EE2S1 Signals and Systems

(3rd ed) Ch. 11 / (4th ed) Ch. 9 — The Discrete-Time Fourier Transform

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Contents

- definition of the DTFT
- relation to the z-transform, region of convergence, stability
- frequency plots
- inverse DTFT
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(3rd ed) Skip sections 11.2.5 (decimation/interpolation), 11.3 (Fourier Series), 11.4 (DFT).
(4th ed) Skip sections 9.2.5 (decimation/interpolation), 9.3 (Fourier Series), 9.4 (DFT).
These are covered in EE3S1.

The Discrete-time Fourier transform (DTFT)

The DTFT is defined as

$$X(\omega) = \mathcal{F}\{x[n]\} := \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Continuous function of ω (while $x[n]$ is a time series)
- $X(\omega + 2\pi) = X(\omega)$: periodic in ω, period 2π:
  It sufficies to consider the interval $\omega \in [-\pi, \pi]$.
- $X(\omega)$ is called "the spectrum"; it measures the frequency content
- Sufficient condition for convergence of the infinite sum:

$$|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}| \le \sum_{n=-\infty}^{\infty} |x[n]|\,|e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  i.e., $x[n]$ is absolutely summable ($x \in \ell_1$).

Relation to the z-transform

The DTFT is obtained from the z-transform by setting $z = e^{j\omega}$ (assuming that the unit circle $|z| = 1$ is in de ROC).

We often write $X(\omega)$ as $X(e^{j\omega})$, cf. the book. (The book's notation avoids confusion between $X(\omega)$ and $X(z)$, different functions.)

We immediately obtain (LTI systems):

- $y[n] = h[n] * x[n] \quad \Leftrightarrow \quad Y(\omega) = H(\omega)X(\omega)$ (filters!)
- $H(\omega) = \sum h[n]e^{-j\omega n}$ exists if the system is BIBO stable ($h \in \ell_1$, i.e., the unit circle is in the ROC of $H(z)$).
- $$\begin{array}{lcl} \delta[n] & \Leftrightarrow & 1 \\ u[n] & \Leftrightarrow & \text{(no ordinary DTFT because of ROC)} \\ \delta[n-N] & \Leftrightarrow & e^{-j\omega N} \\ a^n u[n] \quad (|a| < 1) & \Leftrightarrow & \dfrac{1}{1 - ae^{-j\omega}} \end{array}$$

Exercise [trial exam 2016]

Given $X(\omega) = \cos(\omega)$, determine $x[n]$.

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$$X(\omega) = \frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega} \quad \Rightarrow \quad x[n] = \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1]$$

Exercise [exam January 2023]

Given the DTFT $X(e^{j\omega}) = e^{-j2\omega}\cos^2(\omega)$. Determine $x[n]$.

Hint: first determine the z-transform.

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Hint: first determine the z-transform.

Rewrite (using $z = e^{j\omega}$) as a function of z:

$$X(z) = z^{-2}\frac{1}{4}(z + z^{-1})^2 = \frac{1}{4}(1 + 2z^{-2} + z^{-4})$$

Hence

$$x[n] = \frac{1}{4}\left(\delta[n] + 2\delta[n-2] + \delta[n-4]\right)$$

Frequency plots

$X(\omega)$ is complex. To make a plot, write $X(\omega) = |X(\omega)|e^{j\phi(\omega)}$, where $|X(\omega)|$: amplitude spectrum, $\phi(\omega)$: phase spectrum

Example

plot the amplitude and phase spectrum of $X(\omega) = \dfrac{1}{1 - ae^{-j\omega}}$

- Amplitude spectrum $|X(\omega)|$ is found via

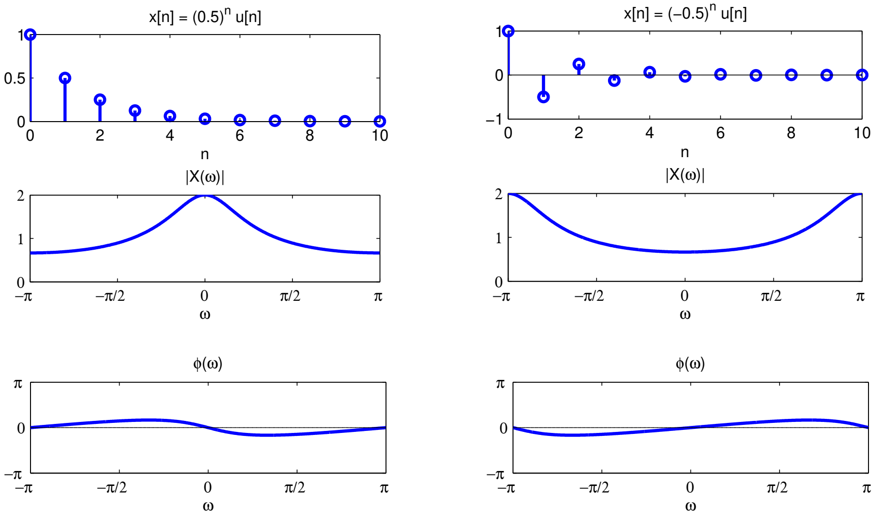
$$|X(\omega)|^2 = X(\omega)X^*(\omega) = \frac{1}{1 - ae^{-j\omega}} \frac{1}{1 - ae^{j\omega}} = \frac{1}{1 + a^2 - 2a\cos(\omega)}$$

- Phase spectrum

$$\frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - a\cos(\omega)) + ja\sin(\omega)}$$

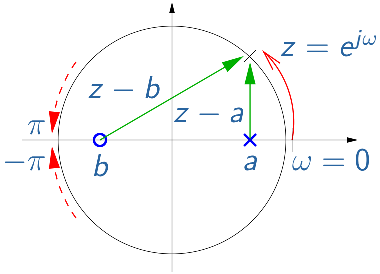
$$\Rightarrow \qquad \phi(\omega) = -\tan^{-1}\left(\frac{a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

Discrete-time Fourier Transform



Estimating frequency plots using phasors

Given a rational transfer function, e.g. $X(z) = \dfrac{z-b}{z-a}$, we can sketch a plot of $|X(\omega)|$ and $\phi(\omega)$ using phasors.

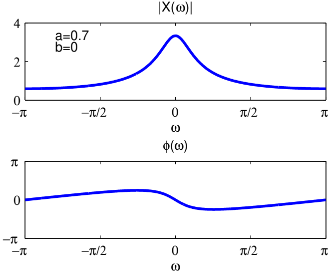


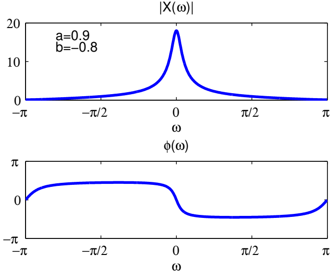
$$|X(\omega)| = \frac{|z-b|}{|z-a|}$$

$$\phi(\omega) = \angle(z-b) - \angle(z-a) \mod 2\pi$$

Estimating frequency plots using phasors

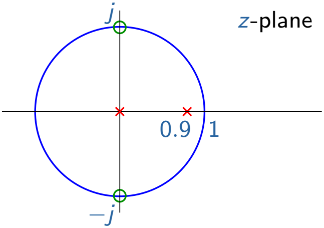
To gain some insight: compute this for a number of values of ω.





Exercise [exam January 2021]

Sketch the magnitude spectrum $|H(e^{j\omega})|$ corresponding to the pole-zero plot:



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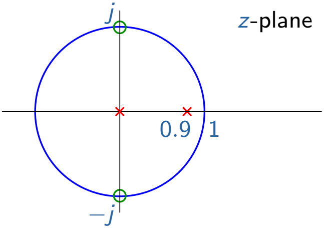
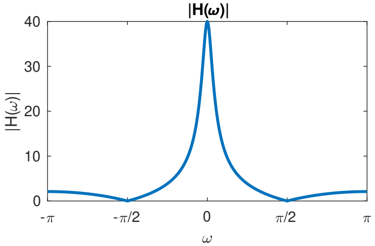
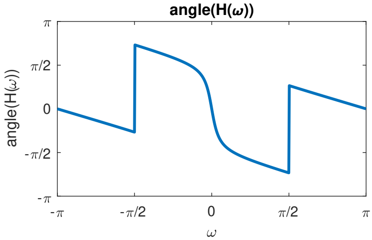
Sketch the magnitude spectrum $|H(e^{j\omega})|$ corresponding to the pole-zero plot:



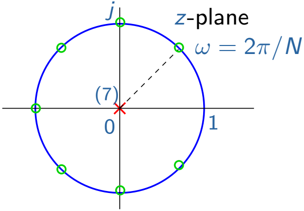




Example: DTFT of a pulse

$p[n] = u[n] - u[n-N]\,,$ pulse of length N

$$P(z) = 1 + z^{-1} + \cdots + z^{-(N-1)} = \frac{1 - z^{-N}}{1 - z^{-1}}$$



$$\begin{aligned} P(\omega) &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} e^{-j\omega(N-1)/2} \\ &= \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2} \end{aligned}$$

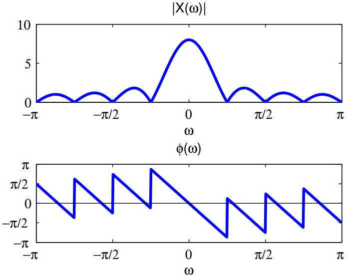
DTFT of a pulse (cont'd)

- The amplitude spectrum is

$$A(\omega) = |P(\omega)| = \left| \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right|$$

"periodic sinc-function" (Dirichlet-function) with $A(0) = N$

- The phase spectrum is $\phi(\omega) = -\omega(N-1)/2$ (lineair phase) plus phase jumps of π due to sign changes of $\sin(\omega N/2)$.



($N = 8$)

zero crossings for $\omega = \pm\frac{2\pi}{N}k$ ($k \neq 0$)

Phase slope $\Leftrightarrow$ delay
Phase jumps (π) $\Leftrightarrow$ change of sign

DTFT of a pulse (cont'd)

- The linear phase corresponds to a delay $z^{-(N-1)/2}$, half the duration of the pulse.
- The first zero in the amplitude spectrum (right of the peak at $\omega = 0$) gives an indication of the "bandwidth"

$$\Delta\omega = \frac{2\pi}{N}$$

  The "bandwidth" is inversely proportional to the duration of the pulse.
  In many applications where we collect N samples (or have N uniformly spaced sensors), this is related to the resolution of a system.

Example: DTFT of a non-causal signal

Determine the spectrum of the non-causal signal $x[n] = a^{|n|}$ with $|a| < 1$.

Example: DTFT of a non-causal signal

Determine the spectrum of the non-causal signal $x[n] = a^{|n|}$ with $|a| < 1$.

The z-transform of $x[n]$ is

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} a^n z^n - 1 = \frac{1}{1 - az^{-1}} + \frac{1}{1 - az} - 1 = \frac{1 - a^2}{1 - a(z + z^{-1}) + a^2}$$

with as ROC the intersection of the ROC of the causal and anti-causal part:

$$\text{ROC: } |a| < |z| < \frac{1}{|a|}$$

The ROC contains the unit circle. Hence

$$X(\omega) = X(z = e^{j\omega}) = \frac{1 - a^2}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2} = \frac{1 - a^2}{1 + a^2 - 2a\cos(\omega)}$$

Note that $x[n]$ is even and $X(\omega)$ is real-valued ($\phi(\omega) = 0$).

Relation to the continuous-time Fourier Transform

Consider a signal $x(t)$ and sample it with period T_s,

$$x_s(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

The (continous-time) Fourier transform is

$$X_s(\Omega) = \mathcal{F}\{x_s(t)\} = \sum_n x(nT_s)\mathcal{F}\{\delta(t - nT_s)\} = \sum x(nT_s)e^{-jn\Omega T_s}$$

Set $\omega = \Omega T_s$ and $x[n] = x(nT_s)$. Then

$$X_s(\Omega) = \mathcal{F}\{x_s(t)\} = \sum_n x[n]e^{-jn\omega} =: X(\omega)$$

The definition of $X(\omega)$ (spectrum of a time series) is consistent to that of $X_s(\Omega)$ (spectrum of a continuous-time signal).

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\mathrm{d}\omega = \int_{-1/2}^{1/2} X(f)e^{j2\pi fn}\mathrm{d}f \quad (\text{with } \omega = 2\pi f)$$

The integral runs over 1 period of the spectrum.

Proof

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\right]e^{j\omega n}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\left[\int_{-\pi}^{\pi} e^{j\omega(n-k)}\mathrm{d}\omega\right] \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\cdot 2\pi\delta[n-k] \quad = \quad x[n]
\end{aligned}$$

Energy (Parseval)

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \mathrm{d}\omega$$

$S_x(\omega) := |X(\omega)|^2$ is called the energy spectrum ("energy spectral density": energy per radial)

Proof

$$\begin{aligned}
E_x &= \sum_n |x[n]|^2 = \sum_n x[n]x^*[n] \\
&= \sum_n x[n]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)e^{-j\omega n}\mathrm{d}\omega\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_n x[n]e^{-j\omega n}\right]\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \mathrm{d}\omega
\end{aligned}$$

Extensions

A *sufficient* condition for the existence of the DTFT was that the signal is absolutely summable. But also for some other signals we can define the DTFT.

Extension to signals with finite energy

Signals with finite energy ($x \in \ell_2$) are not always absolutely summable (the reverse does hold: $\ell_1 \subset \ell_2$). Due to Parseval, the spectrum has equal energy: also finite. We can define a DTFT pair (signal/spectrum) based on the Inverse DTFT (integral over a finite interval).

Example

Ideal low-pass filter:

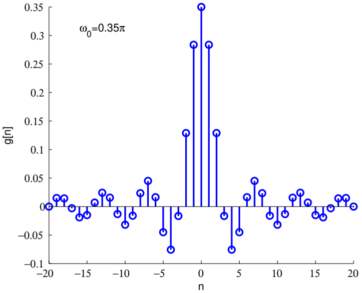
$$G(\omega) = \begin{cases} 1, & -\omega_0 \leq \omega \leq \omega_0 \,, \\ 0, & \text{elsewhere} \end{cases} \qquad \text{with copies every } 2\pi k$$

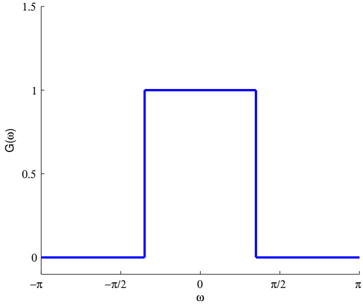
Ideal low-pass filter

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\omega)e^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{\sin(\omega_0 n)}{\pi n}$$

$g[n]$ has finite energy but is not absolutely summable (because $\frac{1}{n}$ converges to 0 very slowly)





Further extension to non-absolutely summable signals

According to the equation, the Inverse DTFT of $2\pi\delta(\omega-\omega_0)$ equals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega-\omega_0)\, e^{j\omega n}\mathrm{d}\omega \;=\; e^{j\omega_0 n}$$

Hence

$$e^{j\omega_0 n} \qquad \Leftrightarrow \qquad 2\pi\delta(\omega-\omega_0)$$

This can be used to compute the DTFT of some signals which are not absolutely summable nor have finite energy (with impulses in the frequency domain), e.g., periodic signals.

- $x[n] = A \quad (-\infty < n < \infty)$ (constant signal) is not absolutely summable. The DTFT is

  $$X(\omega) = 2\pi A\delta(\omega)\,, \qquad -\pi \leq \omega < \pi$$

  Outside this interval: periodic (period 2π), or $X(\omega) = 2\pi A\sum_k \delta(\omega - 2\pi k)$.

Cosine signals

The DTFT of $x[n] = \cos(\omega_0 n + \theta) = \frac{1}{2}\left[e^{j(\omega_0 n+\theta)} + e^{-j(\omega_0 n+\theta)}\right]$ is

$$X(\omega) = \pi\left[e^{j\theta}\delta(\omega - \omega_0) + e^{-j\theta}\delta(\omega + \omega_0)\right], \qquad -\pi \le \omega < \pi$$

Outside this interval: periodic (period 2π).

Periodic signals

More in general, consider

$$x[n] = \sum_{\ell} A_\ell \cos(\omega_\ell n + \theta_\ell)$$

$$\Leftrightarrow \qquad X(\omega) = \sum_{\ell} \pi A_\ell \left[e^{j\theta_\ell} \delta(\omega - \omega_\ell) + e^{-j\theta_\ell} \delta(\omega + \omega_\ell) \right]$$

for $-\pi \leq \omega < \pi$ (periodic outside this interval).

A periodic signal $x[n]$ has harmonically related frequencies: $\omega_\ell = \ell\omega_0$, with $\omega_0 = \frac{2\pi}{N}$, where N is the period (in samples). We obtain a line spectrum, just like with the Fourier Series.

Unit step

The z-transform of a unit step $u[n]$ is

$$\frac{1}{1 - z^{-1}}\,, \qquad \text{ROC: } |z| > 1$$

The unit cicle is not in the ROC, thus the DTFT can only be defined in generalized sense. ($u[n]$ is not absolutely summable and does not have finite energy.)

Define the discrete-time "sign" function:

$$\text{sgn}[n] = \begin{cases} 1\,, & n \geq 0 \\ -1\,, & n < 0 \end{cases}$$

Then

$$\begin{aligned} \text{sgn}[n] \quad &\Leftrightarrow \quad \frac{2}{1 - e^{-j\omega}} \\ u[n] = \frac{1}{2} + \frac{1}{2}\text{sgn}[n] \quad &\Leftrightarrow \quad \pi \sum_k \delta(\omega - 2\pi k) + \frac{1}{1 - e^{-j\omega}} \end{aligned}$$

Unit step (cont'd)

Proof (indication)

Using the Fourier transform of
$\delta[n] = u[n] - u[n-1] = \frac{1}{2}(\text{sgn}[n] - \text{sgn}[n-1])$:

$$1 = \frac{1}{2}\mathcal{F}\{\text{sgn}[n]\} - \frac{1}{2}\mathcal{F}\{\text{sgn}[n-1]\} = \frac{1}{2}\mathcal{F}\{\text{sgn}[n]\} - \frac{1}{2}e^{-j\omega}\mathcal{F}\{\text{sgn}[n]\}$$

$$\Rightarrow \mathcal{F}\{\text{sgn}[n]\} = \frac{2}{1-e^{-j\omega}} \qquad \text{for } \omega \neq \cdots, 0, 2\pi, 4\pi, \cdots$$

For $\omega = \cdots, 0, 2\pi, 4\pi, \cdots$ we consider the DC component of the function, which equals 0 (in contrast to $u[n]$, which motivates why we looked at $\text{sng}[n]$).

$\mathcal{F}\{u[n]\}$ has impulses at these frequencies.

Compare this to the Fourier transform of a continuous-time step function:

$\mathcal{F}\{u(t)\} = \frac{1}{j\Omega} + \pi\delta(\Omega)$.

Shift in time

If $y[n] = x[n-N]$ is a delay by N samples, then

$$Y(\omega) = \sum_n x[n-N]e^{-j\omega n} = e^{-j\omega N}X(\omega)$$

- The delay only affects the phase, which drops by $-\omega N$.
- Generally, a filter $H(\omega)$ that shows a linear phase term ($-\omega N$) inserts a delay of N samples.

  This is called the *phase delay*: the delay that a sinusoid of frequency ω would experience.

  Also used is *group delay*: the derivative (slope) of the phase response.

Shift in frequency

If

$$Y(\omega) = X(\omega - \omega_0)$$

is a frequency shift of $X(\omega)$ by ω_0, then

$$y[n] = x[n] \cdot e^{j\omega_0 n}$$

$y[n]$ equals $x[n]$ modulated by a complex exponential function $e^{j\omega_0 n}$.

- Likewise:

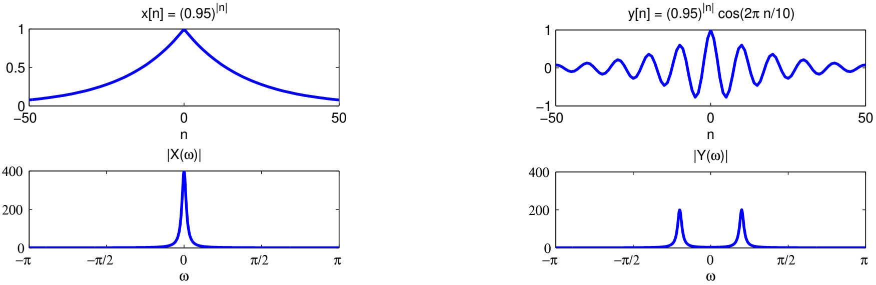
$$x[n] \cdot \cos(\omega_0 n) \quad \Leftrightarrow \quad \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$$

$$x[n] \cdot \sin(\omega_0 n) \quad \Leftrightarrow \quad -\frac{j}{2}\left[X(\omega - \omega_0) - X(\omega + \omega_0)\right]$$

  The modulation shifts the spectrum of $x[n]$ to frequency ω_0.

Example of modulation

$$x[n] = a^{|n|} \cos(\omega_0 n) \qquad \text{with } a = 0.95,\ \omega_0 = \frac{2\pi}{10}$$



More generally: product of two signals

The DTFT of the product $x[n]y[n]$ is

$$\begin{aligned}
\sum x[n]y[n]e^{-j\omega n} &= \sum \left[\frac{1}{2\pi}\int X(\theta)e^{j\theta n}\mathrm{d}\theta\right] y[n]e^{-j\omega n} \\
&= \frac{1}{2\pi}\int X(\theta)\left[\sum y[n]e^{-j(\omega-\theta)n}\right]\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int X(\theta)Y(\omega-\theta)\mathrm{d}\theta
\end{aligned}$$

Hence: a product in time becomes a convolution in frequency domain

$$x[n]y[n] \quad \Leftrightarrow \quad (X * Y)(\omega) := \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)Y(\omega-\theta)\mathrm{d}\theta$$

- Special case (modulation):
  $y[n] = x[n]e^{j\omega_0 n} \quad \Leftrightarrow \quad Y(\omega) = X(\omega-\omega_0)$
  because $e^{j\omega_0 n} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$.

Exercise [trial exam 2016]

Determine the DTFT $X(\omega)$ of

$$x[n] = (-1)^n u[n]$$

Exercise [trial exam 2016]

Determine the DTFT $X(\omega)$ of

$$x[n] = (-1)^n u[n]$$

For $y[n] = u[n]$ we have seen that $Y(\omega) = \frac{1}{1-e^{-j\omega}} + \pi \sum_k \delta(\omega - 2\pi k)$.

We also saw that for a modulation:

$$(-1)^n y[n] \quad \leftrightarrow \quad \frac{1}{2}[Y(\omega - \pi) + Y(\omega + \pi)] = Y(\omega - \pi)$$

(due to periodicity of the spectrum with period 2π, both shifts exactly coincide).

Together, we obtain

$$X(\omega) = \frac{1}{1 - e^{-j(\omega-\pi)}} + \pi \sum_k \delta(\omega - \pi - 2\pi k) = \frac{1}{1 + e^{-j\omega}} + \pi \sum_k \delta(\omega - \pi - 2\pi k$$

Real-valued signals

For real-valued signals, $x[n] = x^*[n]$. Hence

$$X^*(\omega) = X(-\omega)$$

and thus

$$|X(-\omega)| = |X(\omega)| \ : \quad \text{even in } \omega; \qquad \phi(-\omega) = -\phi(\omega) \ : \quad \text{odd in } \omega$$

It suffices to consider the spectrum on the interval $0 \leq \omega \leq \pi$.

Even real-valued signals

If moreover $x[n] = x[-n]$, then $X(\omega)$ is real-valued:

$$X^*(\omega) = \sum_{n=-\infty}^{\infty} x^*[n] e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[-n] e^{j\omega n} = X(\omega)$$

The phase spectrum $\phi(\omega)$ is 0 except for jumps of π due to sign changes in $X(\omega)$.

Summary of properties (cf Table in Chaparro)

$$
\begin{array}{lcl}
ax[n] + by[n] & \Leftrightarrow & aX(\omega) + bY(\omega) \\
x[n-N] & \Leftrightarrow & e^{-j\omega N}X(\omega) \\
x[-n] & \Leftrightarrow & X(-\omega) \\
x^*[n] & \Leftrightarrow & X^*(-\omega) \\
(x_1 * x_2)[n] & \Leftrightarrow & X_1(\omega)X_2(\omega) \\
x[n]y[n] & \Leftrightarrow & (X * Y)(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)Y(\omega-\theta)\mathrm{d}\theta \\
e^{j\omega_0 n} & \Leftrightarrow & 2\pi\delta(\omega-\omega_0) \\
e^{j\omega_0 n}x[n] & \Leftrightarrow & X(\omega-\omega_0) \\
x[n]\cos(\omega_0 n) & \Leftrightarrow & \frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]
\end{array}
$$

Parseval: $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \mathrm{d}\omega$$

EE3S1 preview: Discrete Fourier Transform (DFT)

Suppose $x[n]$ has a finite length of N samples (support $0 \le n \le N-1$), or $x[n]$ is periodic with period N, and we consider only 1 period.

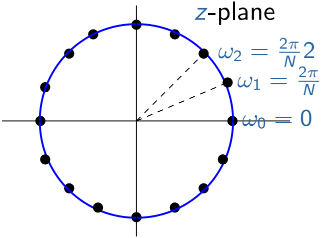
The DTFT $X(\omega)$ is a continuous function of ω, with $-\pi \le \omega < \pi$.

We sample $X(\omega)$ with N samples:

$$X[k] := X(\omega_k) \qquad \text{with} \qquad \omega_k = \frac{2\pi}{N}k, \quad k = 0, \cdots, N-1\,.$$

We obtain

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$



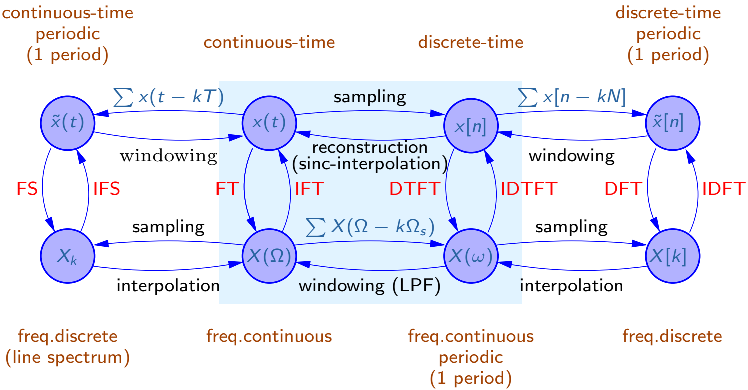
$X[k]$ is called the Discrete Fourier Transform (DFT).

EE3S1 preview: Discrete Fourier Transform (DFT)

- N samples in frequency suffice to recover $x[n]$, $0 \leq n \leq N-1$ (outside this interval: periodic or zero)
- Computationally efficient due to the Fast Fourier Transform (FFT)

The DFT and its properties are explored in the course lab, and fully covered in EE3S1 *Signal Processing*.

Relations



Generally:

- periodic $\Leftrightarrow$ discrete
- short $\Leftrightarrow$ long
- product $\Leftrightarrow$ convolution