EE2S1 Signals and Systems
Chapter 5: The Fourier Transform

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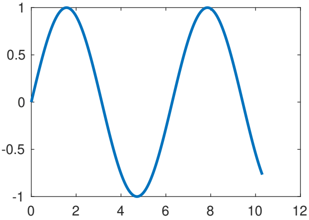
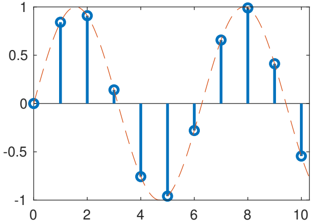
EE2S1 Signals and Systems, part 2





Continuous-time signals

- Laplace transform
- Fourier Series
- Fourier Transform

⇒ **Sampling and reconstruction**

⇒ **Discrete-time signals**

- z-Transform
- Discrete-Time Fourier Transform
- Realizations
- Analog and digital filter design

Course labs

The lectures are alternated with 3 course labs, Wednesday in week 6, 7, 8. Topics:

- Convolution
- Frequency domain plots (FFT)
- Filter design

The course labs are in the form of Python Colab scripts.

- Enroll yourself in a group in Brightspace, before the start of the first course lab.
- Groups of 2; you select your teammate
- Weekly deadlines (1 week after the lab, but you can submit earlier...)

Passing the course labs is a mandatory prerequisite for EE2L1 IP3.

Chapter 5: The Fourier Transform

- Given $x(t)$, consider its Laplace transform, $X(s)$.

$$X(s) = \int x(t)e^{-st}\mathrm{d}t \qquad \Leftrightarrow \qquad x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\mathrm{d}s$$

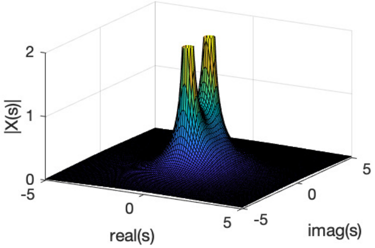
with $\sigma + j\Omega \in \mathrm{ROC}$

- How would you plot $X(s)$?

$$x(t) = \sin(t)u(t)$$

$$X(s) = \frac{1}{1+s^2}$$

$(\mathrm{ROC: Re}(s) > 0)$

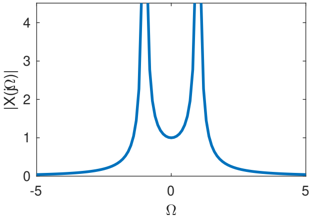


The Fourier Transform

We will define the Fourier Transform as $X(j\Omega)$, that is $X(s)$ with $s = j\Omega$.

- Since Ω is real, we can plot $|X(j\Omega)|$ (magnitude response) and $\arg(X(j\Omega))$ (phase response). Much more clear than a plot of $X(s)$!

$$
\begin{aligned}
x(t) &= \sin(t)u(t) \\
X(j\Omega) &= \frac{1}{1-\Omega^2}
\end{aligned}
$$



[Actually, this result is wrong... why?]

The Fourier Transform

- We can still recover $x(t)$ from $X(j\Omega)$ using the Inverse Laplace Transform (with $\sigma = 0$): no loss of information!

$$X(j\Omega) = \int x(t)e^{-j\Omega t}\mathrm{d}t \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\mathrm{d}j\Omega$$

From Laplace to Fourier

Laplace Transform $X(s) = \int x(t)e^{-st}\mathrm{d}t\,,$ with $s \in \mathrm{ROC}$

Fourier Transform $X(\Omega) = \int x(t)e^{-j\Omega t}\mathrm{d}t$

(Note change in notation, we should have written $X(j\Omega)$.)

- This assumes the $j\Omega$ axis is in the ROC of $X(s)$. But usually, we don't talk about the ROC anymore!
- Many properties of the FT follow from those of the LT.
- This integral can easily be evaluated numerically.
- Ω is in rad/s. In EE we also often use $F = \frac{\Omega}{2\pi}$, in Hz.

From Laplace to Fourier

The FT exists at least if $x(t) \in L_1$, i.e. is absolutely integrable: $\int |x(t)| \mathrm{d}t < \infty$.

Proof If $x(t) \in L_1$, then

$$|X(\Omega)| = |\int x(t) e^{-j\Omega t} \mathrm{d}t| \leq \int |x(t) e^{-j\Omega t}| \mathrm{d}t = \int |x(t)| \mathrm{d}t < \infty$$

so that the Fourier integral converges.

- Signals in L_1 taper off to zero as $t \to \pm\infty$. We will want to consider more general signals, e.g., $x(t) = 1$. This gives rise to distributions in frequency domain, e.g. $\delta(\Omega)$.

Example

Does the Fourier transform of the following signals exist?

- $x(t) = u(t)$
- $x(t) = e^{-2t}u(t)$
- $x(t) = e^{-|t|}$

Example

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- $x(t) = u(t)$
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- $x(t) = e^{-|t|}$

Answer: The Fourier transform exists if the ROC of the Laplace transform $X(s)$ contains the $j\Omega$-axis.

- No: $X(s) = \frac{1}{s}$, ROC $\{\mathrm{Re}(s) > 0\}$.
- Yes: $X(s) = \frac{1}{s+2}$, ROC $\{\mathrm{Re}(s) > -2\}$, so $X(\Omega) = \frac{1}{2+j\Omega}$.
- Yes: $X(s) = \frac{2}{1-s^2}$, ROC $\{-1 < \mathrm{Re}(s) < 1\}$, so $X(\Omega) = \frac{2}{1+\Omega^2}$.

Inverse Fourier transform

The Fourier transform is

$$X(\Omega) = \int x(t)\, e^{-j\Omega t}\, \mathrm{d}t$$

The corresponding inverse Fourier transform is

$$x(t) = \frac{1}{2\pi} \int X(\Omega)\, e^{j\Omega t}\, \mathrm{d}\Omega$$

Proof

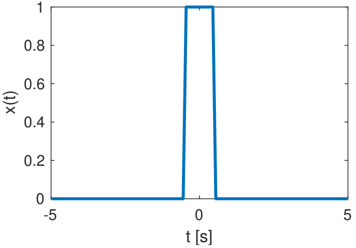
$$\begin{aligned} \frac{1}{2\pi} \int X(\Omega)\, e^{j\Omega t} \mathrm{d}\Omega &= \frac{1}{2\pi} \int \left[\int x(\tau)\, e^{-j\Omega\tau} \mathrm{d}\tau \right] e^{j\Omega t} \mathrm{d}\Omega \\ &= \frac{1}{2\pi} \int x(\tau) \underbrace{\left[\int e^{j\Omega(t-\tau)} \mathrm{d}\Omega \right]}_{2\pi\delta(t-\tau)} \mathrm{d}\tau = x(t) \end{aligned}$$

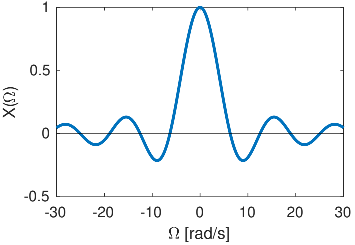
(This dirac property was shown in Lecture 1: *completeness relation*)

Example

Consider a pulse, $x(t) = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$, then

$$X(\Omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\Omega t} \mathrm{d}t = \frac{1}{j\Omega}\left[e^{j\Omega/2} - e^{-j\Omega/2}\right] = \frac{\sin(\Omega/2)}{\Omega/2} \quad =: \mathrm{sinc}(\Omega/2)$$





- In this case, $X(\Omega)$ happens to be real, but generally it is complex
- Careful: several definitions of the sinc function exist

Spectra with delta spikes

The Inverse Fourier Transform shows:

$$X(\Omega) = 2\pi\,\delta(\Omega) \qquad \Rightarrow \qquad x(t) = \frac{1}{2\pi}\int 2\pi\,\delta(\Omega)e^{j\Omega t}\mathrm{d}\Omega = 1$$

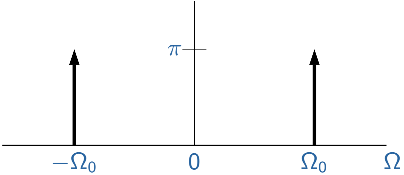
and more generally

$$X(\Omega) = 2\pi\,\delta(\Omega - \Omega_0) \qquad \Rightarrow \qquad x(t) = e^{j\Omega_0 t}$$

- These signals $x(t)$ are not in L_1, and do not have finite energy. Still, we can define their Fourier transform using dirac distributions.

Example

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} \qquad \Rightarrow \qquad \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$



Link to Fourier Series

- If $x(t)$ is periodic with period T_0, then we can express it as

$$x(t) = \sum X_k e^{jk\Omega_0 t}\,, \qquad \Omega_0 = \frac{2\pi}{T_0}$$

  where the X_k are the Fourier series coefficients.

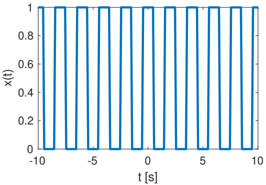
- The Fourier transform of $x(t)$ is $X(\Omega)$:

$$X(\Omega) = \sum X_k \mathcal{F}\{e^{jk\Omega_0 t}\} = \sum X_k 2\pi\, \delta(\Omega - k\Omega_0)$$

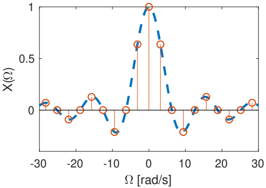
  Thus, $X(\Omega)$ has a *line spectrum*. The harmonic frequencies are $\Omega_k = k\Omega_0$.

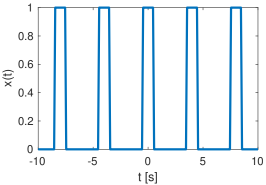
- The Fourier transform is also obtained as a limit of the Fourier series, for $T_0 \to \infty$.

Link to Fourier Series

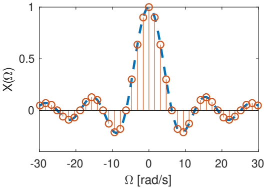


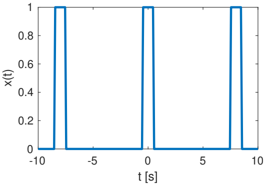
$T_0 = 2$



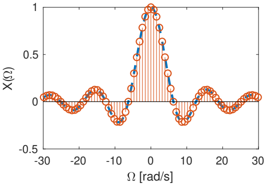


$T_0 = 4$





$T_0 = 8$



Convolution

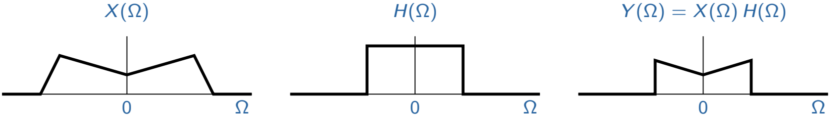
Directly from the Laplace Transform, we know

$$y(t) = x(t) * h(t) \qquad \Leftrightarrow \qquad Y(\Omega) = X(\Omega)\, H(\Omega)$$

This defines the concept of filtering in frequency domain.

(The book writes $H(j\Omega)$, perhaps to maintain the link to the Laplace transform?)

Example: lowpass filter



Duality

We have seen:

$$x(t) = \delta(t) \quad \Leftrightarrow \quad X(\Omega) = 1$$

$$x(t) = 1 \quad \Leftrightarrow \quad X(\Omega) = 2\pi\,\delta(\Omega)$$

This generalizes:

$$x(t) \quad \Leftrightarrow \quad X(\Omega)$$

$$X(t) \quad \Leftrightarrow \quad 2\pi\,x(-\Omega)$$

Duality

Proof Follows from the definition of the FT, with two changes of variables: $\Omega \to \tau$, and $t \to -\Omega$:

$$\begin{aligned} X(\Omega) &= \int x(t)e^{-j\Omega t}\mathrm{d}t \\ X(\tau) &= \int x(t)e^{-j\tau t}\mathrm{d}t \\ X(\tau) &= \int x(-\Omega)e^{j\tau\Omega}\mathrm{d}\Omega = \frac{1}{2\pi}\int 2\pi x(-\Omega)e^{j\Omega\tau}\mathrm{d}\Omega \end{aligned}$$

showing that the inverse FT of $2\pi x(-\Omega)$ is $X(t)$.

Scaling

$$x(at) \quad \Leftrightarrow \quad \frac{1}{|a|} X\left(\frac{\Omega}{a}\right)$$

Proof For $a > 0$, use the definition:

$$\int x(at)e^{-j\Omega t}\mathrm{d}t = \frac{1}{a}\int x(at)e^{-j\frac{\Omega}{a}(at)}\mathrm{d}(at) = \frac{1}{a}X\left(\frac{\Omega}{a}\right)$$

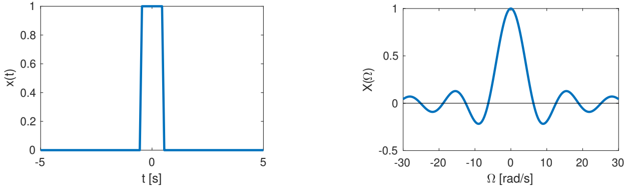
For $a < 0$,

$$\int_{-\infty}^{\infty} x(at)e^{-j\Omega t}\mathrm{d}t = \frac{1}{a}\int_{\infty}^{-\infty} x(at)e^{-j\frac{\Omega}{a}(at)}\mathrm{d}(at) = \frac{1}{-a}X\left(\frac{\Omega}{a}\right)$$

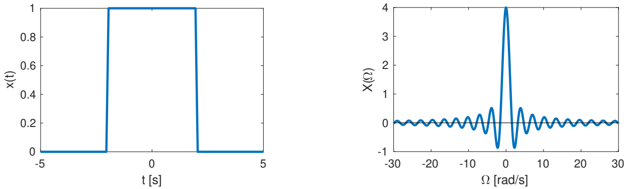
and the result follows.

Scaling

Interpretation For $a < 1$, we stretch $x(t)$, and then $X(\Omega)$ is shrunk correspondingly.



With $a = 1/4$:



Example (problem 5.2)

Find the Fourier transform of $\dfrac{\sin(t)}{t}$.

Hint: recall the FT pair

$$x(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2}) \qquad \Leftrightarrow \qquad X(\Omega) = \frac{\sin(\frac{1}{2}\Omega)}{\frac{1}{2}\Omega}$$

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$$x(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2}) \qquad \Leftrightarrow \qquad X(\Omega) = \frac{\sin(\frac{1}{2}\Omega)}{\frac{1}{2}\Omega}$$

Using duality,

$$\frac{\sin(\frac{1}{2}t)}{\frac{1}{2}t} \qquad \Leftrightarrow \qquad 2\pi\left[u(\Omega+\tfrac{1}{2}) - u(\Omega-\tfrac{1}{2})\right]$$

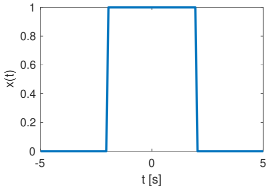
Using the scaling property ($a = 2$):

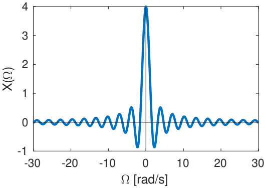
$$\frac{\sin(t)}{t} \qquad \Leftrightarrow \qquad \begin{aligned} &\frac{2\pi}{2}\left[u(\tfrac{1}{2}\Omega+\tfrac{1}{2}) - u(\tfrac{1}{2}\Omega-\tfrac{1}{2})\right] \\ &= \pi\left[u(\Omega+1) - u(\Omega-1)\right] \end{aligned}$$

Modulation

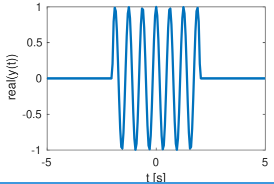
$$x(t)\, e^{j\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega - \Omega_0)$$

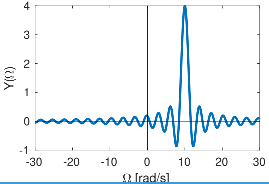
Example





With $y(t) = x(t) \cdot e^{j\Omega_0 t}$, where $\Omega_0 = 10$ [note $y(t)$ is complex]:





Multiplication in time domain [not in book?]

$$x(t)\,y(t) \quad \Leftrightarrow \quad \frac{1}{2\pi} X(\Omega) * Y(\Omega)$$

Proof Apply the inverse Fourier transform to

$$Z(\Omega) = \frac{1}{2\pi} X(\Omega) * Y(\Omega) = \frac{1}{2\pi} \int X(\Omega')\, Y(\Omega - \Omega')\, \mathrm{d}\Omega'$$

then

$$\begin{aligned}
&\frac{1}{2\pi} \int \left[\frac{1}{2\pi} \int X(\Omega') Y(\Omega - \Omega') \mathrm{d}\Omega' \right] e^{j\Omega t} \mathrm{d}\Omega \\
&= \frac{1}{2\pi} \int X(\Omega') e^{j\Omega' t} \left[\frac{1}{2\pi} \int Y(\Omega - \Omega') e^{j(\Omega - \Omega')t} \mathrm{d}\Omega \right] \mathrm{d}\Omega' \\
&= \frac{1}{2\pi} \int X(\Omega') e^{j\Omega' t} \mathrm{d}\Omega' \left[\frac{1}{2\pi} \int Y(\Omega'') e^{j\Omega'' t} \mathrm{d}\Omega'' \right] \\
&= x(t)\, y(t)
\end{aligned}$$

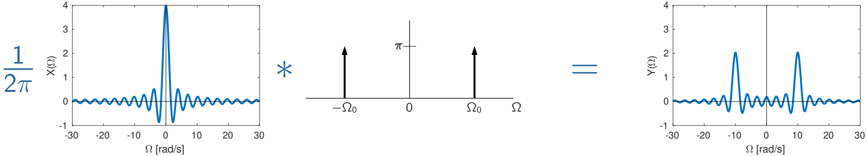
Example

$$x(t)\cos(\Omega_0 t) \Leftrightarrow \frac{1}{2\pi}X(\Omega)*\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)] = \frac{1}{2}[X(\Omega-\Omega_0)+X(\Omega+\Omega$$

- This is consistent with the earlier result [modulation]:

$$x(t)\, e^{j\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega-\Omega_0)$$

$$x(t)\,\frac{e^{j\Omega_0 t}+e^{-j\Omega_0 t}}{2} \quad \Leftrightarrow \quad \frac{1}{2}[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)]$$



Exercise (problem 5.6)

Consider the signal $x(t) = \cos(t), 0 \leq t \leq 1$, and 0 otherwise.

Find $X(\Omega)$.

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Consider the signal $x(t) = \cos(t), 0 \leq t \leq 1$, and 0 otherwise.

Find $X(\Omega)$.

$$x(t) = \cos(t)\,[u(t) - u(t-1)] = \cos(t)\,p(t)$$

so

$$X(\Omega) = \frac{1}{2}\,[P(\Omega+1) + P(\Omega-1)]$$

with

$$P(\Omega) = e^{-s/2} \cdot \frac{e^{s/2} - e^{-s/2}}{s}\Big|_{s=j\Omega} = e^{-j\Omega/2}\frac{\sin(\Omega/2)}{\Omega/2}$$

Energy (Parseval)

$$E_x = \int |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int |X(\Omega)|^2 \mathrm{d}\Omega$$

where E_x is the energy of the signal: the Fourier transform preserves the energy.

Proof Write $|x(t)|^2 = x(t)x^*(t)$, and use the Inverse FT

$$\begin{aligned}
\int |x(t)|^2 \mathrm{d}t &= \frac{1}{2\pi} \int \int x^*(t) X(\Omega) e^{j\Omega t} \mathrm{d}\Omega \mathrm{d}t \\
&= \frac{1}{2\pi} \int X(\Omega) \left[\int x(t) e^{-j\Omega t} \mathrm{d}t \right]^* \mathrm{d}\Omega \\
&= \frac{1}{2\pi} \int X(\Omega) [X(\Omega)]^* \mathrm{d}\Omega
\end{aligned}$$

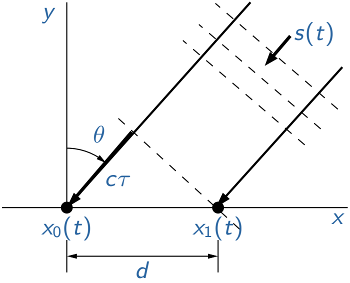
- If $x(t)$ is in L_2, then $X(\Omega)$ is in L_2. This gives rise to many nice properties (Hilbert space).

Time shift

$$x(t-\tau) \quad \Leftrightarrow \quad X(\Omega)\, e^{-j\Omega\tau}$$

The time shift does not influence the amplitude spectrum, but causes a linear "phase delay" $-j\Omega\tau$.

Application Direction estimation using two antennas [plane wave]:



$$\begin{aligned} x_0(t) &= x_1(t-\tau) \\ \Rightarrow \quad X_0(\Omega) &= X_1(\Omega)\, e^{-j\Omega\tau} \\ \Rightarrow \quad e^{-j\Omega\tau} &= \frac{X_0(\Omega)}{X_1(\Omega)} \quad \Rightarrow \quad \tau = \cdots \\ \text{and } \tau &= \frac{d}{c}\sin(\theta) \quad \Rightarrow \quad \theta = \cdots \end{aligned}$$

Applications

Radio astronomy

Phased array processing uses the phase differences in the received signal to estimate the received power from each corresponding direction. This results in an image of the sky.

Similar: ultrasound, MRI, phased array radar, synthetic aperture, $\cdots$

The same concepts are used in IP3 to locate a toy car using a microphone array, or to locate the heart valves using a stethoscope array.

Symmetry

- If $x(t)$ is real, then $X(\Omega) = X^*(-\Omega)$, so

$$|X(\Omega)| = |X(-\Omega)|\,, \qquad \angle X(\Omega) = -\angle X(-\Omega))$$

  The magnitude spectrum is even, the phase spectrum is odd.

- If $x(t)$ is also even, i.e., $x(t) = x(-t)$, then $X(\Omega)$ is real.

Differentiation

Recall for the Laplace transform: $\frac{\mathrm{d}x(t)}{\mathrm{d}t} \Leftrightarrow s\,X(s)$.

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} \quad \Leftrightarrow \quad (j\Omega)^n\,X(\Omega)$$

Integration

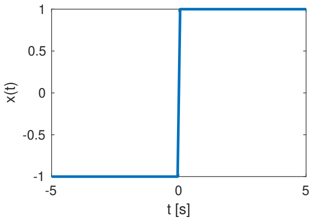
$$\int_{-\infty}^{t} x(t')\,\mathrm{d}t' \quad \Leftrightarrow \quad \frac{X(\Omega)}{j\Omega} + \pi\,X(0)\,\delta(\Omega)$$

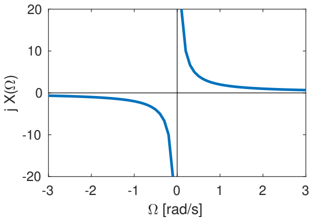
Example

$$\delta(t) \quad \Leftrightarrow \quad 1$$

$$u(t) \quad \Leftrightarrow \quad \frac{1}{j\Omega} + \pi\delta(\Omega)$$

$$\text{sign}(t) = 2[u(t) - 0.5] \quad \Leftrightarrow \quad \frac{2}{j\Omega}$$





Example

Compute the FT of $x(t) = \sin(t)u(t)$:

$$\begin{aligned}
\sin(t) \quad &\Leftrightarrow \quad \frac{\pi}{j}\left(\delta(\Omega-1)-\delta(\Omega+1)\right) \\
u(t) \quad &\Leftrightarrow \quad \frac{1}{j\Omega}+\pi\delta(\Omega) \\
\sin(t)u(t) \quad &\Leftrightarrow \quad \frac{1}{2\pi}\cdot\frac{\pi}{j}\left(\delta(\Omega-1)-\delta(\Omega+1)\right) * \left(\frac{1}{j\Omega}+\pi\delta(\Omega)\right) \\
&= \frac{1}{2(\Omega+1)}-\frac{1}{2(\Omega-1)} + \frac{\pi}{2j}\left(\delta(\Omega-1)-\delta(\Omega+1)\right) \\
&= \frac{1}{1-\Omega^2} + j\frac{\pi}{2}\left(\delta(\Omega+1)-\delta(\Omega-1)\right)
\end{aligned}$$

- Cf. slide 5: the result there was incorrect because $j\Omega$ is not in the ROC. As a result, the two delta spikes at $\Omega = \pm 1$ were missed.

Existence of the Fourier transform [extra]

Sufficient conditions for the Fourier integral to exist (Dirichlet conditions):

- $x(t) \in L_1$
- $x(t)$ has finitely many extrema
- $x(t)$ has finitely many discontinuities

It can be shown that:

- If $x(t) \in L_1$, then $X(\Omega)$ is bounded and continuous, and

$$\lim_{\Omega \to \pm\infty} X(\Omega) = 0 \qquad \text{(Riemann-Lebesgue lemma)}$$

- If the Dirichlet conditions are satisfied, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t_0} \mathrm{d}t = \frac{1}{2} \left(x(t_0^-) + x(t_0^+) \right)$$

Regularity and the Fourier transform [extra]

The decay of $X(\Omega)$ depends on the worst singular behavior of $x(t)$

- If $x(t)$ is p times differentiable and all derivatives are in L_1, then

$$\lim_{\Omega \to \pm\infty} |\Omega|^p X(\Omega) = 0$$

  so that regularity of $x(t)$ translates to rapid decay of $X(\Omega)$

If $x(t) \in L_1$ has compact support (e.g., a pulse), then

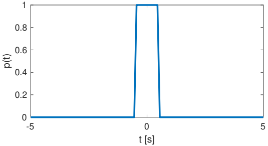
- $X(\Omega) \in C^\infty$, i.e., is infinitely many times continuously differentiable
- $X(\Omega)$ cannot have a compact support

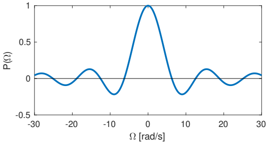
Similarly for $X(\Omega) \in L_1$, by duality

Example

- Rectangular pulse (discontinuous; not differentiable):

$$p(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2}) \quad \Leftrightarrow \quad P(\Omega) = \frac{\sin(\Omega/2)}{\Omega/2}$$

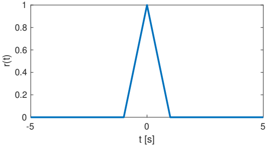


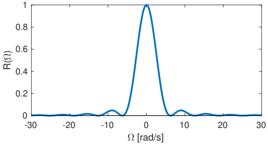


$P(\Omega)$ decays as $\frac{1}{\Omega}$

- Triangular pulse ($1\times$ differentiable; derivative discontinuous):

$$r(t) = p(t) * p(t) \quad \Leftrightarrow \quad R(\Omega) = \left(\frac{\sin(\Omega/2)}{\Omega/2}\right)^2$$





$R(\Omega)$ decays as $\frac{1}{\Omega^2}$

Summary

Table 5.1 Basic Properties of Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1$, integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$ $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)dt$, real
Sine transform	$x(t)$ odd	$X(\Omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\Omega t)dt$, imaginary

Summary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
(1)	$\delta(t)$	1
(2)	$\delta(t - \tau)$	$e^{-j\Omega\tau}$
(3)	$u(t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$
(4)	$u(-t)$	$\frac{-1}{j\Omega} + \pi\delta(\Omega)$
(5)	$\text{sign}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\Omega}$
(6)	$A, -\infty < t < \infty$	$2\pi A\delta(\Omega)$
(7)	$Ae^{-at}u(t), a > 0$	$\frac{A}{j\Omega+a}$
(8)	$Ate^{-at}u(t), a > 0$	$\frac{A}{(j\Omega+a)^2}$
(9)	$e^{-a\|t\|}, a > 0$	$\frac{2a}{a^2+\Omega^2}$
(10)	$\cos(\Omega_0 t), -\infty < t < \infty$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
(11)	$\sin(\Omega_0 t), -\infty < t < \infty$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$
(12)	$p(t) = A[u(t + \tau) - u(t - \tau)], \tau > 0$	$2A\tau \frac{\sin(\Omega\tau)}{\Omega\tau)}$
(13)	$\frac{\sin(\Omega_0 t)}{\pi t}$	$P(\Omega) = u(\Omega + \Omega_0) - u(\Omega - \Omega_0)$
(14)	$x(t)\cos(\Omega_0 t)$	$0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$