# EE2S1 Signals and Systems

**(3rd ed) Ch. 8.1, 8.2**
**(4th ed) Ch. 6.1, 6.2**
} **Sampling theory**

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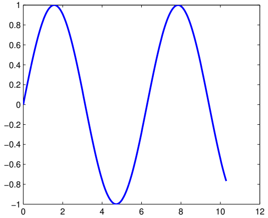
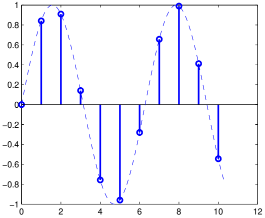
# Sampling

$x(t)$  $\Rightarrow$ $x[n]$ 

From a given signal $x(t)$ in continuous time domain, we take samples

$$x[n] = x(nT_s)\,, \qquad n = \cdots, -1, 0, 1, 2, \cdots$$

$T_s$ is called the sampling period. Sampling usually leads to loss of information.

- What does the sequence $\{x[n]\}$ tell about $x(t)$? Can we recover $x(t)$?
- What is a suitable definition for the “spectrum” of $x[n]$?

# Sampling

Given an analog signal $x(t)$. After uniform sampling with period $T_s$, we obtain

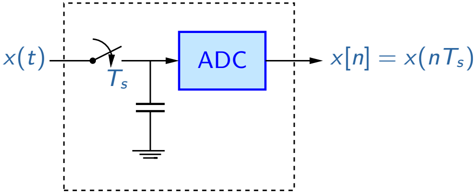
$$x[n] = x(nT_s)\,, \qquad n = \cdots, -1, 0, 1, 2, \cdots$$

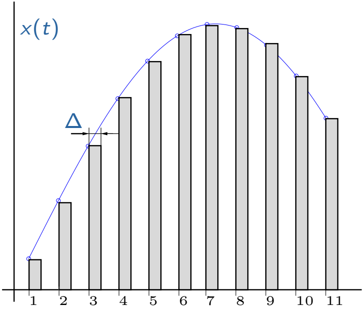
We define

| | | |
|---|---|---|
| $T_s$ | [s] | Sampling period |
| $F_s = 1/T_s$ | [Hz] | Sampling frequency |
| $\Omega_s = 2\pi F_s = 2\pi/T_s$ | [rad/s] | Sampling angular frequency |

# Sampling

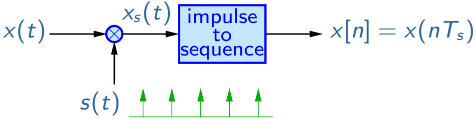
Sampling is done using a switch, a memory element that keeps the voltage for a while, and an analog-to-digital converter (ADC) which measures the amplitude and quantizes it into a number of bits.

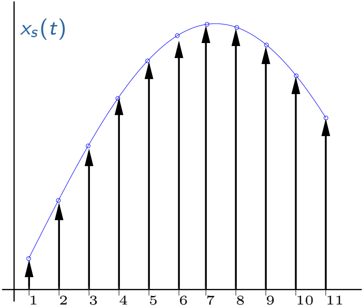


# Ideal sampling

If $\Delta \to 0$, we obtain "ideal sampling", a model in which the sampled signal is represented by a train of delta pulses.




$x_s(t)$ and $x[n]$ have a one-to-one relation (contain the same information). Therefore we first study $x_s(t)$.

# Sampling in time domain

We represent sampling as pointwise multiplication of $x(t)$ by a "delta impulse train" $s(t)$,

$$s(t) = \sum_n \delta(t - nT_s)$$

$s(t)$ is also called a "delta comb". The book writes $\delta_{T_s}(t)$.

We obtain

$$\begin{aligned} x_s(t) = x(t)\, s(t) \; = \; \sum_n x(t)\delta(t - nT_s) \; &= \; \sum_n x(nT_s)\delta(t - nT_s) \\ &= \; \sum_n x[n]\delta(t - nT_s) \end{aligned}$$

# Spectrum of $x_s(t)$

Use the FT of a shifted delta, and linearity,

$$x_s(t) = \sum_n x(nT_s)\delta(t - nT_s) \quad \Leftrightarrow \quad X_s(\Omega) = \sum_n x(nT_s)e^{-j\Omega T_s n}$$

This motivates the definition of the Discrete-Time Fourier Transform which we will study later (Ch.11). Let $\omega = \Omega T_s$, then

$$\text{DTFT:} \qquad X_d(\omega) := \sum_n x[n]e^{-j\omega n}$$

This is the spectrum of a discrete sequence $x[n]$. It matches $X_s(\Omega)$.

But how is $X_s(\Omega)$ related to the spectrum of $x(t)$?

# Impulse train as sum of sinusoids

The impulse train $s(t)$ is periodic with $\Omega_s = 2\pi/T_s$. Write the Fourier Series:

$$s(t) = \sum_{k=-\infty}^{\infty} D_k e^{jk\Omega_s t}$$

The Fourier Series coefficients are given by:

$$\begin{aligned} D_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} s(t)e^{-jk\Omega_s t}\mathrm{d}t &= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-jk\Omega_s t}\mathrm{d}t \\ &= \frac{1}{T_s}e^{-j\cdot 0}\cdot 1 = \frac{1}{T_s} \end{aligned}$$

Hence

$$s(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\Omega_s t}$$

The impulse train is now written as an infinite sum of sinusoids.

# Effect of sampling in the frequency domain

Compute the Fourier Transform and apply the “frequency shift” (modulation) property:

$$x_s(t) = x(t)s(t) = \frac{1}{T_s}\sum_k x(t)e^{jk\Omega_s t} \quad \Leftrightarrow \quad X_s(\Omega) = \frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)$$

Thus, $X_s(\Omega)$ is a sum of shifted spectra of $X(\Omega)$. The shifts are multiples of $\Omega_s$.

This results in a spectrum that is periodic with period $\Omega_s$.

# Alternative derivation

Use table 5.1 of Chapparo:

$$s(t) = \sum_k \delta(t - kT_s) = \frac{1}{T_s} \sum_k e^{jk\Omega_s t} \quad \Leftrightarrow \quad S(\Omega) = \frac{2\pi}{T_s} \sum_k \delta(\Omega - k\Omega_s)$$

$s(t)$ is an impulse train in time domain, and corresponds to $S(\Omega)$: an impulse train in frequency domain.

$$x_s(t) = x(t)s(t) \quad \Leftrightarrow \quad X_s(\Omega) = \frac{1}{2\pi} X(\Omega) * S(\Omega) = \frac{1}{T_s} \sum_k X(\Omega - k\Omega_s)$$

Modulation of $x(t)$ with the impulse train $s(t)$ corresponds to a convolution of the spectrum $X(\Omega)$ with an impulse train $S(\Omega)$.

(The result is the same as before.)

# Interpretation

$$X_s(\Omega) = \frac{1}{T_s} \sum_k X(\Omega - k\Omega_s)$$

The spectrum of the sampled signal, $X_s(\Omega)$, is a sum of shifted spectra of the original signal, $X(\Omega)$. Shifts are multiples of $\Omega_s$.

- $X_s(\Omega)$ is periodic, with period $\Omega_s$.
  It is sufficient to know only one period of $X_s(\Omega)$, e.g. the interval $-\frac{1}{2}\Omega_s < \Omega < \frac{1}{2}\Omega_s$.
  This interval is also called the *fundamental interval*. $\frac{1}{2}\Omega_s$ is the *folding frequency*.
- Sampling is not an LTI operator (because it is time varying), hence the sampled signal can have frequencies which the original signal did not have – this can never happen with an LTI operator.
- The summation of shifted spectra can lead to *aliasing*.

# Exercise

Given $x(t) = \dfrac{\sin(\Omega_0 t)}{t}$.

a) What is $X(\Omega)$? Make a drawing of $|X(\Omega)|$.
b) We sample with period $T_s = \pi/\Omega_0$. Determine $X_s(\Omega)$ and make a drawing.

# Exercise

Given $x(t) = \dfrac{\sin(\Omega_0 t)}{t}$.

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b) We sample with period $T_s = \pi/\Omega_0$. Determine $X_s(\Omega)$ and make a drawing.

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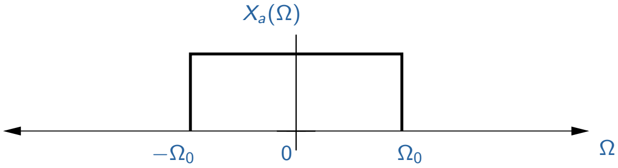
a) Last lecture, we derived that

$$X(\Omega) = \pi\left[u(\Omega + \Omega_0) - u(\Omega - \Omega_0)\right]$$

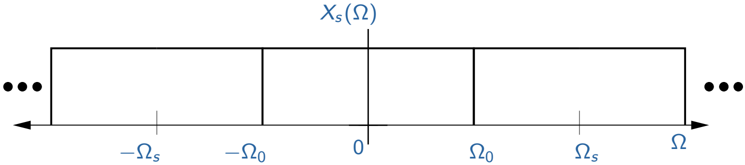



The maximal frequency is $\Omega_0$.

# Exercise (cont'd)

b) $T_s = \frac{\pi}{\Omega_0} \Rightarrow \Omega_s = \frac{2\pi}{T_s} = 2\Omega_0$, this is twice the highest frequency.

$$X_s(\Omega) = \frac{1}{T_s} \sum_k X_a(\Omega - k\Omega_s)$$




The spectrum is flat, this would match with $x_s(t) = A\delta(t)$, for a certain amplitude $A$. Check:

$$x(nT_s) = \frac{\sin(\Omega_0 \cdot n\frac{\pi}{\Omega_0})}{n\frac{\pi}{\Omega_0}} = \Omega_0 \frac{\sin(n\pi)}{n\pi} = \begin{cases} \Omega_0 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

# Aliasing

Example signal: $x(t) = e^{j\Omega_0 t}$, where $\Omega_0$ is an angular frequency [rad/s].

- After sampling with period $T_s$:

$$x[n] = e^{j\Omega_0 T_s n} = e^{j\omega_0 n}$$

- $\omega_0 = \Omega_0 T_s$ is the "normalized" angular frequency [rad].
- We can distinguish only $-\pi < \omega_0 < \pi$. Higher frequencies result in the same $x[n]$.

$$-\frac{\pi}{T_s} < \Omega_0 < \frac{\pi}{T_s}$$

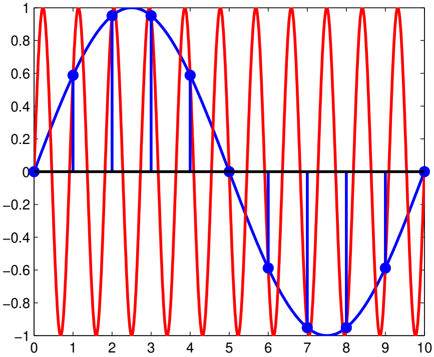
$$-\tfrac{1}{2}\Omega_s < \Omega_0 < \tfrac{1}{2}\Omega_s$$

  This is the fundamental interval.

- Higher frequencies are also mapped onto this interval: aliasing.

# Aliasing

We cannot distinguish between a sinusoid with frequency $\Omega_0$ and one with $\Omega_0 + k\Omega_s$, for $k = \cdots, -2, -1, 0, 1, 2, \cdots$.




The red signal has $\Omega_0 > \frac{1}{2}\Omega_s$, after sampling the signal is identical to that of the blue signal for which $\Omega_0 < \frac{1}{2}\Omega_s$.

# Aliasing

In EE, we often work with frequencies $F_0$ in Hz:

$$x(t) = e^{j2\pi F_0 t}$$

- After sampling with period $T_s$:

$$x[n] = e^{j2\pi F_0 T_s n} = e^{j2\pi f_0 n}$$

  $F_s = \frac{1}{T_s}$ is the sampling frequency [Hz].

- $f_0 = F_0 T_s = \frac{F_0}{F_s}$ is the "normalized" frequency (no dimension).
- We can distinguish only $-\frac{1}{2} < f_0 < \frac{1}{2}$. The fundamental interval is

$$-\tfrac{1}{2} F_s < F_0 < \tfrac{1}{2} F_s$$

# Aliasing

We call a signal $x(t)$ *band limited* if

$$X(\Omega) = 0\,, \qquad |\Omega| > \Omega_{\text{max}}$$

Regarding the sampling of $x(t)$, we have 3 cases:

- $\Omega_s > 2\Omega_{\text{max}}$: this is the *Nyquist sample rate condition*.
  The shifted spectra $X(\Omega - k\Omega_s)$ do not overlap. The fundamental interval only contains a single copy:

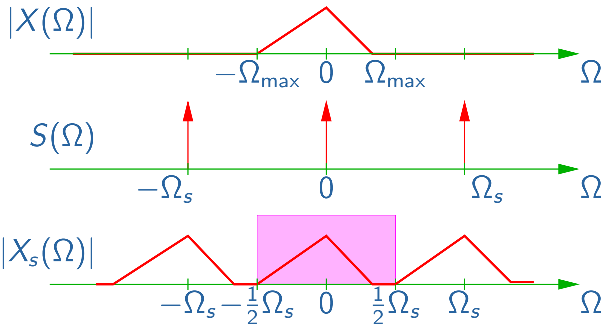
  $$X_s(\Omega) = X(\Omega), \qquad \text{for } |\Omega| < \tfrac{1}{2}\Omega_s$$

  The aliasing is not destructive.
- The Nyquist condition does not hold. The aliasing is destructive.
- The signal is not band-limited. The Nyquist condition does not hold. Hence the aliasing is destructive.

# Aliasing

## Case 1

- $\Omega_s > 2\Omega_{\text{max}}$ (no destructive aliasing)

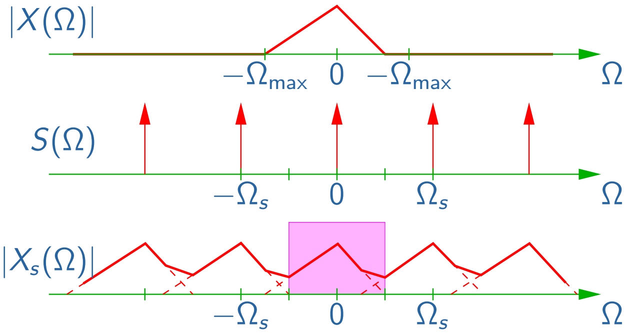



The pink box indicates the fundamental interval: $[-\frac{1}{2}\Omega_s, \frac{1}{2}\Omega_s]$. Outside this interval, everything is periodic with period $\Omega_s$.

# Aliasing

## Case 2

- $\Omega_s < 2\Omega_{max}$ (destructive aliasing)




The part of the spectrum beyond $\pm\frac{1}{2}\Omega_s$ is apparently "folded back".

# Aliasing

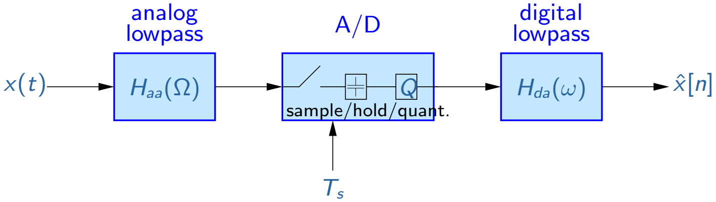
## Case 3

- $x(t)$ not band limited. E.g.,

  $$x(t) = u(t+0.5) - u(t-0.5) \quad \Leftrightarrow \quad X(\Omega) = \frac{\sin(0.5\Omega)}{0.5\Omega}.$$

To prevent aliasing, practical Analog to Digital Converters (ADCs) employ an anti-aliasing filter which cuts off all frequencies above $\frac{1}{2}\Omega_s$.

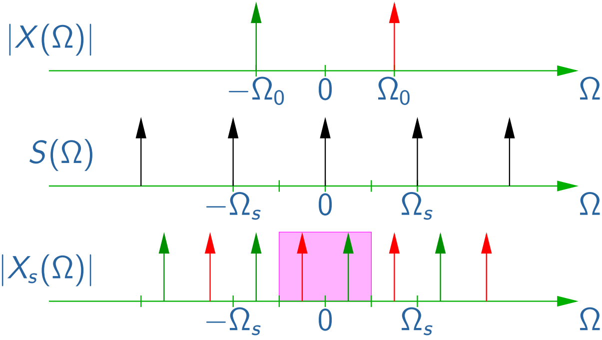
Often, the analog filter is not perfect, and after sampling another (digital) filter is used to correct for this.



# Example

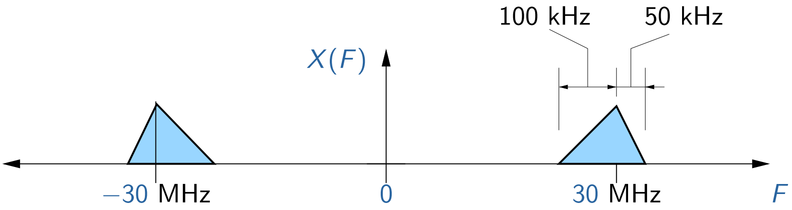
$x(t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$ is a sinusoid with $\Omega_0 > \frac{1}{2}\Omega_s$.

$X(\Omega)$ has two components (red/green) that are each shifted with multiples of $\Omega_s$.

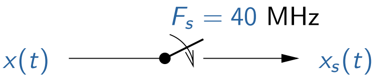


# Exercise

Given an analog real-valued signal $x(t)$ with frequency spectrum:




This signal is sampled with a sample frequency of 40 MHz:




a) Is the Nyquist criterion satisfied?

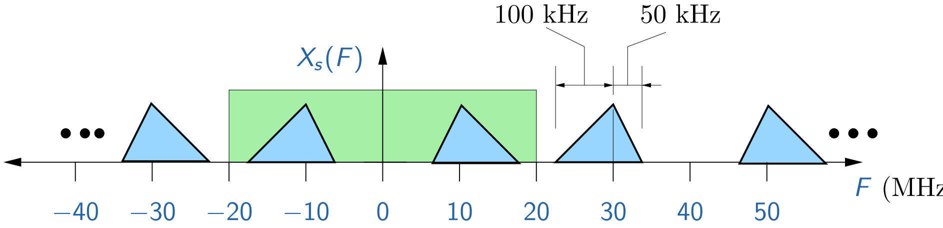
b) Make a drawing of $X_s(F)$.

# Exercise

## Answer

a) No: the highest frequency in the signal is 30.05 MHz, the sample frequency should be twice as large (60.1 MHz), which is not the case here.

b) Due to the sampling at 40 MHz, the original spectrum is repeated with period 40 MHz. Therefore, the part at 30 MHz is seen at 70, 110 MHz etc and also at −10, −50, ⋯ MHz. The part of the spectrum at −30 MHz (not shown in the graph but the signal was assumed real-valued) is repeated and is seen at 10, 50, ⋯ MHz.



# Reconstruction

Suppose that $x(t)$ is band limited ($\Omega_{\max}$), and $\Omega_s > 2\Omega_{\max}$ (Nyquist condition holds: no destructive aliasing).

Can we recover $x(t)$ from its samples $x[n] = x(nT_s)$?

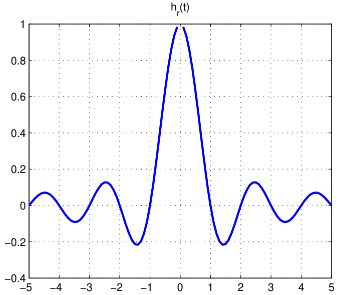
**Shannon's sampling theorem:** A band limited signal can be reconstructed from its samples if the Nyquist condition holds.

# Reconstruction (frequency domain)

- Define an ideal lowpass filter:

$$H_r(\Omega) = \begin{cases} T_s, & |\Omega| \le \Omega_s/2 \\ 0, & |\Omega| > \Omega_s/2 \end{cases}$$

$$\begin{aligned} h_r(t) &= \frac{T_s}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} e^{j\Omega t} \mathrm{d}\Omega \\ &= \frac{\sin(\pi t/T_s)}{\pi t/T_s} =: \mathrm{sinc}(t/T_s) \end{aligned}$$



# Reconstruction (frequency domain)

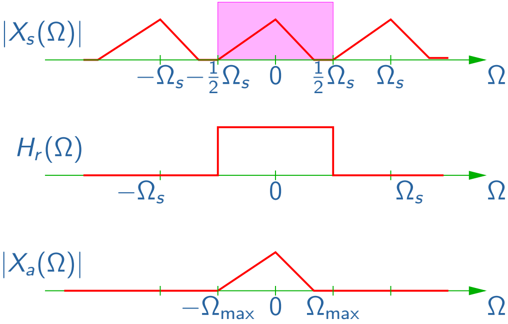
- Apply $H_r(\Omega)$ to $X_s(\Omega)$:

$$X_a(\Omega) = X_s(\Omega)H_r(\Omega) \quad \Leftrightarrow \quad x_a(t) = x_s(t) * h_r(t) = \sum_n x[n]h_r(t - nT_s)$$

- Recall that $X_s(\Omega) = \frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)$
  The lowpass filter selects the term $k = 0$. The result is
  $X_a(\Omega) = X(\Omega)$ and hence $x_a(t) = x(t)$: perfect reconstruction.

This is called *ideal band limited signal reconstruction*, and $h_r(t)$ is called the *reconstruction filter* or *interpolation filter*.

# Reconstruction (frequency domain)




The reconstruction filter eliminates the extra copies; the original signal is recovered.

Note that this operation can only be done using filtering in the analog time domain. A sequence of samples $\{x[n]\}$ must first be made analog (a series of impulse spikes).

# Reconstruction (interpretation in time domain)

- In time domain, we have $x_a(t) = x_s(t) * h_r(t) = \sum_n x[n] h_r(t - nT_s)$.

  $x_a(t)$ is a sum of weighted and shifted impulse responses $h_r(t)$:

$$x_a(t) = \cdots + x[0]h_r(t) + x[1]h_r(t - T_s) + x[2]h_r(t - 2T_s) + \cdots$$

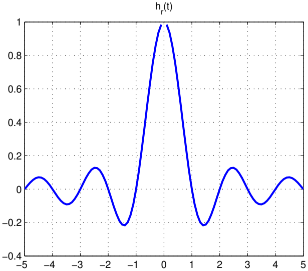
- Consider the zero crossings of $h_r(t) = \dfrac{\sin(\pi t / T_s)}{\pi t / T_s} = \mathrm{sinc}(t/T_s)$:

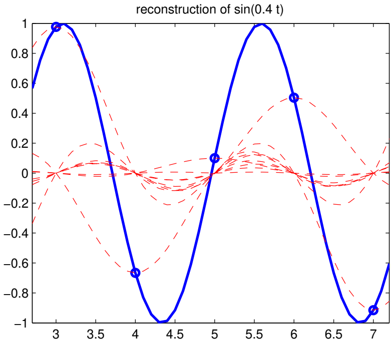
$$h_r(kT_s) = \mathrm{sinc}(k) = \begin{cases} 1\,, & k = 0 \quad \text{(use L'Hopital)} \\ 0\,, & k = \pm 1, \pm 2, \cdots \end{cases}$$

  Using $x_a(t) = \sum_n x[n] h_r(t - nT_s)$, it follows that

$$x_a(kT_s) = x[k] \;: \qquad \text{interpolation of the given samples}$$

# Reconstruction (interpretation in time domain)




The sinc-functions take care of the interpolation. Note the locations of the zero crossings of the shifted sinc-functions.

# Reconstruction (interpretation in time domain)

Digital-to-Analog Conversion (DAC) using ideal low-pass filters cannot be implemented:

- $h_r(t)$ is not causal,
- $h_r(t)$ has an impulse response of infinite duration,
- We cannot convert samples $x[n]$ into analog impulses.

In practice, approximations are used (low-pass filters). We will return to this issue in EE3S1 Signal Processing.

# Exercise (cf. Problem 8.5)

Let $x(t)$ be a bandlimited signal with maximal frequency $\Omega_{\max}$. We sample $x(t)$ with period $T_s = 1$. The Nyquist condition is satisfied.

a) What is the largest value for $\Omega_{\max}$?

b) Specify the passband/stopband conditions on the reconstruction filter. (Make plots.)

# Exercise (cf. Problem 8.5)

Let $x(t)$ be a bandlimited signal with maximal frequency $\Omega_{\max}$. We sample $x(t)$ with period $T_s = 1$. The Nyquist condition is satisfied.

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a) $\Omega_s = 2\pi/T_s = 2\pi$, and we need $\Omega_{\max} < \frac{1}{2}\Omega_s$, so $\Omega_{\max} < \pi$.

b) The lowpass filter $H_r(\Omega)$ should have its passband from $0$ until $\Omega_{\max}$, and its stopband from $\Omega_s - \Omega_{\max}$ until infinity.