EE2S1 Signals and Systems

(3rd ed) Ch. 10
(4th ed) Ch. 8 } **The z-transform**

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Contents

- definition of the z-transform
- region of convergence
- convolution property, transfer function
- causality and stability
- inverse z-transform [postponed till Lecture 18]

(3rd ed) Skip sections 10.5.3, 10.6, 10.7
(4th ed) Skip sections 8.5.3, 8.6, 8.7

The Laplace transform for sampled sequences

Suppose that we have a sampled signal:

$$x_s(t) = \sum x[n]\delta(t - nT_s)\,, \qquad x[n] := x(nT_s)$$

The Laplace transform $\mathcal{L}\{x_s(t)\}$ is

$$X_s(s) = \sum x[n]\mathcal{L}\{\delta(t - nT_s)\} = \sum x[n]\, e^{-sT_s n} \quad = \quad \sum x[n]\, z^{-n}$$

where $z := e^{sT_s}$.

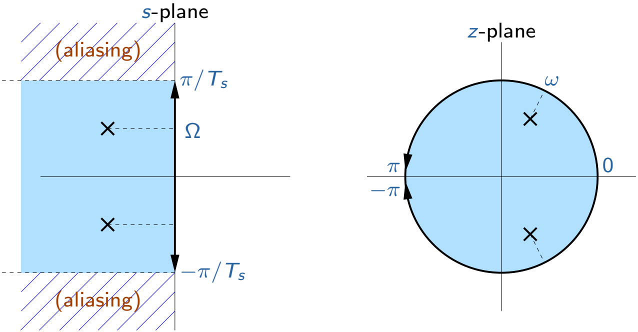
- For $s = j\Omega$ we obtain $z = e^{j\Omega T_s} = e^{j\omega}$, with $\omega = \Omega T_s$.
- More generally: $s = \sigma + j\Omega$ becomes $z = e^{\sigma T_s} e^{j\Omega T_s} = r\, e^{j\omega}$.

Aliasing

The mapping $s \to z = e^{sT_s}$ is not one-to-one.

For a given $z = e^{j\omega}$ we can take $-\pi \leq \omega \leq \pi$, this corresponds to $-\frac{\pi}{T_s} \leq \Omega \leq \frac{\pi}{T_s}$: the fundamental interval.

Complex numbers $s = j\Omega$ with Ω outside this interval are mapped onto the same z. Left half-plane is mapped to the inside of the unit circle.



The z-transform

From now on, we will work with z and apply this transform to time series, even if there is no connection to continuous-time signals.

The (two-sided) z-transform of a time series $x[n]$ is defined as

$$X(z) = \mathcal{Z}(x[n]) := \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \qquad z \in \mathrm{ROC}$$

We also need to indicate the region of convergence (ROC).

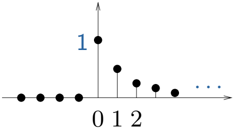
For example:

$$\begin{aligned} x &= [\cdots, 0, 1, 2, \boxed{3}, 4, 5, 0, \cdots] \\ \Rightarrow \quad X(z) &= z^2 + 2z^1 + 3 + 4z^{-1} + 5z^{-2} \\ &\quad \text{ROC: } z \in \mathbb{C} \backslash \{0, \infty\} \end{aligned}$$

Exercise

Determine the z-transform (and ROC) of the exponential series:

$$x[n] = a^n\, u[n]$$



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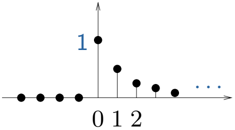


$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^n z^{-n} = 1 + az^{-1} + a^2 z^{-2} + \cdots \\ &= \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \end{aligned}$$

ROC: $|az^{-1}| < 1$, hence $|z| > a$

Delay

$$
\begin{aligned}
x[n] \quad &\Leftrightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
x[n-k] \quad &\Leftrightarrow \quad \sum_{n=0}^{\infty} x[n-k] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[n-k] z^{-(n-k)} z^{-k} \\
&= z^{-k} X(z)
\end{aligned}
$$

A unit delay corresponds to multiplication by z^{-1}.

The z-transform

A few properties:

$$
\begin{array}{lcl}
ax[n] + by[n] & \Leftrightarrow & aX(z) + bY(z) \\
x[n-k] & \Leftrightarrow & z^{-k}X(z) \\
a^n x[n] & \Leftrightarrow & X(\frac{z}{a}) \\
x[-n] & \Leftrightarrow & X(z^{-1})
\end{array}
$$

often $a = e^{j\omega_0}$ (modulation)

$$
\begin{array}{lcl}
x[n] = \delta[n] & \Leftrightarrow & X(z) = 1 \\
x[n] = u[n] & \Leftrightarrow & X(z) = \dfrac{1}{1 - z^{-1}}
\end{array}
$$

ROC: $z \in \mathbb{C}$

ROC: $|z| > 1$

See Chaparro for tables and more properties.

Region of convergence

The region of convergence (ROC) of the z-transform of a signal $x[n]$ contains those values of z for which the summation converges.

With $z = re^{j\omega}$ we find

$$\text{ROC:} \quad |X(z)| = |\sum x[n]z^{-n}| \leq \sum |x[n]|\, r^{-n} < \infty$$

- The ROC is the area where $|X(z)| < \infty$, this depends on r but not on ω. Hence, the ROC is limited by circles.
- $X(z)$ and the ROC together uniquely determine $x[n]$.
- Poles p_k are the locations where $X(p_k) \to \infty$: these are never in the ROC.

  Zeros z_k are the locations where $X(z_k) = 0$.

Example

- Determine the poles and zeros of

$$X(z) = 1 + 2z^{-1} \quad = \quad \frac{z+2}{z}$$

  Answer: 1 pole at $z = 0$; 1 zero at $z = -2$.

- Same for

$$X(z) = \frac{1 + 2z^{-1}}{1 + z^{-2}} \quad = \quad \frac{z(z+2)}{z^2 + 1}$$

  Answer: poles at $z = \pm j$; 1 zero at $z = -2$, 1 zero at $z = 0$.

Theory says that for rational functions, the number of poles equals the number of zeros (also taking into account those at $z = 0$ and $z = \infty$).

If $X(z)$ is a rational function with real-valued coefficients, then the complex poles and zeros appear in conjugated pairs: if p_k is a complex pole, then so is p_k^*.

ROC for a finite sequence

If $x[n] = 0$ outside an interval $-\infty < N_0 \leq n \leq N_1 < \infty$, i.e.

$$X(z) = x[N_0]z^{-N_0} + \cdots + x[N_1]z^{-N_1}$$

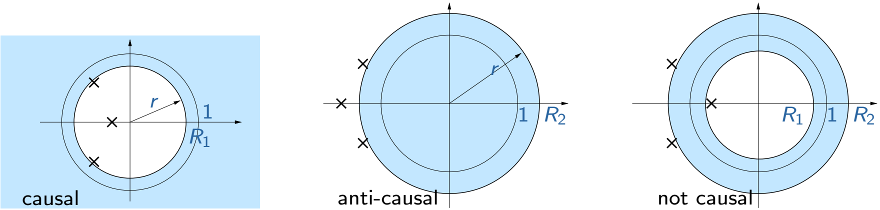
then the sum has a finite number of terms, and the ROC is all of $\mathbb{C}$, except perpaps at $z = 0$ or $|z| = \infty$:

$$\begin{array}{ll} 0\,, & \text{if } N_1 \geq 0\,, \\ \infty\,, & \text{if } N_0 \leq 0 \end{array} \qquad \text{e.g.: } X(z) = z + 1 + z^{-1}$$

ROC of an infinite sequence

Split the sequence $x[n]$ into the sum of a causal and an anti-causal term, and use the linearity of the z-transform.

- The causal part $X_c(z)$ has ROC containing $|z| = \infty$, therefore it is $|z| > R_1$, the largest radius of the poles.
- The anti-causal part $X_{ac}(z)$ has ROC containing $z = 0$, therefore it is $|z| < R_2$, the smallest radius of the poles.
- Hence, the ROC of $X(z)$ is the intersection: $R_1 < |z| < R_2$. All poles are outside the ROC.

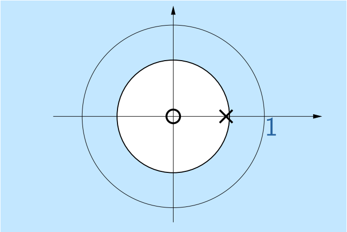


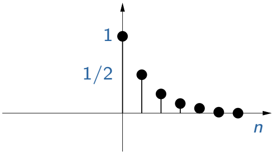
Example

- **Causal signal**: consider

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \quad \Leftrightarrow \quad X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

ROC: $|z| > \frac{1}{2}$





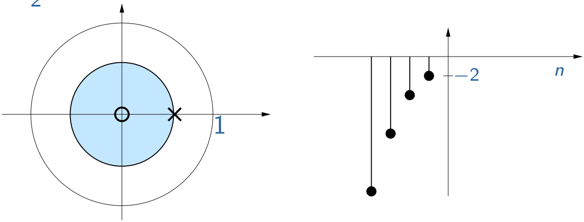
Example

- **Anti-causal signal**: consider

$$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

$$X_2(z) = -\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^n z^{-n} = -\sum_{m=0}^{\infty}(2z)^m + 1 = \frac{-1}{1-2z} + 1 = \frac{z}{z-\frac{1}{2}}$$

ROC: $|z| < \frac{1}{2}$



The same $X(z)$ corresponds to different $x[n]$ depending on the ROC.

Example

Compute the z-transform of the two-sided signal:

$$x[n] = \left(\frac{1}{2}\right)^{|n|}$$

- For the causal part ($n \geq 0$) we find:

$$x_c[n] = \left(\frac{1}{2}\right)^n u[n] \;\Leftrightarrow\; X_c(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}},$$

ROC: $|z| > \frac{1}{2}$

- For the anti-causal part ($n \leq 0$):

$$x_{ac}[n] = \left(\frac{1}{2}\right)^{-n} u[-n] \;\Leftrightarrow\; X_{ac}(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n = \frac{1}{1 - \frac{1}{2}z},$$

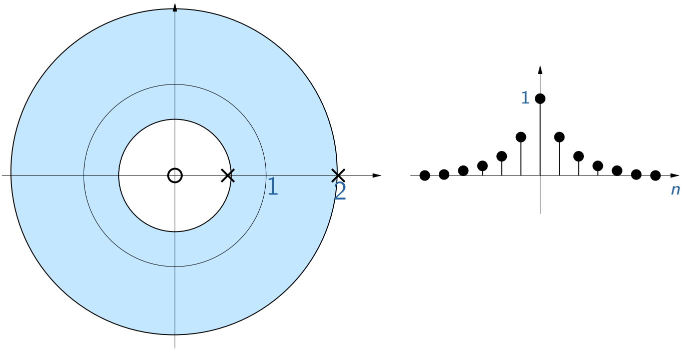
ROC: $|z| < 2$

Example (cont'd)

- For $x[n]$ we find

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{1}{1-\frac{1}{2}z} - 1 = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2} = \frac{-1\frac{1}{2}\,z}{(z-\frac{1}{2})(z-2)}$$

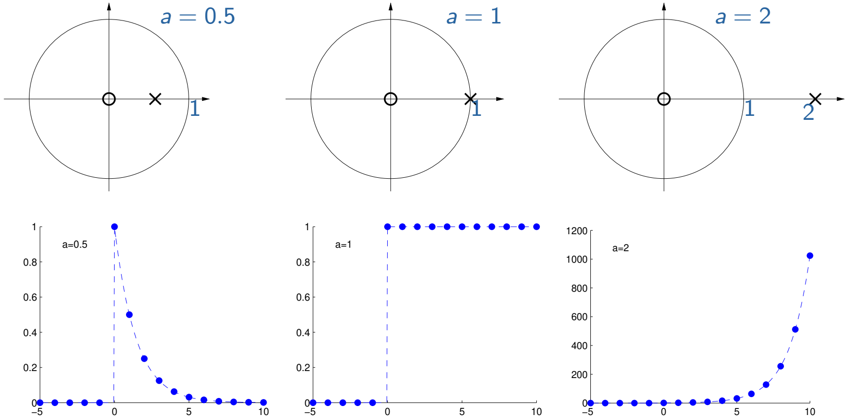
with ROC: $\frac{1}{2} < |z| < 2$.



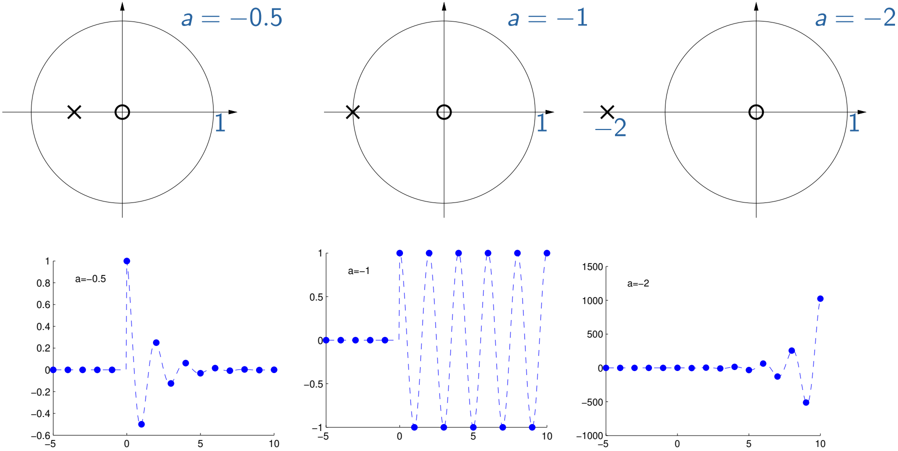
Exponential signals

$$x[n] = a^n u[n] \qquad \Leftrightarrow \qquad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}\,, \quad \text{ROC: } |z| > a$$

Pole at $z = a$, zero at $z = 0$.



Exponential signals



Harmonic (exponentially damped) signals

$$\begin{aligned} x[n] &= r^n \cos(\omega_0 n + \theta) u[n] = \left[\frac{e^{j\theta}}{2} r^n e^{j\omega_0 n} + \frac{e^{-j\theta}}{2} r^n e^{-j\omega_0 n} \right] u[n] \\ &= \left[\gamma \alpha^n + \gamma^* (\alpha^*)^n \right] u[n] \end{aligned}$$

with $\alpha = re^{j\omega_0}$ and $\gamma = \frac{e^{j\theta}}{2}$ both complex.

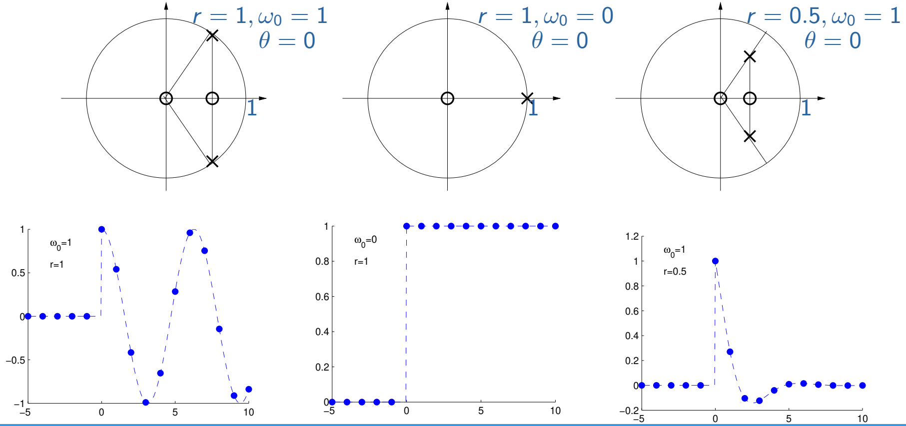
$$X(z) = \frac{\gamma z}{z - \alpha} + \frac{\gamma^* z}{z - \alpha^*} = \cdots = \frac{z(z\cos(\theta) - r\cos(\omega_0 - \theta))}{(z - re^{j\omega_0})(z - re^{-j\omega_0})}, \text{ ROC: } |z| > |\alpha|$$

This is a second-order rational function with real-valued coefficients.

- Poles at $z = re^{j\omega_0}$ and $z = re^{-j\omega_0}$.
  Special case: $r = 1$, now $x[n]$ is an undamped (causal) sinusoid, with its two poles on the unit circle.
- Zeros at $z = 0$ and $z = \dfrac{r\cos(\omega_0 - \theta)}{\cos(\theta)}$.

Harmonic signals

- $r = 1$, $\omega_0 = 1$, $\theta = 0$
- $r = 1$, $\omega_0 = 0$ (one pole and zero cancel each other)
- $r = 0.5$, $\omega_0 = 1$



Double poles

For a causal $x[n]$:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$\frac{\mathrm{d}X(z)}{\mathrm{d}z} = \sum_{n=0}^{\infty} x[n]\frac{\mathrm{d}z^{-n}}{\mathrm{d}z} = -z^{-1}\sum_{n=0}^{\infty} nx[n]\,z^{-n}$$

Hence

$$nx[n]u[n] \quad \Leftrightarrow \quad -z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$$

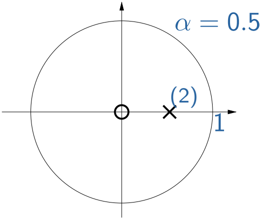
Taking a derivative often leads to double poles.

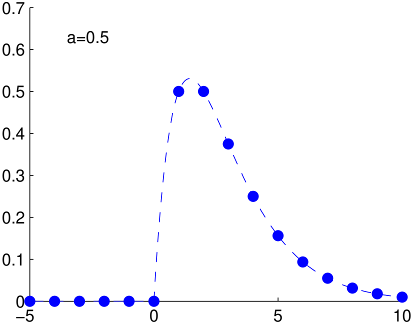
Example

Taking $x[n] = \alpha^n u[n]$ so that $X(z) = \dfrac{z}{z-\alpha}$, then

$$n\alpha^n u[n] \qquad \Leftrightarrow \qquad \frac{\alpha z}{(z-\alpha)^2}$$

Double pole at $z = \alpha$, zero at $z = 0$ and $z = \infty$.





The transfer function

Consider an LTI system $\mathcal{S}$ with impulse response $h[n]$. Earlier we found

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Define

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Then

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k]z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x[k]z^{-k}H(z) \;=\; H(z)X(z) \end{aligned}$$

Computing the convolution

Given $x[n] = [\boxed{1}, 2, 0, \cdots]$ and $h[n] = [\boxed{3}, 2, 4, 0, \cdots]$.

Compute $y[n] = x[n] * h[n] = \sum_{k=0}^{\infty} x[k]h[n-k]$:

$$
\begin{array}{lccccl}
x[0]h[n]: & 3 & 2 & 4 & 0 & 0\cdots \\
x[1]h[n-1]: & 0 & 2\cdot 3 & 2\cdot 2 & 2\cdot 4 & 0\cdots \\
\hline
y[n]: & 3 & 8 & 8 & 8 & 0\cdots
\end{array}
$$

Alternatively, compute using $Y(z) = X(z)H(z)$:

$$
\begin{aligned}
Y(z) &= (1+2z^{-1})(3+2z^{-1}+4z^{-2}) \\
&= (3+2z^{-1}+4z^{-2}) + 2z^{-1}(3+2z^{-1}+4z^{-2}) \\
&= 3 + (2+2\cdot 3)z^{-1} + (4+2\cdot 2)z^{-2} + (2\cdot 4)z^{-3} \\
&= 3 + 8z^{-1} + 8z^{-2} + 8z^{-3}
\end{aligned}
$$

Lineair difference equations

The equivalent of a differential equation in discrete time is a linear difference equation, e.g.

$$y[n]+a_1 y[n-1]+\cdots+a_N y[n-N] \;=\; b_0 x[n]+b_1 x[n-1] = \cdots+b_M x[n-M]$$

Take left and right the z-transform:

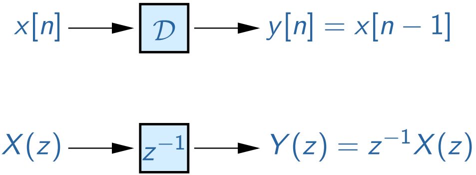
$$Y(z)\underbrace{\left(1+a_1 z^{-1}+\cdots+a_N z^{-N}\right)}_{A(z)} = X(z)\underbrace{\left(b_0+b_1 z^{-1}+\cdots+b_M z^{-M}\right)}_{B(z)}$$

Therefore,

$$H(z) := \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$H(z)$ is a rational transfer function.

Realizations



- The delay-element is a memory (clocked D-flip-flop): It shows at the output what was the input at the previous clock cycle.
- Block schemes ("realizations") consist of delays, multipliers and adders.

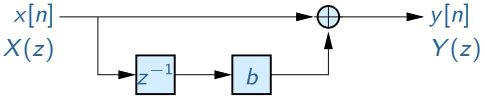
  In block schemes, $\mathcal{D}$ is usually written as z^{-1}. Therefore, $x[n]$ and $X(z)$ are often interchangeably used in block schemes.
- The impulse response $h[n]$ follows for $n = 1, 2, \cdots$ by inserting an input signal $x[n] = \delta[n]$ into the realization, and recursively computing the signals in the scheme sample by sample (assuming initial conditions of the delays are zero).

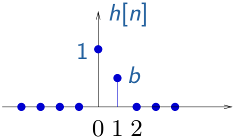
Realizations

A rational transfer function $H(z)$ corresponds to a realization using delays, multipliers and adders.

Examples:

- $H(z) = 1 + bz^{-1} \qquad \Rightarrow \qquad h[n] = \delta[n] + b\delta[n-1]$





Insert $x[n] = \delta[n]$ to find $h[n]$. Insert $X(z) = 1$ to find $H(z)$.

Realizations

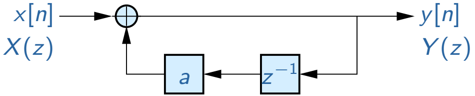
- $H(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$, ROC: $|z| > a$

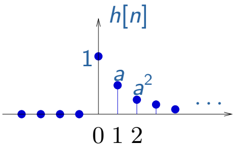
  $h[n] = a^n u[n]$

- Derivation of a realization:

  $Y(z) = H(z)X(z) \;\Rightarrow\; Y(z)(1-az^{-1}) = X(z) \;\Rightarrow$

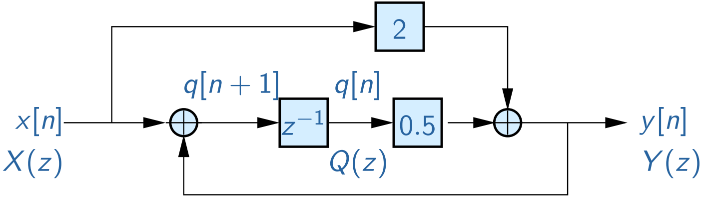
  $Y(z) = X(z) + az^{-1}Y(z)$





Exercise

Determine the transfer function of the following system:



Exercise

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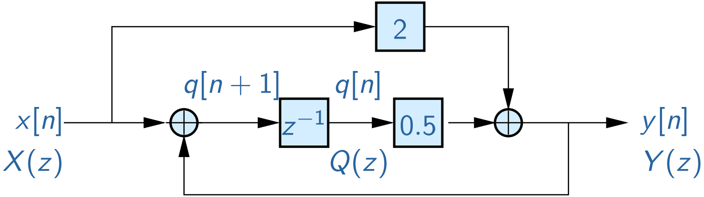


$$\begin{cases} Y(z) = 2X(z) + 0.5Q(z) \\ Q(z) = z^{-1}Y(z) + z^{-1}X(z) \end{cases}$$

$$Y(z) = 2X(z) + 0.5[z^{-1}Y(z) + z^{-1}X(z)]$$

$$(1 - 0.5z^{-1})Y(z) = (2 + 0.5z^{-1})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + 0.5z^{-1}}{1 - 0.5z^{-1}} = \frac{2z + 0.5}{z - 0.5}$$ ROC: $|z| > 0.5$

Causality

For a causal LTI system, we have $h[n] = 0, \; n < 0$.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \quad = \quad \sum_{n=0}^{\infty} h[n] z^{-n}$$

Hence, an LTI system is causal iff the ROC of $H(z)$ contains the outside of a circle, including $z = \infty$.

Stability

Earlier: A system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

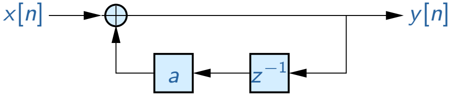
Note:

$$|H(z)| \;\leq\; \sum |h[n]z^{-n}| \;=\; \sum |h[n]|\,|z^{-n}|$$

On the unit circle, a BIBO stable system satisfies: $|H(z)| < \infty$: the unit circle is contained in the ROC.

- An FIR system is always BIBO stable (finite sum).
- A *causal and stable* LTI system has $H(z)$ with ROC containing the unit circle and its outside: $|z| \geq 1$.
  All poles are strictly inside the unit circle.

Example



- $a = 0.5 \quad \Rightarrow \quad H(z) = \dfrac{1}{1 - 0.5z^{-1}} = 1 + 0.5z^{-1} + 0.25z^{-2} + \cdots$
  ROC: $|z| > 0.5$, causal and stable
- $a = 2 \quad \Rightarrow \quad H(z) = \dfrac{1}{1 - 2z^{-1}} = 1 + 2z^{-1} + 4z^{-2} + \cdots$
  ROC: $|z| > 2$, causal but non-stable
- $H(z) = \dfrac{1}{1 - 2z^{-1}} = -\dfrac{0.5z}{1 - 0.5z} = -0.5z - 0.25z^2 - 0.125z^3 - \cdots$
  ROC: $|z| < 2$, non-causal but stable.
  This series (impulse response) does not correspond to the realization (which is causal by construction).

Conclusions

- A causal stable system has all poles within the unit circle. The ROC contains at least the unit circle and the area outside it.
- Along with $H(z)$, we also must indicate the ROC.
- Often the ROC is omitted. In that case, depending on the situation, assume either
  - the system is stable: the unit circle is within the ROC
  - the system is causal: ROC contains infinity.

Initial value and final value

If $x[n]$ is causal, then

Initial value: $x[0] = \lim_{z\to\infty} X(z)$

Final value: $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$ (if ROC $\supset \{|z| \geq 1\} \backslash \{1\}$).

Proof:

- $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n} = x[0]$
- $$\begin{aligned} \lim_{z\to 1}(z-1)X(z) &= \lim_{z\to 1} x[0]z + \sum_{n=0}^{\infty}(x[n+1]-x[n])z^{-n} \\ &= x[0] + \sum_{n=0}^{\infty}(x[n+1]-x[n]) \\ &= \lim_{n\to\infty} x[n] \end{aligned}$$

The properties can be used to check the correctness of a computed $x[n]$.

Examples

- $X(z) = 1 \quad \Rightarrow \quad x[n] = \delta[n]$.
  Initial value: $\lim_{z\to\infty} 1 = 1$. Final value: $\lim_{z\to 1}(z-1)\cdot 1 = 0$

- $X(z) = \dfrac{1}{1-z^{-1}} \quad \Rightarrow \quad x[n] = u[n]$.
  Initial value: $\lim_{z\to\infty} \dfrac{1}{1-z^{-1}} = 1$.
  Final value: $\lim_{z\to 1} \dfrac{z-1}{1-z^{-1}} = \lim_{z\to 1} z = 1$.

- $X(z) = \dfrac{z^{-1}}{(1-z^{-1})^2} \quad \Rightarrow \quad x[n] = n\,u[n]$.
  Initial value: $\lim_{z\to\infty} \dfrac{z^{-1}}{(1-z^{-1})^2} = 0$.
  Final value: $\lim_{z\to 1} \dfrac{(z-1)z^{-1}}{(1-z^{-1})^2} = \lim_{z\to 1} \dfrac{1}{1-z^{-1}} = \infty$.