EE2S1 Signals and Systems

(3rd ed) Ch. 7.3 / (4th ed) Ch. 5.9.2 } Analog filter design

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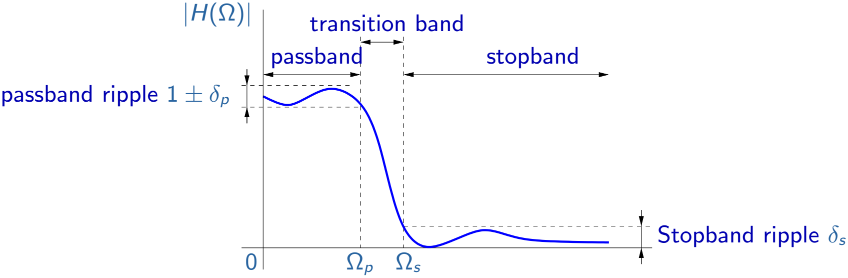
Contents

How can I design an analog filter $H(s)$ that meets certain specifications?



Note differences in notation. We often write $H(\Omega)$ instead of $H(j\Omega)$.

Continuous-time filter functions

- General form:

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \cdots + b_n s^n}{1 + a_1 s + \cdots + a_n s^n}$$

- Stability and causality:

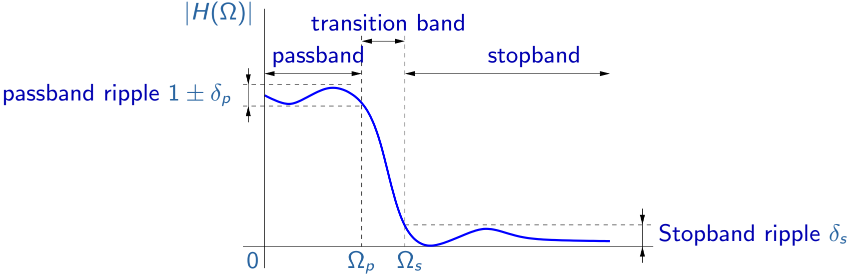
$$\begin{aligned} & \text{poles of } H(s) \text{ in left half plane} \\ \Leftrightarrow \quad & \text{zeros of } A(s) \text{ in left half plane} \end{aligned}$$

- Frequency spectrum: $|H(\Omega)|^2 = |H(j\Omega)|^2 = H(s)H(-s)\Big|_{s=j\Omega}$
- Damping (loss): $\alpha(\Omega) = \frac{1}{|H(\Omega)|^2}$, usually specified in dB:

$$\alpha(\Omega)[\text{dB}] = -10\log(|H(\Omega)|^2) = -20\log(|H(\Omega)|)$$

Filter specifications

- Example: specification of a low-pass filter



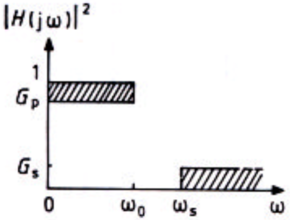
- A causal filter has a finite number of zeros and cannot be an ideal filter (Paley-Wiener): $|H(\Omega)|$ cannot be constant over an interval.
- Usually, only the amplitude spectrum is specified, because the phase spectrum is (almost) completely determined by this (cf. the *Hilbert transform* or the causality requirement)

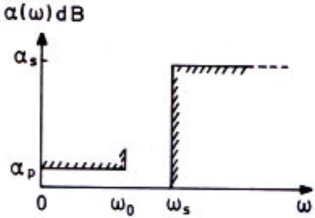
Practical design

We limit ourselves to design techniques based on amplitude specifications.

Specs for low-pass filters (the other types are derived from these)

- Ω_p also written as Ω_0: pass-band frequency
- G_p: minimal squared-amplitude in the pass-band (or α_p in dB)
- Ω_s: stop-band frequency
- G_s: maximal squared-amplitude (or α_s in dB)



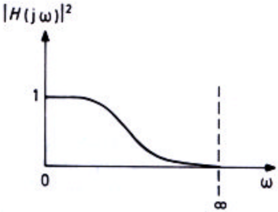


Butterworth filter

We start from the following characteristics:

- $|H(j\Omega)|^2$ is an even function
- $H(s)$ is rational, order n
- $\lim_{\Omega \to 0} |H(j\Omega)|^2 = 1$
- $\lim_{\Omega \to \infty} |H(j\Omega)|^2 = 0$

so that



$$|H(j\Omega)|^2 = \frac{1 + \sum_{r=1}^{n-1} b_r \Omega^{2r}}{1 + \sum_{r=1}^{n} a_r \Omega^{2r}}$$

Butterworth filter

The "Butterworth filter" is obtained if we require $|H(j\Omega)|^2$ to be maximally flat for $\Omega = 0$ and $\Omega = \infty$:

- $2n - 1$ derivatives equal to zero at $\Omega = 0$

$$\Rightarrow \quad a_r = b_r \qquad r = 1, 2, \ldots, n-1$$

- $2n - 1$ derivatives equal to zero at $\Omega = \infty$

$$\Rightarrow \quad b_r = 0 \qquad r = 1, 2, \ldots, n-1$$

This results in

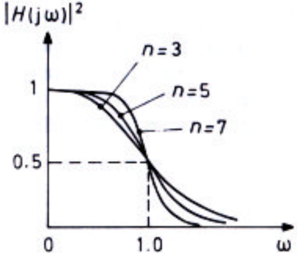
$$|H(j\Omega)|^2 = \frac{1}{1 + a_n \Omega^{2n}} \qquad \text{or} \qquad |H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 \Omega^{2n}}$$

Butterworth filter

Filter parameters are ϵ and n. How to design them?

- Example ($\epsilon = 1$):

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2n}}$$



- Larger $n \Rightarrow$ steeper roll-off (smaller transition band)
- Independent of n, these filters have a cutoff frequency (3 dB damping) at $\Omega_c = 1$:

$$|H(\Omega_c = 1)|^2 = \frac{1}{1+(1)^{2n}} = \frac{1}{2} \quad \Rightarrow \quad \alpha(\Omega_c) = -10\log\left(\frac{1}{2}\right) = 3\text{ dB}$$

Butterworth filter

- What if we want a 3 dB point at some other Ω_c? Use as template

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2n}}$$

- What if Ω_0 is specified, and a corresponding damping $\alpha(\Omega_0)$? Use

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2(\Omega/\Omega_0)^{2n}}$$

For this template, independent of n, we have at Ω_0

$$|H(j\Omega_0)|^2 = \frac{1}{1+\epsilon^2} \qquad \Rightarrow \qquad \alpha_p = \alpha(\Omega_0) = 10\log(1+\epsilon^2)$$

$$\Rightarrow \qquad \epsilon = \sqrt{10^{\alpha_p/10} - 1}$$

Next, find the minimal n from the damping condition at Ω_s.

Example 1: Design Butterworth filter

Determine the minimal order of the Butterworth filter with pass-band frequency $F_0 = 1.2$ kHz, maximal damping in the pass-band $\alpha_p = 0.5$ dB, stop-band frequency $F_s = 1.92$ kHz, and minimal damping in the stop-band $\alpha_s = 23$ dB

Solution

- We start from

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega/\Omega_0)^{2n}} \qquad \text{with} \qquad \Omega_s/\Omega_0 = F_s/F_0 = 1.6$$

- From $\alpha_p = \alpha(\Omega_0)$ we derive ϵ:

$$|H(\Omega_0)|^2 = \frac{1}{1+\epsilon^2} = 10^{-\alpha_p/10} \quad \Rightarrow \quad \epsilon = \sqrt{10^{\alpha_p/10} - 1} = 0.3493$$

Example 1 (cont'd)

- From ϵ, Ω_s/Ω_0 and α_s we derive the minimal n:

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\Omega_s/\Omega_0)^{2n}} = 10^{-\alpha_s/10}$$

$$\Rightarrow \qquad n \geq \frac{\log[(10^{\alpha_s/10}-1)/\epsilon^2]}{2\log(\Omega_s/\Omega_0)} = 7.87$$

- The derivation of $H(s)$ from $|H(j\Omega)|^2$ is called *spectral factorization*, you'll need a computer for this.

  Use $|H(j\Omega)|^2 = H(j\Omega)H(-j\Omega) = H(s)H(-s)\Big|_{s=j\Omega}$

  Analytic extension to the entire complex plane: substitute $\Omega = -js$

$$H(s)H(-s) = |H(j\Omega)|^2\Big|_{\Omega=-js}$$

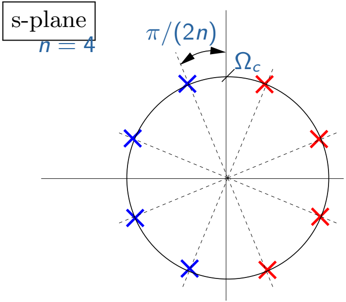
What is $H(s)$ for the Butterworth filter?

$$H(s)H(-s) = |H(j\Omega)|^2\Big|_{\Omega=-js} = \frac{1}{1+\epsilon^2(-js)^{2n}} = \frac{1}{1+\epsilon^2(-s^2)^n}$$

- The poles of $H(s)H(-s)$ follow as

  $(-js_k)^{2n} = -1/\epsilon^2 \;\Rightarrow\; s_k = (1/\epsilon)^{1/n} e^{j[(2k-1)\pi/(2n)+\pi/2]},\; k = 1, 2, \ldots, 2n$

  These are located on a circle with radius $(1/\epsilon)^{1/n} = \Omega_c$



Stable: poles of $H(s)$ are the n poles in the left-half plane
Then $H(-s)$ will have the remaining n poles in the right-half plane

Chebyshev filter

The Butterworth filter has maximal error in the pass-band at Ω_0, elsewhere the error is smaller. Perhaps the filter order can be made smaller (or the response sharper for the same filter order) by distributing the error more uniformly over the pass-band?

- We keep the maximal flatness in $\Omega = \infty$:
  $2n-1$ derivatives zero at $\Omega = \infty \;\Rightarrow\; b_r = 0, r = 1, 2, \ldots, n-1$

$$|H(j\Omega)|^2 = \frac{1}{1 + \sum_{r=1}^{n} a_r \Omega^{2r}} \;=:\; \frac{1}{1 + \epsilon^2 [T_n(\Omega)]^2}$$

  where $T_n(\Omega)$ is an even or odd polynomial of order n (because $T_n^2(\Omega)$ has to be even)

- In the pass-band we must have: $|T_n(\Omega)| \leq 1$.
  Elsewhere: $|T_n(\Omega)| \to \infty$

Chebyshev filter

From now on, normalize the pass-band to $-1 \leq \Omega \leq 1$:

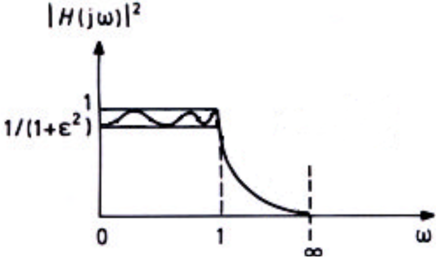
$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)}$$

with

$$\begin{array}{ll} |T_n(\Omega)| \leq 1 & (|\Omega| < 1) \\ |T_n(\Omega)| \to \infty & (|\Omega| \to \infty) \end{array}$$

Example polynomials:

$$\begin{aligned} T_0(\Omega) &= 1 \\ T_1(\Omega) &= \Omega \\ T_2(\Omega) &= 2\Omega^2 - 1 \\ T_3(\Omega) &= 4\Omega^3 - 3\Omega \end{aligned}$$



Chebyshev polynomials

Idea: $T_n(\Omega)$ has to oscillate between -1 and 1 in the pass-band, hence set

$$T_n(\Omega) = \cos(n\,\theta(\Omega)) \qquad -1 \leq \Omega \leq 1$$

How do we design $\theta(\Omega)$ such that $T_n(\Omega)$ is an even or odd polynomial of order n?

- From the property $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$ we obtain the recursion

  $$T_n(\Omega) = 2\,T_{n-1}(\Omega)\cos(\theta(\Omega)) - T_{n-2}(\Omega)$$

  with $T_0(\Omega) = 1$ and $T_1(\Omega) = \cos(\theta(\Omega))$.

- Repeat this to obtain

  $$T_n(\Omega) = c_n[\cos(\theta(\Omega))]^n + c_{n-2}[\cos(\theta(\Omega))]^{n-2} + c_{n-4}[\cos(\theta(\Omega))]^{n-4} + \cdots$$

Chebyshev polynomials

- $T_n(\Omega)$ is an even or odd polynomial in Ω of order n if $\theta(\Omega) = \cos^{-1}\Omega$ $(|\Omega| \leq 1)$. This gives $\cos(\theta(\Omega)) = \Omega$.

  Hence, the function $T_n(\Omega) = \cos(n\cos^{-1}(\Omega))$ satisfies

  $$T_n(\Omega) = 2\Omega T_{n-1}(\Omega) - T_{n-2}(\Omega)$$

  and is an even or odd polynomial of order n.

- Also valid: $\cosh(\alpha+\beta) + \cosh(\alpha-\beta) = 2\cosh\alpha\cosh\beta$
  If we use this to expand $\cosh(n\cosh^{-1}(\Omega))$, $(|\Omega| > 1)$ we obtain the same recursion, so the same polynomials!

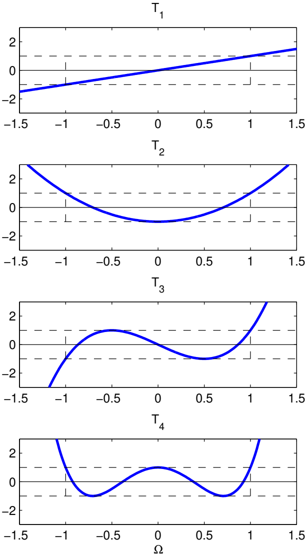
Chebyshev polynomials

The recursion

$T_n(\Omega) = 2\Omega T_{n-1}(\Omega) - T_{n-2}(\Omega)$ gives

$$\begin{aligned} T_0(\Omega) &= 1 \\ T_1(\Omega) &= \Omega \\ T_2(\Omega) &= 2\Omega^2 - 1 \\ T_3(\Omega) &= 4\Omega^3 - 3\Omega \\ T_4(\Omega) &= 8\Omega^4 - 8\Omega^2 + 1 \\ T_5(\Omega) &= 16\Omega^5 - 20\Omega^3 + 5\Omega \\ &\vdots \end{aligned}$$

$$T_n(\Omega) = \begin{cases} \cos(n\cos^{-1}(\Omega)), & (|\Omega| \le 1) \\ \cosh(n\cosh^{-1}(\Omega)), & (|\Omega| > 1) \end{cases}$$



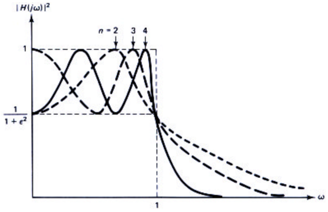
Chebyshev filters

- Resulting filters:

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)}$$

or more general

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2\left(\frac{\Omega}{\Omega_0}\right)}$$



Design of Chebyshev filters

For the design of ϵ and n, we usually start from

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2\left(\frac{\Omega}{\Omega_0}\right)}$$

with

$$T_n(\Omega) = \begin{cases} \cos(n\cos^{-1}(\Omega)), & (|\Omega| \leq 1) \\ \cosh(n\cosh^{-1}(\Omega)), & (|\Omega| > 1) \end{cases}$$

- If Ω_0 and a damping $\alpha(\Omega_0)$ is specified:
  Use that $T_n^2(1) = 1$ for any n

$$|H(j\Omega_0)|^2 = \frac{1}{1+\epsilon^2} \qquad \Rightarrow \qquad \alpha_p = \alpha(\Omega_0) = 10\log(1+\epsilon^2)$$

$$\Rightarrow \qquad \epsilon = \sqrt{10^{\alpha_p/10} - 1}$$

  Same as for Butterworth! Use this to determine ϵ from the specs.

Design of Chebyshev filters

- Next, find n from the damping condition at Ω_s. You will need to evaluate $T_n(\Omega_s/\Omega_0)$.
  Since $\Omega_s > \Omega_0$, use the "cosh" formula.

- If needed, calculate the cut-off frequency (3 dB level) $\Omega_c > \Omega_0$:

$$[\cosh(n \cosh^{-1}(\Omega_c/\Omega_0))]^2 = 1/\epsilon^2$$

$$\Rightarrow \qquad \Omega_c = \Omega_0 \cosh(1/n \cdot \cosh^{-1}(1/\epsilon))$$

Note: $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \Rightarrow \cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$

Example 2: Design Chebyshev filter

Determine the minimal order of a Chebyshev filter with pass-band frequency $F_0 = 1.2$ kHz, maximal damping in the pass-band $\alpha_p = 0.5$ dB, stop-band frequency $F_s = 1.92$ kHz, and minimal damping in the stop-band $\alpha_s = 23$ dB.

Solution:

- We start from

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega/\Omega_0)} \qquad \text{with} \qquad \Omega_s/\Omega_0 = F_s/F_0 = 1.6$$

- From $\alpha_p = \alpha(\Omega_0)$ we derive ϵ:

$$|H(\Omega_0)|^2 = \frac{1}{1+\epsilon^2} = 10^{-\alpha_p/10} \quad \Rightarrow \quad \epsilon = \sqrt{10^{\alpha_p/10} - 1} = 0.3493$$

Example 2 (cont'd)

- From ϵ, Ω_s/Ω_0 and α_s we derive the minimal order n:

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2[\cosh(n\cosh^{-1}(\Omega_s/\Omega_0))]^2} = 10^{-\alpha_s/10}$$

$$\Rightarrow \quad n \geq \frac{\cosh^{-1}(\sqrt{(10^{\alpha_s/10}-1)/\epsilon^2})}{\cosh^{-1}(\Omega_s/\Omega_0)} = 3.82$$

What is $H(s)$ for the Chebyshev filter?

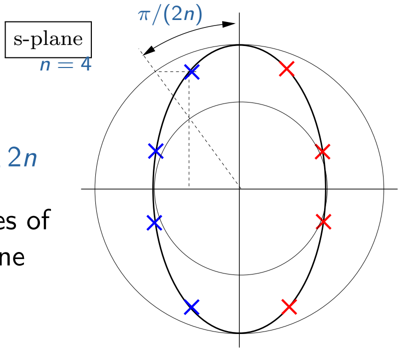
- Like with the Butterworth filter we look for $H(s)$ for which

$$H(s)H(-s) = |H(j\Omega)|^2\Big|_{\Omega=-js} = \frac{1}{1+\epsilon^2 T_n^2(-js)}$$

- Poles of $H(s)H(-s)$ satisfy

$$\begin{aligned} T_n^2(-js_k) &= -1/\epsilon^2 \\ \Rightarrow \quad s_k &= \sigma_k + j\Omega_k,\ k = 1, \cdots, 2n \end{aligned}$$

  These turn out to lie on an ellipse. Poles of $H(s)$ are the n poles in the left-half plane

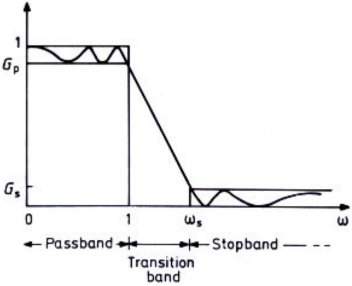


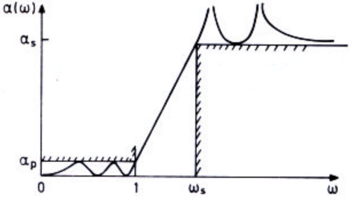
Elliptic filter

Generalization of the Chebyshev filter:

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 R_n^2(\Omega)}$$

with $R_n(\Omega)$ an arbitrary rational function in Ω.





We will not discuss this any further.

Frequency transformations: lowpass to lowpass

To transform a prototype filter into a desired filter we use transformations of the frequency axis:

- low-pass to low-pass : shift a frequency from $\Omega = 1$ to $\Omega = \Omega_0$:

$$\text{substitute:} \qquad \Omega \to \frac{\Omega}{\Omega_0} \qquad s \to \frac{s}{\Omega_0}$$

This maps

$$|H(\Omega)|^2 = \frac{1}{1 + \Omega^{2n}}$$

to

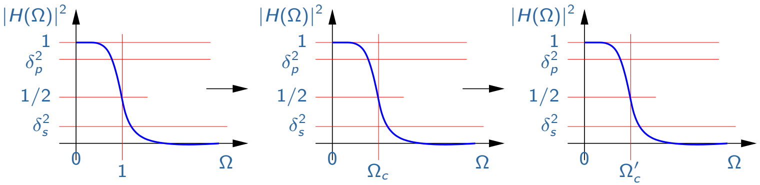
$$|H(\Omega)|^2 = \frac{1}{1 + (\frac{\Omega}{\Omega_0})^{2n}}$$

Low-pass to low-pass

- More generally: shift a frequency of $\Omega = \Omega_0$ to $\Omega = \Omega_0'$:

$$\text{substitute:} \qquad \Omega \to \Omega \frac{\Omega_0}{\Omega_0'} \qquad s \to s \frac{\Omega_0}{\Omega_0'}$$

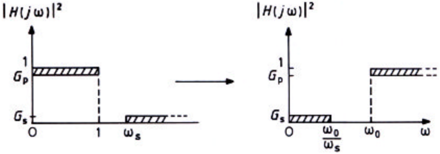
$$|H(\Omega)|^2 = \frac{1}{1 + (\frac{\Omega}{\Omega_0})^{2n}} \qquad \Rightarrow \qquad |H(\Omega)|^2 = \frac{1}{1 + (\frac{\Omega}{\Omega_0'})^{2n}}$$



Frequency transforms (2)

- low-pass to high-pass: mapping $1 \to \Omega_0$, and $\Omega_s \to \Omega_0/\Omega_s$

$$\Omega \to \frac{\Omega_0}{\Omega} \qquad s \to \frac{\Omega_0}{s}$$



- More generally: mapping $\Omega_0 \to \Omega_0'$ with reversal of the frequency axis

$$\Omega \to \frac{\Omega_0 \Omega_0'}{\Omega}, \qquad s \to \frac{\Omega_0 \Omega_0'}{s}$$

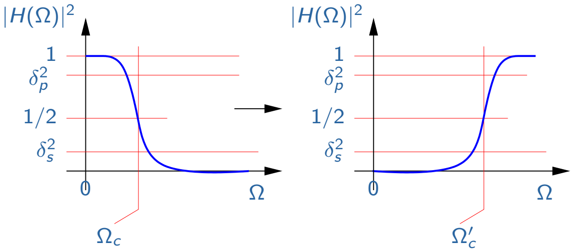
Example: low-pass to high-pass

Suppose the template low-pass filter has cut-off frequency $\Omega = \Omega_c$:

$$|H(\Omega)|^2 = \frac{1}{1 + (\frac{\Omega}{\Omega_c})^{2n}}$$

Transform $\Omega \to \frac{\Omega_c \Omega'_c}{\Omega}$ gives a high-pass filter with cut-off frequency Ω'_c:

$$|H(\Omega)|^2 = \frac{1}{1 + (\frac{\Omega'_c}{\Omega})^{2n}}$$



Example 3: use of the low-to-high transform

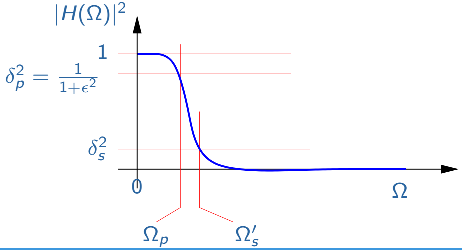
Design an analog *high-pass* filter design with specifications:

– Pass-band: starting at $F_p = 50$ Hz; ripple in the pass-band: $\leq$ 1 dB
– Stop-band: until $F_s = 40$ Hz; stop-band damping: $\geq$ 30 dB.

- We start with a Butterworth low-pass filter structure of the form

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 (\Omega/\Omega_p)^{2n}}$$

which we design such that $\Omega_p = 2\pi \cdot 50$, and $|H(\Omega_p)|^2$ equal to -1 dB.

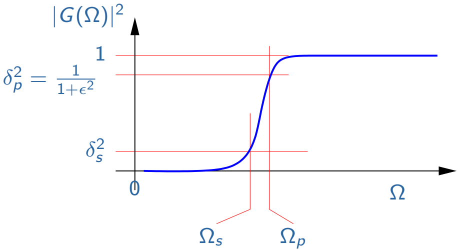


Example 3 (cont'd)

- Next, we apply to $H(\Omega)$ a low-to-high transform:

$$\Omega \to \frac{\Omega_p^2}{\Omega} \qquad \text{gives} \qquad |G(\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega_p/\Omega)^{2n}}$$

This is a high-pass filter with pass-band Ω_p.



Instead of first designing $H(\Omega)$, we will directly determine ϵ and n for $G(\Omega)$.

Example 3 (cont'd)

So we use as template highpass filter:

$$|G(\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega_p}{\Omega}\right)^{2n}}$$

- Determine ϵ by evaluation at $\Omega_p = 2\pi \cdot 50$:

$$|G(\Omega_p)|^2 = \frac{1}{1+\epsilon^2} = 10^{-1/10} \quad \Rightarrow \quad \epsilon = \sqrt{10^{1/10}-1} = 0.5088.$$

- Determine n by evaluation at $\Omega_s = 2\pi \cdot 40$:

$$|G(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\frac{2\pi\cdot 50}{2\pi\cdot 40})^{2n}} = 10^{-30/10} \Rightarrow (\frac{50}{40})^{2n} = \frac{10^{30/10}-1}{\epsilon^2} = 3858$$

$$\Rightarrow \qquad n = \frac{1}{2}\frac{\log(3858)}{\log(5/4)} = 18.5$$

We take filter order $n = 19$.