# Detection and Estimation with Redundant Range Differences

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***Abstract*— In this paper, we explore the use of redundant range differences in signal estimation and detection. Redundant range differences are known to lie in a certain subspace. This information forms our basis of estimation and detection algorithms. In addition to this information, we also use the configuration of the base stations to check the consistency of range difference estimates. In summary we propose the shrunken estimator as an improvement over the least squares estimator for range difference smoothing. Shrunken estimator is known to give less mean square error compared to the least squares estimator. For detection purposes we propose an method that can passively detect the presence of a signal form redundant range differences which is based on matched subspace detectors.**

## I. Introduction

Positioning with range differences is a popular method in cases when there is either by design or naturally, no synchronization between the receivers and the transmitters. Range differences are obtained from channel measurements by methods like cross correlation and maximum likelihood. Estimated range differences describe hyperboloid in 3D and by intersecting these hyperboloid the location of the source is found. For a given base station configuration we explore to what extent the redundant range differences can be used for estimation and detection purposes. Redundant range differences show a significantly simple property that they lie in a linear subspace. Beyond that range differences are bounded by the triangle inequality where bounds are obtained from the coordinates of the base stations. We make use of these properties for detection and estimation purposes.

The first problem that we encounter is range difference smoothing. In this case, the redundancy between range differences and the locations of base stations are put together to obtain an enhanced estimate of range differences. This has the effect of improving the accuracy of range differences which in return will provide an improved estimate of source location. In the literature various forms of range difference smoothing were used in [1] and [2] however in these approaches the coordinates of the base stations were not taken into account. We add the coordinates of base stations as a prior knowledge into our problem and generalize the range difference smoothing as a constrained norm minimization problem.

The second problem that we encounter is detection. Consider a case where we estimate range differences from the channel without deciding whether a signal is present or we observe only noise. In such a case, we consider the problem of signal detection at range difference level. This approach uses the fact that if there is a signal present, then the estimated range difference should have a strong energy in a certain linear subspace. On the other hand if we do crosscorrelations only on noise we will get some random numbers which do not lie in the specified linear subspace with high probability. We consider various cases where we either have or lack the knowledge of the signal location and channel characteristics. We observe that in general the problem of signal detection is dependent on the configuration of base stations and the source location. The configuration of base stations puts an upper bound on the maximum possible detection performance. Indeed to have good detection it is better to separate the base stations from each other.

## II. Range difference smoothing

We consider source location algorithms with $L \geq 3$ number of base stations. The coordinates of the $k^{th}$ base station are denoted as $x_k, y_k$ and are assumed to be precisely known. $x_m, y_m$ denotes the coordinates of the mobile station. $r_k$ is the distance between mobile station and base station $k$, and $r_{kl} = r_k - r_l$ denotes the range difference. We assume that $r_{kl} = -r_{lk}$ and $r_{kk} = 0$. For simplicity, all the developments are in 2D. Generalization to 3D is easy.

In the case of $L$ base stations there are $L(L-1)/2$ possible range difference measurements. An example with 4 base stations follows

$$\mathbf{\Delta} = \begin{bmatrix} r_{21} \\ r_{31} \\ r_{41} \\ r_{32} \\ r_{42} \\ r_{43} \end{bmatrix} = \begin{bmatrix} -1 & 1 & & \\ -1 & & 1 & \\ -1 & & & 1 \\ & -1 & 1 & \\ & -1 & & 1 \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}. \quad (1)$$

In practice, however, these range difference measurements $\mathbf{\Delta}$ will be corrupted by noise. We can write the previous equations in matrix form as follows

$$\tilde{\mathbf{\Delta}} = \mathbf{D}\mathbf{r} + \mathbf{w} \quad (2)$$

Projecting the noisy range difference measurements to the column space of $\mathbf{D}$ was used in [1] and called as *feasible*

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*bivector*. The rationale for this was to obtain range difference vectors that are in the column space of $\mathbf{D}$ and satisfy consistency.

Another point of view is to use the fact that in the noiseless case $r_{31}-r_{21} = r_{32}$. This is indeed an example of an incidence graph that has applications in geodesy [3]. When we have noise of course this equation will no longer be satisfied so it is reasonable to force the range differences to satisfy it. Schmidt describes the range differences as $\mathbf{\Delta} = 1 \wedge \mathbf{r}$ and a trivector equation $1 \wedge 1 \wedge \mathbf{r} = 0$ where $\wedge$ denotes the wedge product.

In this section we generalise the idea used by Schmidt [1]. The subject is studied from a multilinear algebra point of view by Schmidt [1]. In this point of view Schmidt proves two things. The equality of circuital sum trivector condition and the fact that range differences must be in the column space of matrix $\mathbf{D}$. We will develop the subject in the context of statistics and linear algebra. We believe that this point of view will generalise and simplify the ideas.

The idea of using the redundant combinations of range differences appears to be somehow (in a restricted sense) used by different authors. Schmidt [1] takes the subject from an algebraic (multi-linear algebra) point of view. Hahn and Tretter [2] deal with redundant range differences but their channel model assumes that there are no attenuation differences in the propagation to each sensor. Our main contribution is to add the coordinates of the base stations to the problem. This information was ignored by previous authors who dealed with the problem. As it is clear from simple plane geometry, we may put an explicit bound on the absolute value of the range differences $\mathbf{\Delta}$. Let $p_{kl}$ denote the distance between base station $k$ and base station $l$. From here we can set the following bound

$$p_{kl} \geq |r_{kl}| \qquad (3)$$

This follows directly from the triangle inequality.

Now we are ready to formulate our range difference smoothing problem.

$$\text{minimize } \|\tilde{\mathbf{\Delta}} - \mathbf{\Delta}\| \qquad (4)$$

$$\text{subject to } -p_{ij} \leq r_{ij} \leq p_{ij} \qquad (5)$$

$$\mathbf{\Delta} \in \mathcal{R}(\mathbf{D}) \qquad (6)$$

We did not specify any norm yet. Indeed we can use $\ell_1$ or $\ell_2$ norms. The advantage of using $\ell_1$ norm is that it can provide robustness to outliers.

The fact that $\mathbf{\Delta}$ is in the column space of $\mathbf{D}$ can be written as $\mathbf{\Delta}$ is orthogonal to the orthogonal complement of the colum space of $\mathbf{D}$. These expressions become very clear if we use singular value decomposition (SVD). Let the SVD of $\mathbf{D}$ be

$$\mathbf{D} = \mathbf{U\Sigma V}^T \qquad (7)$$

And let $\mathbf{U} = [\mathbf{U}_1\mathbf{U}_2]$. In this formulation $\mathbf{U}_1$ spans the colum space of $\mathbf{D}$. But we also see from the properties of SVD that $\mathbf{\Delta}$ is orthogonal to $\mathbf{U}_2$. This can be written as

$$\mathbf{U}_2^T\mathbf{\Delta} = 0 \qquad (8)$$

Hence the fact that a vector lies in a column space can be simply written into a fact that it is orthogonal to the space spanned by the orthogonal complements. In other words the equality of the feasible bivector approach and circuital sums approach is a simple subspace duality. An alternative derivation can be found in [1].

Now we aim to provide two special cases which are interesting from an estimation theory perspective. The first is the Best Linear Unbiased Estimate (BLUE) of range differences in the case of Gaussian noise with some known covariance matrix. In this scenario we ignore the knowledge that range differences should obey the triangle inequality. This approach reduces to the solution provided by Schmidt. We however provide a statistical analysis rather then an algebraic one.

The next category is min-max regret estimators. The question is whether we can improve the Least Squares estimator in mean squared error sense. The answer is positive. We show that shrunken estimator suits very well to our problem.

### A. *Classical statistical case*

It is assumed that the noise is zero mean and the covariance matrix $\mathbf{C_w}$ is known. The observation -measurement- equations take the following form:

$$\tilde{\mathbf{\Delta}} = \mathbf{Dr} + \mathbf{w} \qquad (9)$$

Note that the matrix $\mathbf{D}$ is not full rank. If we assume that the range vector $\mathbf{r}$ is deterministic and unknown, then it is not estimable. However, the range difference vector $\mathbf{\Delta}$ is estimable. In general, for any linear transformation of the form $\boldsymbol{a}^T\mathbf{r}$, for which $\boldsymbol{a}$ is in the column space of $\mathbf{D}^T$, $\boldsymbol{a}^T\mathbf{r}$ is estimable [4]. Note that we are not necessarily interested in the actual range values $\mathbf{r}$. The BLUE of $\mathbf{\Delta}$ is

$$\hat{\mathbf{\Delta}} = \mathbf{DD}^-\tilde{\mathbf{\Delta}} \qquad (10)$$

where

$$\mathbf{D}^- = (\mathbf{D}^T\mathbf{C}_\mathbf{w}^{-1}\mathbf{D})^\dagger\mathbf{D}^T\mathbf{C}_\mathbf{w}^{-1} \qquad (11)$$

and $\dagger$ denotes Moore-Penrose pseudo inverse.

For the case where $\mathbf{C_w} = \sigma^2\mathbf{I}$, the BLUE reduces to the *feasible bivector* solution discussed in [1]. By using the closed form for the projection operator given in [1], we obtain the covariance of the estimate as:

$$\mathbf{P} = \frac{1}{L}\mathbf{DD}^T, \qquad \mathbf{C}_{\hat{\mathbf{\Delta}}} = \sigma^2\mathbf{P} = \frac{\sigma^2}{L}\mathbf{DD}^T \qquad (12)$$

From here it is easy to see that:

$$\text{var}(\hat{r}_{kl}) = \frac{2}{L}\sigma^2 < \sigma^2 \qquad (13)$$

This inequality justifies the advantage of smoothing the range differences. We reduce the variance on range differences significantly by using range difference smoothing.

### *B. Min-max regret approach to range difference smoothing*

In this section we apply the min-max regret estimators [5] to the problem. Min-max regret estimators are known to provide smaller mean square error compared to least squares estimators. However these estimators require that a certain norm bound can be put on the estimated parameters. Surprisingly this fits to our case of range difference smoothing.

Since our initial matrix is not full rank we will introduce the following parameterization. This is necessary in order to apply the min-max regret estimators.

$$\tilde{\mathbf{\Delta}} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w} \tag{14}$$

In the above notation $\mathbf{H}$ is a full rank $N \times p$ matrix spanning the column space of $\mathbf{D}$. This matrix can simply be obtained from the SVD of the matrix $\mathbf{D}$. We assume now that noise $\mathbf{w}$ is white Gaussian with some known variance. Since the range differences are bounded by the coordinates of the base stations, we can impose the following inequality

$$\boldsymbol{\theta}^T \mathbf{H}^T \mathbf{C}_{\mathbf{w}}^{-1} \mathbf{H}\boldsymbol{\theta} \leq K^2 \tag{15}$$

From here the shrunken estimator is

$$\hat{\boldsymbol{\theta}} = \Big(1 - \frac{1}{\sqrt{1+K^2}}\Big)\boldsymbol{\theta}_{LS}, \qquad K^2 \geq a \tag{16}$$

where $a = (p-1)^2 - 1$ and $p$ is the rank of $\mathbf{H}$ and $\boldsymbol{\theta}_{LS}$ is the LS estimator of $\boldsymbol{\theta}$. The estimate of range differences is simply obtained as $\mathbf{H}\hat{\boldsymbol{\theta}}$.

Note that if the coordinates of the base stations are known precisely and we know the noise covariance then the minimax estimator will have less mean square error compared to the LS estimator. When the noise variance is unknown it can be estimated as

$$\hat{\sigma}^2 = \frac{1}{N-p}\tilde{\mathbf{\Delta}}^T(\mathbf{I} - \mathbf{P}_{\mathbf{H}})\tilde{\mathbf{\Delta}} \tag{17}$$

However we observed in simulations that this approach does not provide improvement compared to least squares estimator.

We must note that it is possible to add the uncertainty in the estimates of noise variance into the problem by using the results in [6]. It was shown in [6] that the min-max regret estimator can be extended to the case where the covariance of noise is uncertain. However exploring this extension is beyond the scope of the current paper.

## III. Signal detection based on range differences

In this section, we discuss to what extent the fact that range differences lie in a subspace can be used to perform signal detection. For simplicity and analytical derivations, we assume that the noise is white Gaussian. We will base our results on matched subspace detectors [7]. Our linear model is:

$$\tilde{\mathbf{\Delta}} = \mathbf{D}\mathbf{r} + \mathbf{w} \tag{18}$$

where we assume that $\mathbf{w} \sim \mathcal{N}(\mathbf{1}\phi, \sigma^2\mathbf{I})$. Our purpose in providing the mean of noise as $\mathbf{1}\phi$ is to remove the effects of unknown common bias. If there is a bias in the range difference estimates that will influence the performance of the detector significantly.

Now lets think the case where we do crosscorrelations on noise. What we will receive is random range estimates. We aim to model this as white Gaussian noise with some large unknown variance. On the other hand what happens when there is a signal and we are performing crosscorrelations. Then again our range difference estimates will be noise. We again assume that the noise on them is white Gaussian.

Under these conditions the hypothesis testing problem can be written as follows

$$\mathcal{H}_0 : \tilde{\mathbf{\Delta}} \sim \mathcal{N}(\mathbf{S}\phi, \sigma^2\mathbf{I}) \tag{19}$$

$$\mathcal{H}_1 : \tilde{\mathbf{\Delta}} \sim \mathcal{N}(\mathbf{H}\boldsymbol{\theta} + \mathbf{S}\phi, \sigma^2\mathbf{I}) \tag{20}$$

where $\mathbf{S} = \mathbf{1}$ and the matrix $\mathbf{H}$ is chosen such that it is full rank and spans the space $\mathbf{R}(\mathbf{D})$. The test statistics becomes

$$L(\tilde{\mathbf{\Delta}}) = \frac{s-p}{p}\frac{\tilde{\mathbf{\Delta}}^T \mathbf{P}_{\mathbf{S}}^{\perp}\mathbf{P}_{\mathbf{G}}\mathbf{P}_{\mathbf{S}}^{\perp}\tilde{\mathbf{\Delta}}}{\tilde{\mathbf{\Delta}}^T \mathbf{P}_{\mathbf{S}}^{\perp}\mathbf{P}_{\mathbf{G}}^{\perp}\mathbf{P}_{\mathbf{S}}^{\perp}\tilde{\mathbf{\Delta}}} > \eta \tag{21}$$

where $\mathbf{G} = \mathbf{P}_{\mathbf{S}}^{\perp}\mathbf{H}$. The performance of this detector under gaussian assumption can be obtained explicitly. However, since in practice the noise will be non-gaussian. Our approach is only an approximation. In the above formulation $s$ is the dimension of the space $\langle \mathbf{S} \rangle^{\perp}$ and $p$ is the dimension of $\langle \mathbf{G} \rangle$.

$$\lambda^2 = \frac{1}{\sigma^2}(\mathbf{H}\boldsymbol{\theta})^T \mathbf{P}_{\mathbf{S}}^{\perp}(\mathbf{H}\boldsymbol{\theta}) \tag{22}$$

Under the two hypothesis, the distributions of observations are given as follows

$$\mathcal{H}_0 : L(\tilde{\mathbf{\Delta}}) \sim F_{p,s-p}(0) \tag{23}$$

$$\mathcal{H}_1 : L(\tilde{\mathbf{\Delta}}) \sim F_{p,s-p}(\lambda^2) \tag{24}$$

From here under the Gaussian assumption we have the following detector characteristics.

$$P_{FA} = 1 - P[F_{p,s-p}(0) \leq \eta] \tag{25}$$

$$P_D = 1 - P[F_{p,s-p}(\lambda^2) \leq \eta] \tag{26}$$

## IV. Simulations

We tested the performance of the proposed algorithms via computer simulations. In all the simulations the coordinates of the base stations were fixed at $(0,0), (0,10), (10,0), (10,10)$ And the location of the mobile station is $(3,3)$ for testing the shrunken estimator. Fig. 1 shows the performance of the shrunken estimator and it is better than the LS estimator in MSE sense. We observed that the adaptive algorithm where the variance of noise is estimated does not perform better than LS estimator.

We have also tested the detection performance of the matched subspace detector. Fig. 2, Fig. 3, and Fig. 4 show how the detection performance is influenced by the location. In the simulations the variance of noise is set to 1.

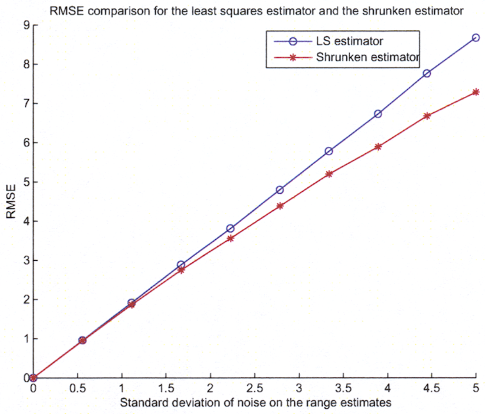


Fig. 1. RMSE comparison for least squares and shrunken estimators

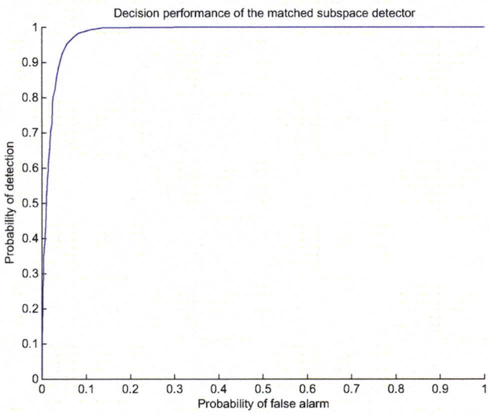


Fig. 2. Detection performance at the location $(3, 3)$

## V. Conclusion

We have explored the potential use of redundant range differences for both estimation and detection purposes. It has been observed that obtaining redundant range differences and using a smoothing procedure improves the accuracy of range difference estimation significantly. In the literature the smoothing approach to range differences was used by Schmidt [1]. We added the information of the base station locations to the smoothing procedure and generalized it as a general constrained norm minimization problem. In the case of known noise covariance, it was shown that by using a minimax regret estimator the LS estimation of Schmidt can be improved in mean square sense.

Finally we observed that range differences themselves can be used for signal detection. There is huge amount redundancy in range differences hence when there are many base stations we will have perfect detection performance.

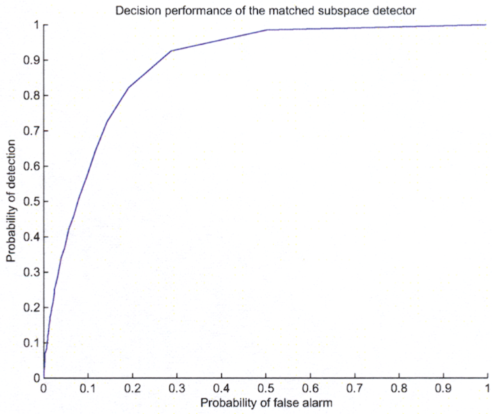


Fig. 3. Detection performance at the location $(4, 4)$

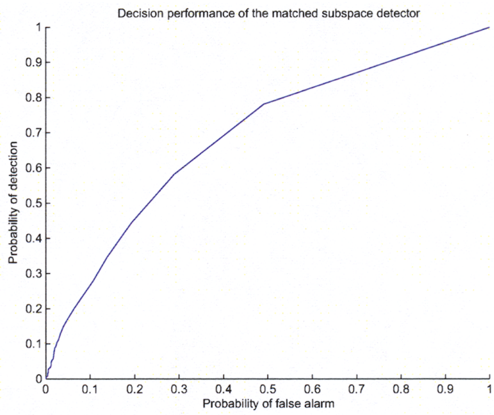


Fig. 4. Detection performance at the location $(4.5, 4.5)$

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