response (squared-amplitude) of the resulting high-pass filter.

f Make a drawing of this frequency response (squared-amplitude); also indicate the stop-band and pass-band frequencies, and the corresponding dampings.

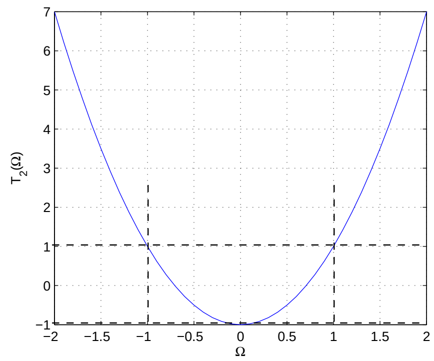
Answer

a $\Omega \rightarrow \frac{6}{\Omega}$.

The pass-band frequency is $\Omega_p = 1$ rad/s, with a damping of 3 dB. The stop-band frequency is $\Omega_s = \frac{6}{3} = 2.0$ rad/s.

Other transforms are also possible, but this version is convenient because the pass-band frequency is now equal to 1.

b $T_2(\Omega) = 2\Omega^2 - 1$.



c

$$|G(\Omega)|^2 = \frac{1}{1+\epsilon^2[T_2(\Omega)]^2} = \frac{1}{1+\epsilon^2(2\Omega^2-1)^2}$$

For $\Omega = 1$, we must have a damping of 3 dB, hence $|G(1)|^2 = \frac{1}{2}$.

$$|G(1)|^2 = \frac{1}{1+\epsilon^2} = \frac{1}{2} \qquad \Rightarrow \quad \epsilon = 1$$

$$|G(\Omega)|^2 = \frac{1}{1+(2\Omega^2-1)^2} = \frac{1}{4\Omega^2 - 4\Omega^2 + 2}$$

d

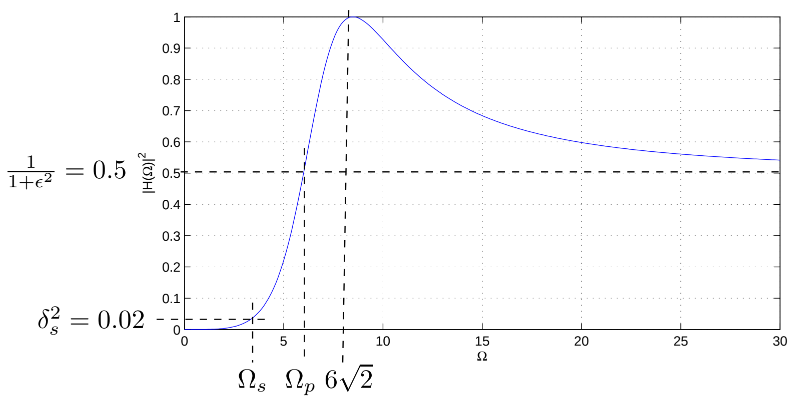
$$|G(\Omega_p)|^2 = \frac{1}{1+(2(2.0)^2-1)^2} = \frac{1}{50}$$

This corresponds to -17 dB.

e

$$\begin{aligned} |H(\Omega)|^2 &= \tfrac{1}{1+(2\left(\frac{6}{\Omega}\right)^2-1)^2} \\ &= \tfrac{1}{1+(\frac{72}{\Omega^2}-1)^2} \\ &= \tfrac{\Omega^4}{2\Omega^4 - 2\cdot 72\Omega^2 + (72)^2} \end{aligned}$$

f



The peak corresponds to the location where $T_2(6/\Omega) = 0$, i.e., for $\Omega = 6\sqrt{2} \approx 8.5$. The shape of $T_2(\Omega)$ in item b indicates that there is only a single peak. For $\Omega \to \infty$ is $T_2(6/\Omega) = T_2(0) = -1$, the transfer function is $H(\Omega) = 1/(1+\epsilon^2) = 1/2$, the same value as for Ω_p. (A higher-order filter would show a number of wiggles between 1/2 and 1.)

7. Question

We would like to design an analog low-pass filter with the following specifications:

– Pass-band: until $F_p = 40$ Hz; ripple in the pass-band : ≤ 1 dB
– Stop-band: from $F_s = 50$ Hz; stop-band damping: ≥ 30 dB.

In class, we studied the “Chebyshev I” function, specified by

$$|H_I(\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_c)}$$

where $T_N(x)$ is the Chebyshev polynomial of order N. This results in a filter with an equiripple in the pass-band. Alternatively, we here consider the “Chebyshev II” function, obtained in two steps:

First we apply a low-pass to high-pass transform, $\Omega \to \Omega_c^2/\Omega$:

$$|H_C(\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega_c/\Omega)}$$

Next, we take the complement of this function,

$$|H_{II}(\Omega)|^2 = 1 - |H_C(\Omega)|^2 = \frac{\epsilon^2 T_N^2(\Omega_c/\Omega)}{1 + \epsilon^2 T_N^2(\Omega_c/\Omega)}$$

a Make neat drawings of $|H_I(\Omega)|^2$ and $|H_C(\Omega)|^2$, and of $|H_{II}(\Omega)|^2$. Indicate Ω_c, and the locations of the pass-band frequency, stop-band frequency, maximal pass-band ripple δ_p, maximal stop-band ripple δ_s. Clearly show the ripples.

b What is $|H_I(\Omega_c)|^2$? How do you choose Ω_c?

c Compute step-by-step the required filter order N for $H_I(\Omega)$ which will meet the given specifications.

(Remark: $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$.)

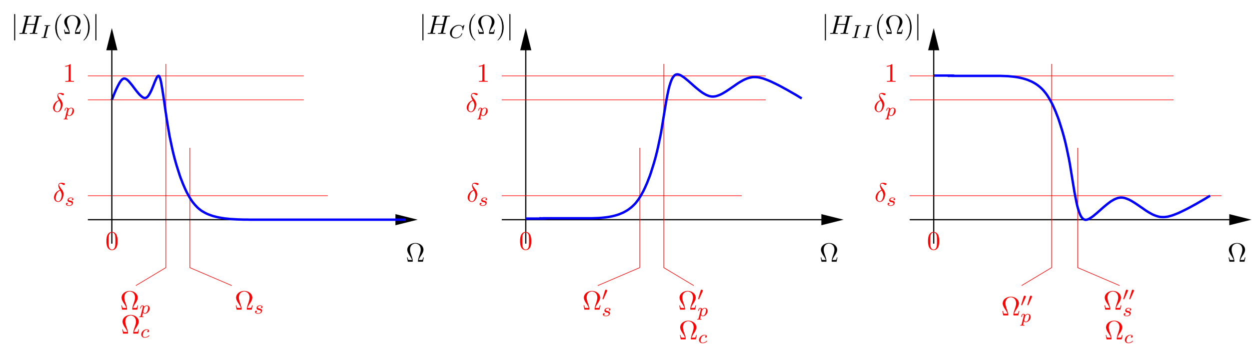
d What is $|H_{II}(\Omega_c)|^2$? How do you choose Ω_c?

e Now compute the required filter order N for $H_{II}(\Omega)$.

f Are there advantages in using $H_{II}(\Omega)$ instead of $H_I(\Omega)$?

Answer

a The transform maps low frequencies to high, and high frequencies to low. The frequency-axis is mirrored (around Ω_c).



b Substitute (using $T_N^2(1) = 1$):

$$|H_I(\Omega_c)|^2 = \frac{1}{1+\epsilon^2}$$

Choose $\Omega_c = \Omega_p = 2\pi \cdot 40$.

(Note: Ω_c is not a cut-off frequency.)

c The pass-band criterion results in:

$$|H_I(\Omega_p)|^2 = \frac{1}{1+\epsilon^2} = \delta_p^2 = 10^{-1/10}$$

$$\epsilon = (\delta_p^{-2} - 1)^{1/2} = (10^{1/10} - 1)^{1/2} = 0.5088$$

The stop-band criterion results in:

$$|H_I(\Omega_s)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega_s/\Omega_c)} = \delta_s^2 = 10^{-30/10}$$

$$T_N(\Omega_s/\Omega_p) = \frac{(\delta_s^{-2} - 1)^{1/2}}{\epsilon} = \frac{(10^{30/10} - 1)^{1/2}}{0.5088} = 62.1206$$

Use the equation $T_N(x) = \cosh(N \cosh^{-1}(x))$ (valid for $|x| > 1$):

$$\cosh(N \cosh^{-1}(\Omega_s/\Omega_p)) = 62.1206 \quad \Rightarrow \quad N = \frac{\cosh^{-1}(62.1206)}{\cosh^{-1}(50/40)} = 6.96$$

Round N upwards, resulting in $N = 7$.

d

$$|H_{II}(\Omega_c)|^2 = \frac{\epsilon^2}{1+\epsilon^2}$$

Choose $\Omega_c = \Omega_s = 2\pi \cdot 50$.

This choice is because for $\Omega > \Omega_s$, the equation has $T_N(x)$ with $x = \Omega_s/\Omega < 1$, i.e., $|T_N(\Omega_s/\Omega)| < 1$: at this point the ripples will start.

e Stop-band criterion results in:

$$|H_I(\Omega_s)|^2 = \frac{\epsilon^2}{1+\epsilon^2} = \delta_s^2 = 10^{-30/10}$$

$$\epsilon^2 = (1+\epsilon^2)\delta_s^2$$

$$\epsilon^2 = \frac{\delta_s^2}{1-\delta_s^2}$$

$$\epsilon = (\frac{\delta_s^2}{1-\delta_s^2})^{1/2} = (\frac{10^{-30/10}}{1-10^{-30/10}})^{1/2} = 0.0316$$

The pass-band criterion results in:

$$|H_{II}(\Omega_p)|^2 = \frac{\epsilon^2 T_N^2(\Omega_s/\Omega_p)}{1+\epsilon^2 T_N^2(\Omega_s/\Omega_p)} = \delta_p^2 = 10^{-1/10}$$

$$T_N(\Omega_s/\Omega_p) = \frac{1}{\epsilon}\left(\frac{\delta_p^2}{1-\delta_p^2}\right)^{1/2} = \frac{1}{0.0316}\left(\frac{10^{-1/10}}{1-10^{-1/10}}\right)^{1/2} = 62.19$$

$$\cosh(N \cosh^{-1}(\Omega_s/\Omega_p)) = 62.19 \quad \Rightarrow \quad N = \frac{\cosh^{-1}(62.19)}{\cosh^{-1}(50/40)} = 6.96$$

We take $N = 7$.

f The filter order turns out to be the same. An advantage is that the pass-band is flat, so that the output signal is less distorted.

8. Question

We would like to design a digital *high-pass* filter with the following specifications:

– Pass-band: starting at 3.0 kHz; ripple in the pass-band : ≤ 1 dB
– Stop-band: below 2.0 kHz; stop-band damping: ≥ 40 dB
– Sample rate: 12 kHz

The digital filter is designed by means of a bilinear transform applied to an analog transfer function.

a What are the pass-band and stop-band frequencies in the digital time-domain?

b What are the filter specifications in the analog time-domain?

c Which transform are you going to use to convert to a low-pass filter? What are the specs for this filter?

d Compute the required filter order N for a Butterworth filter.

e Suppose that the resulting Butterworth filter is of the form

$$H(s) = \frac{1}{D(s)}\,, \qquad D(s) = d_0 + d_1 s + \cdots + d_N s^N$$

with poles $s_1, \cdots, s_N$. How do you obtain the filter coefficienten for the digital *high-pass* filter?

Answer

a

$$\omega_p = \frac{3}{12} 2\pi = \frac{1}{2}\pi\,, \qquad \omega_s = \frac{2}{12} 2\pi = \frac{1}{3}\pi$$

b

$$\Omega_p = \tan(\frac{\omega_p}{2}) = 1\,, \qquad \Omega_s = \tan(\frac{\omega_p}{2}) = .5774$$

c

$$s \longrightarrow \frac{1}{s}\,, \qquad \Omega \longrightarrow \frac{1}{\Omega}$$

(Other options exist: more generally $s \rightarrow \frac{\Omega_0}{s}$, but this should be taken into account later in step e.)

Specs for the low-pass filter:

- $\Omega'_p = 1$, ripple in the pass-band smaller than 1 dB
- $\Omega'_s = 1/0.5774 = 1.7321$, damping in the stop-band larger than 40 dB

d For Butterworth:

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 (\Omega/\Omega'_p)^{2N}}$$

Pass-band:

$$\frac{1}{1+\epsilon^2} = \delta_p^2 = (10^{-1/20})^2 \quad \Rightarrow \quad \epsilon = 0.5087$$

Stop-band:

$$\frac{1}{1+\epsilon^2(\Omega_s'/\Omega_p')^{2N}} = \delta_s^2 = (10^{-40/20})^2 = 10^{-4} \quad \Rightarrow \quad (\Omega_s'/\Omega_p')^{2N} = \frac{10^4-1}{\epsilon^2}$$

Hence

$$N = \frac{\log(9999/(0.5087)^2)}{\log((1.7321)^2)} = 9.6135$$

Round upwards: $N = 10$.

e First back to a high-pass filter: $s \to \frac{1}{s}$, resulting in

$$H_{HP}(s) = \frac{1}{D(\frac{1}{s})} = \frac{s^N}{d_0 s^N + \cdots + d_N}$$

Then apply the bilinear transform $s = \frac{1-z^{-1}}{1+z^{-1}}$, resulting in

$$H_{HP}(z) = \frac{(1-z^{-1})^N}{d_0(1-z^{-1})^N + d_1(1-z^{-1})^{N-1}(1+z^{-1}) + \cdots + d_N(1+z^{-1})^N}$$

The coefficients are found by expanding the numerator and denominator as polynomials in z^{-1}.

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9. Question

We would like to design a digital low-pass filter with the following specifications:

- Ripple in the pass-band : ≤ 1 dB
- Pass-band: until 4 kHz
- Stop-band damping: ≥ 40 dB
- Stop-band: starting at 6 kHz
- Sample rate: 24 kHz

The digital filter is designed using a bilinear transform applied to an analog transfer function.

a What are the pass-band and stop-band frequencise in digital time-domain?

b What are the filter specifications in analog time-domain?

c Compute the required filter order for a Butterworth filter

d Compute the required filter order for a Chebyshev filter
(Remark: $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$.)

e Make a drawing of the transfer function of the resulting two digital filters after the bilinear transform. Also mark the filter specifications in the figure.

Answer

a

$$f_p = \frac{4}{24} = \frac{1}{6} \Rightarrow \omega_p = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$f_s = \frac{6}{24} = \frac{1}{4} \Rightarrow \omega_s = \frac{2\pi}{4} = \frac{\pi}{2}$$

b Apply the bilinear transform: $\omega = 2\arctan(\Omega)\,,\ \Omega = \tan(\frac{\omega}{2})$:

$$\Omega_p = \tan(\frac{\omega_p}{2}) = 0.5774$$

$$\Omega_s = \tan(\frac{\omega_s}{2}) = 1$$

For the ripples: $\delta_p = 10^{-1/20} = 0.8913$, $\delta_s = 10^{-40/20} = 0.01$.

c Use the derivation shown in the book or on the slides.

Butterworth: $|H(\omega)|^2 = \dfrac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$.

For the pass-band, we find $\dfrac{1}{1+\epsilon^2} = \delta_p^2 \quad \Rightarrow \quad \epsilon = \sqrt{\dfrac{1}{0.7943} - 1} = 0.5089$

For the stop-band, we have: $\delta_s = 10^{-40/20} = 0.01$.

For the filter order:

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\Omega_s/\Omega_p)^{2N}} = \delta_s^2 \quad \Rightarrow \quad (\frac{\Omega_s}{\Omega_p})^{2N} = \frac{\frac{1}{\delta_s^2} - 1}{\epsilon^2} =: \frac{\delta^2}{\epsilon^2} \quad \Rightarrow \quad N \geq \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)}$$

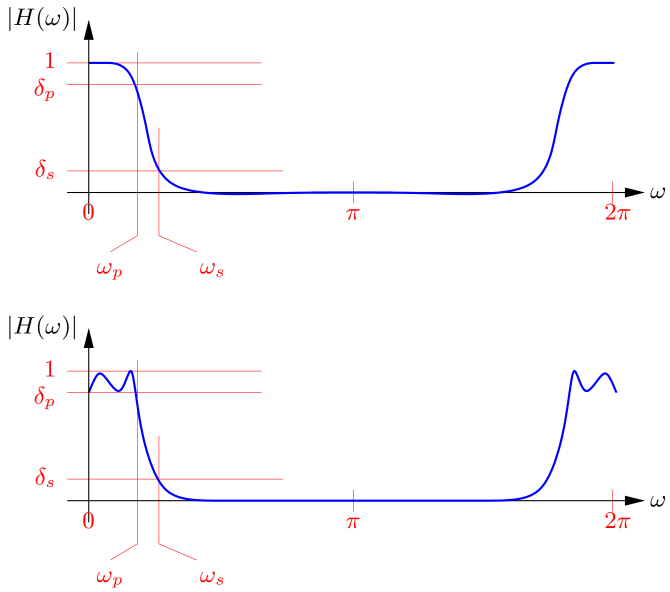
Substitution gives $\delta = 99.995$ and $N \geq 9.618$, i.e., the filter order is $N \geq 10$.

d Similar derivation for Chebyshev results in

$$N \geq \frac{\cosh^{-1}(\delta/\epsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = 5.212$$

Hence a 6th order filter

e



10. Question

We design a digital 2nd order band-stop filter (notch filter) with notch frequency $\omega_0 = \pi/2$, and 3 dB bandwidth of the notch $b = \pi/3$. I.e., if ω_1 and ω_2 are the -3 dB cut-off frequencies to the left and right of the notch frequency, then $b = \omega_2 - \omega_1$.

We use the bilinear transform. A template for an analog 2nd order notch filter is

$$H_a(s) = \frac{s^2 + \Omega_0^2}{s^2 + Bs + \Omega_0^2}$$

In here, Ω_0 is the notch frequency, and $B = \Omega_2 - \Omega_1$ is the 3 dB bandwidth, where Ω_1 and Ω_2 are the cut-off frequencies. We also have $\Omega_1\Omega_2 = \Omega_0^2$.

a What is $H_a(0)$, $H_a(j\Omega_0)$, $H_a(j \cdot \infty)$? Make a drawing of $|H_a(j\Omega)|$.

b How are ω and Ω related, according to the bilinear transform? What is the Ω_0 which corresponds to $\omega_0 = \pi/2$?

We can easily show that if $\omega_1 = \frac{\pi}{2} - \frac{b}{2}$ and $\omega_2 = \frac{\pi}{2} + \frac{b}{2}$, then after the bilinear transform we have $\Omega_1\Omega_2 = 1$.

c Based on this, select suitable values for ω_1 and ω_2, and determine the value of B.

Determine $H_a(s)$ which meets the analog specifications.

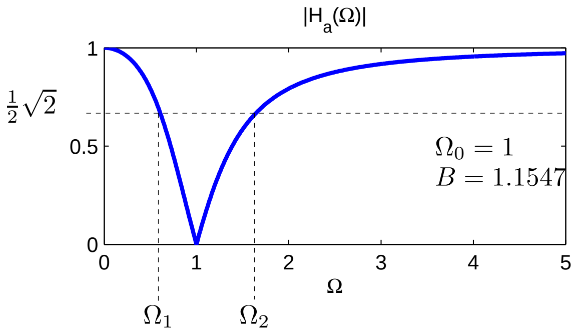
d Determine the digital filter $H(z)$ which meets the specifications.

e Compute the poles and zeros of $H(z)$ and show a pole/zero diagram.

Is this according to your expectations?

Answer

a $H_a(0) = 1$; $H_a(j\Omega_0) = 0$; $H_a(\infty) = 1$.



b $\Omega = \tan(\frac{\omega}{2})$. Hence $\Omega_0 = 1$.

c $\omega_1 = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$. $\omega_2 = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2}{3}\pi$.

$\Omega_1 = \frac{1}{3}\sqrt{3} = 0.5744$. $\Omega_2 = \sqrt{3} = 1.7321$.

$B = \Omega_2 - \Omega_1 = \frac{2}{3}\sqrt{3} = 1.1547$.

$$H_a(s) = \frac{s^2 + 1}{s^2 + (1.1547)s + 1}$$

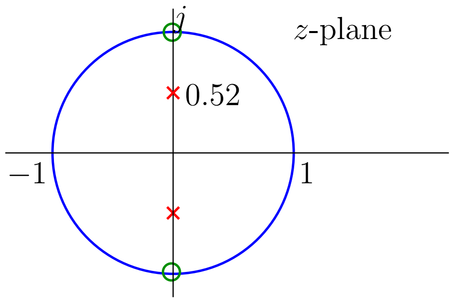
d Substitute $s = \frac{1-z^{-1}}{1+z^{-1}}$. This results in

$$H(z) = \frac{(1-z^{-1})^2 + (1+z^{-1})^2}{(1-z^{-1})^2 + \frac{B}{2}(1-z^{-1})(1+z^{-1}) + (1+z^{-1})^2} = \frac{1+z^{-2}}{(1+\frac{B}{2}) + (1-\frac{B}{2})z^{-2}}$$

Hence

$$H(z) = \frac{1+z^{-2}}{1.5744 + (0.4256)z^{-2}}$$

e Poles: $z = \pm j(0.5199)$; zeros $z = \pm j$.



We see zeros at the desired notch frequency ($\omega = \pm\pi/2$), and poles which, at some distance, will cancel these zeros. In this example, the poles are not very close to the zeros, hence the bandwidth of the notch is rather large (as is also evident from the specifications).