## 17. Exercises for part 2 (first part)

1. Sampling
2. $z$-transform, inverse $z$-transform
3. DTFT (missing...)

# Exercises

## Sampling

Given $x_a(t) = \dfrac{\sin(\Omega_0 t)}{t}$.

a) What is $X_a(\Omega)$? Make a drawing of $|X_a(\Omega)|$.

b) Is $x_a(t)$ band limited? (What is the maximal frequency in the signal?)

c) At which frequency should I at least sample to avoid aliasing (i.e., the Nyquist condition)?

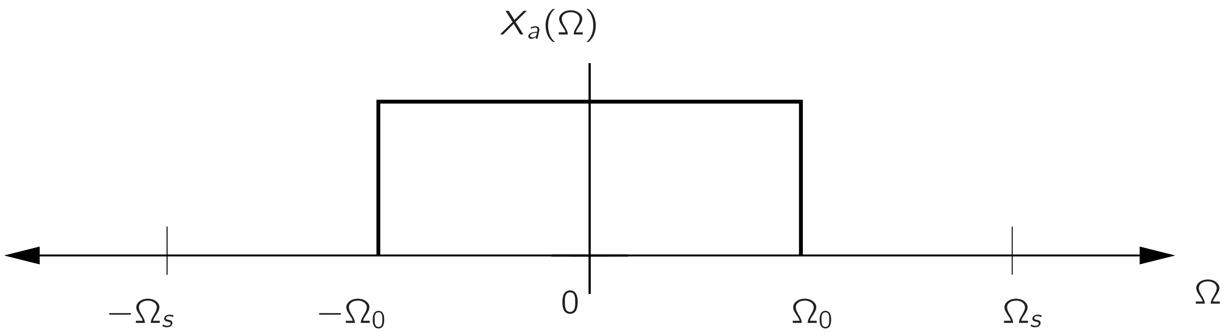
d) We sample at this frequency, resulting in $x[n]$. Determine the corresponding spectrum $X(\omega)$ and make a drawing.

# Exercises

## Answer

a)

$$X_a(\Omega) = \pi\left[u(\Omega + \Omega_0) - u(\Omega - \Omega_0)\right]$$



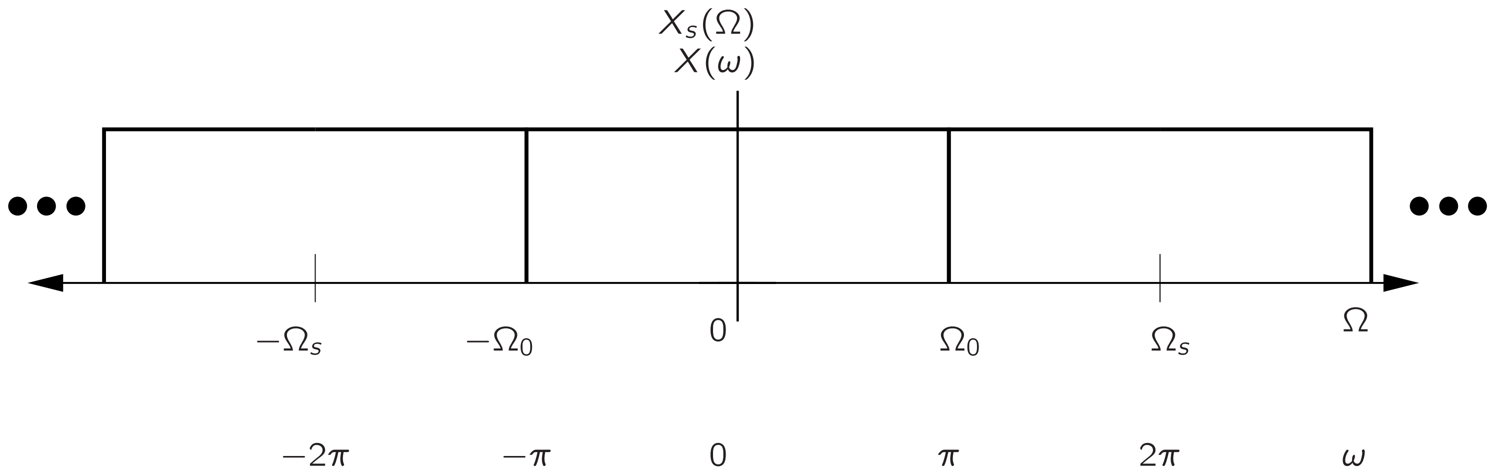

b) Band limited: the maximal frequency is $\Omega_0$.

c) Twice the highest frequency: $\Omega_s = 2\Omega_0$. The sample frequency is

$$F_s = \frac{\Omega_s}{2\pi} = \frac{\Omega_0}{\pi}$$

d) The spectrum of $x_s(t) = \sum_n x[n]\delta(t - nT_s)$ is $X_s(\Omega) = \frac{1}{T_s}\sum_k X_a(\Omega - k\Omega_s)$.

This corresponds to the spectrum $X(\omega)$ of $x[n]$, with $\omega = 2\pi\Omega/\Omega_s$, so that $\Omega_s \to 2\pi$.



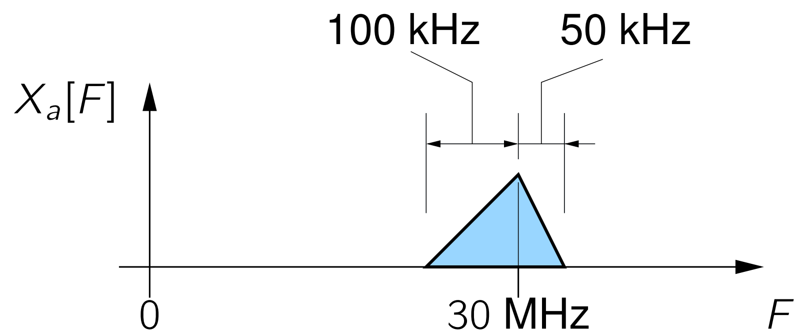

The spectrum is flat, this matches with $x[n] = A\delta[n]$, for a certain amplitude $A$. Check: $x[n] = x_a(nT_s)$ with $T_s = \frac{1}{F_s} = \frac{2\pi}{\Omega_s} = \frac{\pi}{\Omega_0}$, so that

$$x[n] = \frac{\sin(\Omega_0 \cdot n\frac{\pi}{\Omega_0})}{n\frac{\pi}{\Omega_0}} = \Omega_0\frac{\sin(n\pi)}{n\pi} = \Omega_0\delta[n]$$

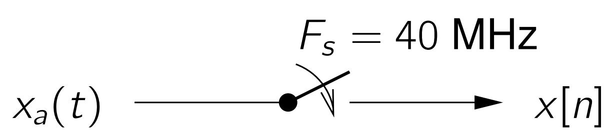
# Exercises

## Sampling

Given an analog real-valued signal $x_a(t)$ with frequency spectrum:




This signal is sampled with a sample frequency of 40 MHz:



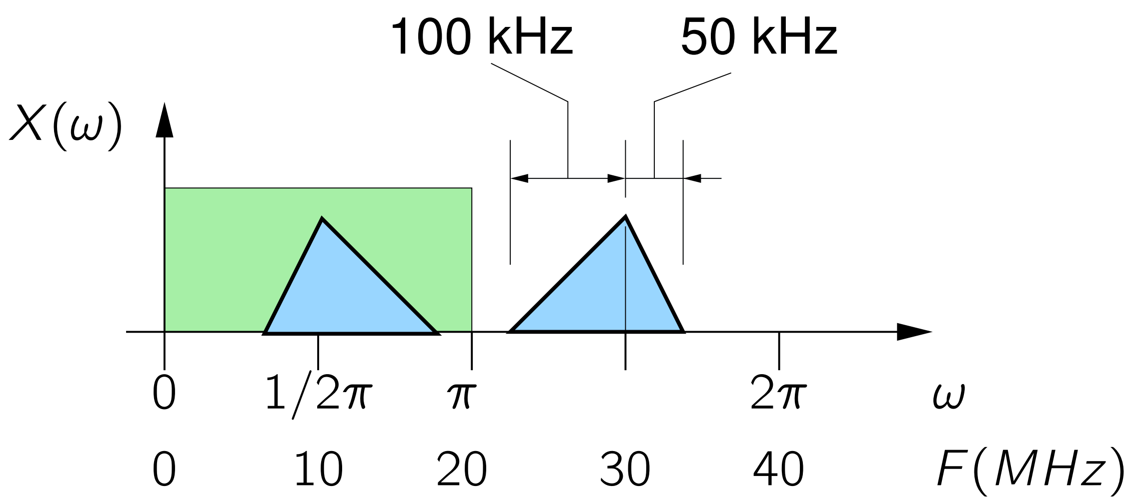

a Is the Nyquist criterion satisfied?

b Make a drawing of the frequency spectrum of the digital signal $x[n]$. Also indicate which frequencies (in Hz) play a role.

**Answer**

a No: the highest frequency in the signal is 30.05 MHz, the sample frequency should be twice as large (60.1 MHz), which is not the case here.

b Due to the sampling, part of the spectrum at 30 MHz is repeated with period 40 MHz and is seen at $70$, $110$ MHz etc and also at $-10$, $-50$, $\cdots$ MHz. The part of the spectrum at $-30$ MHz (not shown in the graph but the signal was assumed real-valued) is repeated and is seen at $10$, $50$, $\cdots$ MHz.

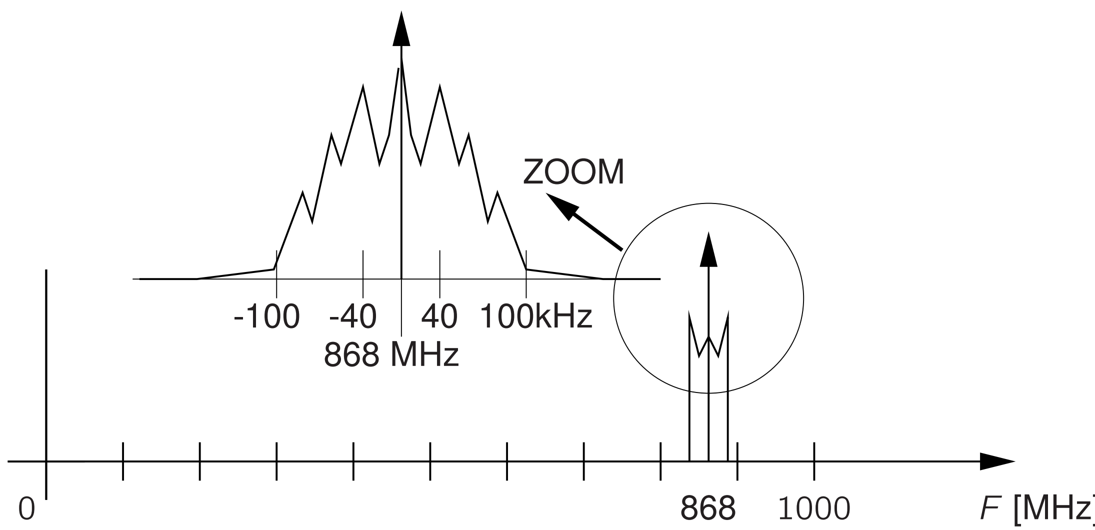


# Exercises

## Sampling

An RFID tag (e.g.. a smart card) modulates the transmitted carrier signal of a reader device. The transmitted carrier is at 868 MHz, the modulation of the tag is a square wave with a frequency around 40 kHz.

Below, the spectrum of the received signal $x_a(t)$ is shown (the transmitted carrier signal is dominantly present):



# Exercises

Make a drawing of the spectrum of $x[n]$ if we sample at 250 MHz (accurately indicate the frequencies; also mark the fundamental interval).

**Answer**

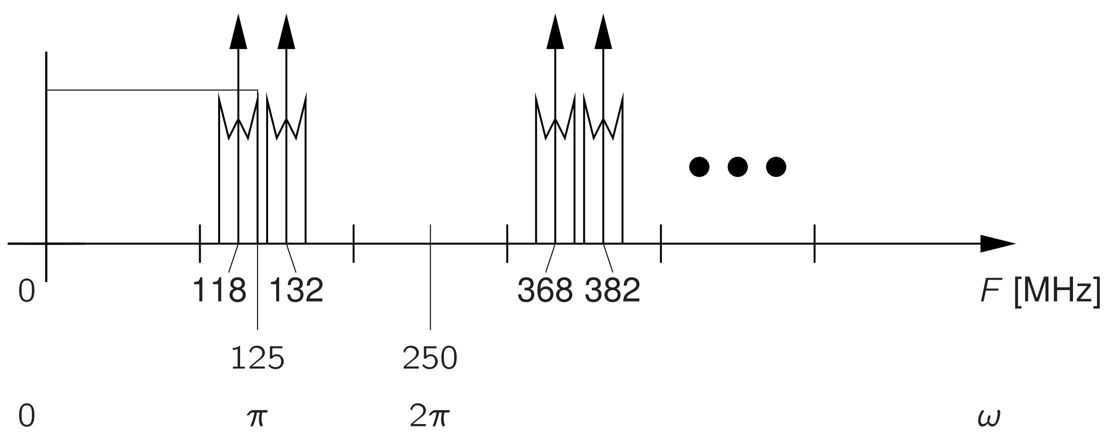
The spectrum becomes periodic with period $250$ MHz. Due to aliasing, the frequencies around $868$ MHz are shifted with multiples of 250 MHz, i.e.,

$$868 + k \cdot 250 = \begin{cases} 618 \text{ MHz}, & k = -1 \\ 368 & k = -2 \\ 118 & k = -3 \end{cases}$$

Etc. The spectrum is symmetric, we also have a part at $-868$ MHz which is shifted with multiples of 250 MHz, i.e.,

$$-868 + k \cdot 250 = \begin{cases} 132 \text{ MHz}, & k = 4 \\ 382 & k = 5 \\ 632 & k = 6 \end{cases}$$

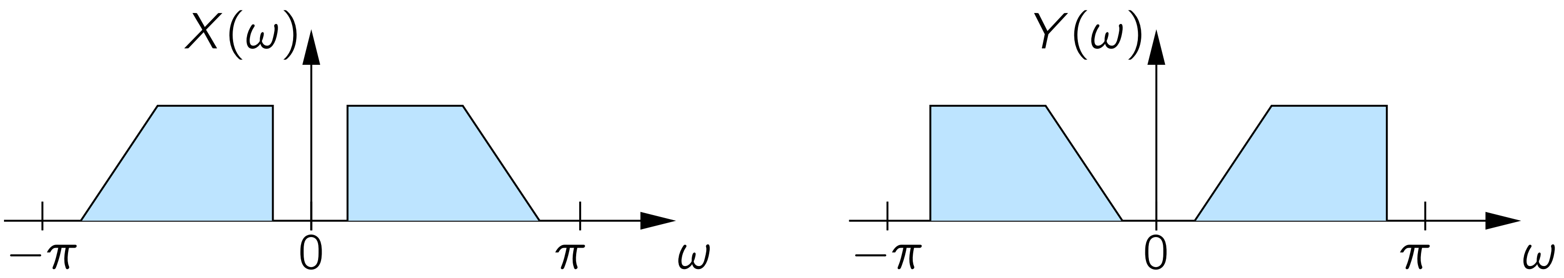
# Exercises



# Exercises

## Spectrum of discrete-time signals

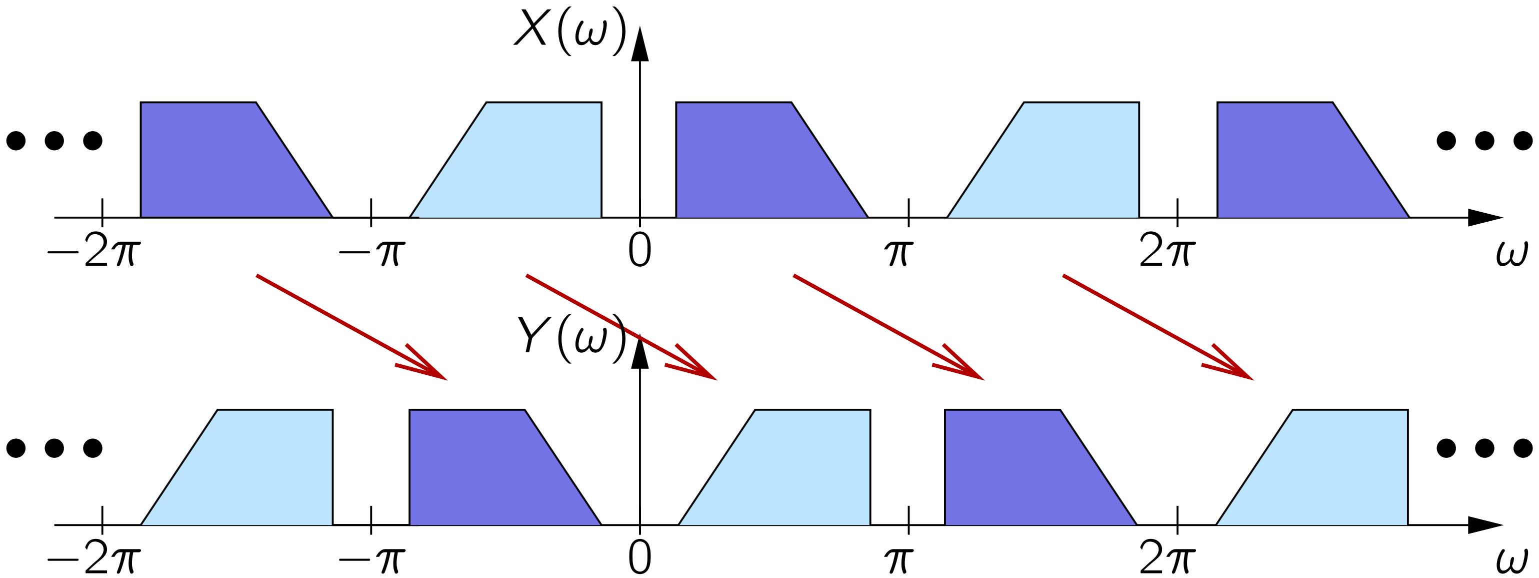
A simple scrambler swaps high and low frequencies in a signal




How can we carry out this operation? And reverse it?

**Oplossing:**

First make a drawing of a larger part of the frequency domain (the spectrum is periodic)




It is seen that the scrambler shifts the signal over $\pi$ rad: $Y(\omega) = X(\omega - \pi)$.

In time domain, a frequency shift corresponds to modulation:

$$Y(\omega) = X(\omega - \omega_0) \qquad \Leftrightarrow \qquad y[n] = x[n] \cdot e^{j\omega_0 n}$$

At $\omega_0 = \pi$ we find $y[n] = x[n] \cdot (-1)^n$. Inverse operation: repeat the modulation.

# Exercises

## Z-transform

Determine the $z$-transform and the ROC of

a) $x[n] = [\cdots, 0, 1, \boxed{2}, 1, 0, \cdots]$

b) $x[n] = \text{sgn}[n] = \begin{cases} 1, & n \geq 0, \\ -1, & n < 0 \end{cases}$

c) $x[n] = -a^n u[-n]$. For which values of $a$ does the DTFT $X(e^{j\omega})$ exist?

# Exercises

**Answer:**

a) $X(z) = z + 2 + z^{-1}$, ROC: $\mathbb{C} \setminus \{0, \infty\}$.

b) For the causal part ($n \geq 0$) we find

$$X_c(z) = \frac{1}{1 - z^{-1}} \qquad \text{ROC: } |z| > 1$$

For the anti-causal part (assume $n < 0$):

$$X_{ac}(z) = -\frac{z}{1 - z} \qquad \text{ROC: } |z| < 1$$

The $z$-transform of the sum is (take the intersection of the ROC's)

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{z}{1 - z} \quad \text{ROC: empty!}$$

Hence the $z$-transform does not exist.

# Exercises

c) The signal exists for $n \leq 0$.

$$X(z) = -1 - \frac{z}{a} - \frac{z^2}{a^2} - \cdots = -\frac{1}{1 - z/a} = \frac{az^{-1}}{1 - az^{-1}} \qquad \text{ROC: } |z| < a$$

The DTFT exists if the ROC contains the unit circle: for $|a| > 1$.

# Exercises

## Z-transform

Determine the $z$-transform and the ROC of $x_1[n] = u[n]$ and of $x_2[n] = -u[-n-1]$.

## Answer:

$$X_1(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} \qquad \text{ROC: } |z| > 1$$

$$X_2(z) = -z - z^2 - \cdots = -\frac{z}{1 - z} = \frac{1}{1 - z^{-1}} \qquad \text{ROC: } |z| < 1$$

The same $z$-transform but a different ROC.

Note the relation between the ROC (inside or outside of a circle) and (anti-)causality of $x[n]$!

# Exercises

## Inverse Z-transform

Given $X(z) = \dfrac{1 - z^{-10}}{1 - z^{-1}}$, determine $x[n]$. (Assume that the ROC is $|z| > 1$.)

## Answer:

For this ROC, $x[n]$ is causal. Split $X(z)$:

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{z^{-10}}{1 - z^{-1}}$$

Using the $z$-transform of $u[n]$ and a delay of 10 samples, we find

$$x[n] = u[n] - u[n - 10]$$

# Exercises

## Difference equations and the $z$-transform

Given the 2nd order LTI system with difference equation

$$y[n] = 0.25y[n-2] + x[n]$$

Here, $x[n]$ is the input and $y[n]$ is the output.

a) If $y[n] = 0.5^n u[n]$, then compute $x[n]$.

b) What is the system impulse response.

# Exercises

**Answer**

$$Y(z) = \frac{X(z)}{1 - 0.25z^{-2}} \qquad \text{and} \qquad Y(z) = \frac{1}{1 - 0.5z^{-1}}$$

Hence

$$X(z) = 1 + 0.5z^{-1} \qquad \text{so that} \qquad x[n] = \delta[n] + 0.5\delta[n-1]$$

The transfer function is (take $X(z) = 1$ so that $Y(z) = H(z)$)

$$H(z) = \frac{1}{1 - 0.25z^{-2}} = 1 + 0.25z^{-2} + (0.25)^2 z^{-4} + \cdots$$

The impulse response is

$$h[n] = [\boxed{1},\ 0,\ 0.25,\ 0,\ (0.25)^2,\ 0 \cdots]$$

This is also found if we carry out the recursion step-by-step taking as input an impulse $x[n] = \delta[n]$.

# Exercises

The impulse response is obtained in closed form by computing the poles of $H(z)$, and splitting them (partial fraction expansion):

$$H(z) = \frac{1}{(1-0.5z^{-1})(1+0.5z^{-1})} = \frac{0.5}{1-0.5z^{-1}} + \frac{0.5}{1+0.5z^{-1}}$$

$$h[n] = 0.5\left\{(0.5)^n u[n] + (-0.5)^n u[n]\right\}$$

Verify that this gives the same result!

# Exercises

Determine the inverse $z$-transform of

$$X(z) = \frac{8 - 4z^{-1}}{z^{-2} + 6z^{-1} + 8}$$

Assume that $x[n]$ is causal.

**Answer**

First write this as a polynomial in $z^{-1}$ and make proper - already the case here.

Determine the poles and split into elementary terms (partial fraction expansion):

$$X(z) = \frac{2 - z^{-1}}{2(1 + 0.25z^{-1})(1 + 0.5z^{-1})} = \frac{A}{1 + 0.25z^{-1}} + \frac{B}{1 + 0.5z^{-1}}$$

where we find $A = -3$ en $B = 4$.

The inverse is found as

$$x[n] = -3(-0.25)^n u[n] + 4(-0.5)^n u[n]$$