Nonlinear Modified Newton Minimization of Reduced-Order Objective Functions for Two-Parameter Inversion Problems

R. F. Remis*

Abstract — **We present a modified Newton minimization scheme for two-parameter inversion problems. Starting point is a so-called reduced-order objective function which measures the discrepancy between measured data and model-order reduction scattered field data. This objective function is minimized on a specified range of permittivity and conductivity values and a nonlinear transformation turns this constraint minimization problem into an unconstraint one. A backtracking procedure to obtain proper step lengths is implemented as well and a numerical example illustrates the performance of the method.**

1 INTRODUCTION

In this paper we present a modified Newton approach for two-parameter electromagnetic inversion problems. We try to find the conductivity and permittivity of an object by inverting measured electromagnetic data through a Newton-type minimization procedure. Starting point of our method is an objective function which measures the discrepancy between the measured and modeled data. Instead of minimizing this objective function directly, we first construct a reduced-order objective function that approximates the original objective function on a domain of permittivity and conductivity values of interest. To find the medium parameters of the object, we subsequently minimize the reduced-order objective function using a modified Newton approach. Gradients and Hessians appearing in Newton's scheme can be computed at low computational costs, since it is the reduced-order objective function that is minimized instead of the full original objective function [1].

The reduced-order objective function approximates the original objective function on a prescribed domain of interest, and we therefore have to make sure that the medium parameter values generated by the Newton's method stay within the domain of interest as iteration proceeds. To this end, we transform our constraint minimization problem to an unconstraint problem using a nonlinear transformation [2]. Furthermore, to obtain a sufficient decrease of the objective function, a backtracking procedure based on the so-called Armijo condition is implemented [3]. The performance of the method is illustrated through a numerical example in which we try to find the effective medium parameters of an inhomogeneous object.

*Circuits and Systems Group, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands, e-mail: `r.f.remis@tudelft.nl`, tel.: +31 152781442.

2 BASIC EQUATIONS

We consider E-polarized electromagnetic fields in a configuration that is invariant in the z-direction. A penetrable object occupies a bounded domain $\mathbb{D}^{\text{obj}}$ in the xy-plane and is illuminated by an incident electromagnetic field. The background medium is lossless and has a permittivity ε_{b} and a permeability μ_0. It is well known that the total electric field strength inside the object satisfies the so-called object equation

$$E_z(\mathbf{x}) - \frac{\mathrm{i}k_{\text{b}}^2}{4}\int_{\mathbf{x}'\in\mathbb{D}^{\text{obj}}} H_0^{(1)}(k_{\text{b}}|\mathbf{x}-\mathbf{x}'|)\chi(\mathbf{x}')E_z(\mathbf{x}')\,\mathrm{d}A = E_z^{\text{inc}}(\mathbf{x}), \tag{1}$$

with $\mathbf{x}\in\mathbb{D}^{\text{obj}}$, while at a receiver location $\mathbf{x}_{\text{r}}\notin\mathbb{D}^{\text{obj}}$, we have the data equation

$$E_z^{\text{sc}}(\mathbf{x}_{\text{r}}) = \frac{\mathrm{i}k_{\text{b}}^2}{4}H_0^{(1)}(k_{\text{b}}|\mathbf{x}_{\text{r}}-\mathbf{x}'|)\chi(\mathbf{x}')E_z(\mathbf{x}')\,\mathrm{d}A. \tag{2}$$

In the above equations, k_{b} is the wave number of the background medium, χ is the contrast function of the object, and $H_0^{(1)}$ is the Hankel function of the first kind and order zero.

Suppose now that the contrast function can be written as

$$\chi(\mathbf{x}) = \zeta\chi_{\text{p}}(\mathbf{x}), \tag{3}$$

where ζ is an unknown complex-valued contrast coefficient and $\chi_{\text{p}}(\mathbf{x})$ is a known contrast profile function. Substitution of Eq. (3) in the object and data equation and subsequently discretizing the resulting expressions on a uniform grid with step size δ, we obtain the discretized object equation

$$(\mathbf{I}-\zeta\mathbf{G}\mathbf{P})\mathbf{u} = \mathbf{u}^{\text{inc}}, \tag{4}$$

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and the discretized data equation

$$u^{\rm sc} = \gamma_{\rm r}\zeta \mathbf{r}^T\mathbf{P}\mathbf{u}, \tag{5}$$

where $\gamma_{\rm r} = {\rm i}(k_{\rm b}\delta)^2/4$. In these equations, $\mathbf{I}$ is the identity matrix, $\mathbf{G}$ is the discretized convolution operator, and $\mathbf{P}$ is the discretized counterpart of the contrast profile function. Furthermore, $\mathbf{u}$ contains the total field approximations within the object, while vector $\mathbf{r}$ is a discretized version of the data operator. Finally, incident electric field strength values are stored in vector $\mathbf{u}^{\rm inc}$. In particular, with an incident field generated by a line source located at $\mathbf{x} = \mathbf{x}_{\rm s}$ and having a source signature $f(\omega)$, the incident field vector is given by

$$\mathbf{u}^{\rm inc} = \gamma_{\rm s}\mathbf{s} \quad \text{with} \quad \mathbf{s} = \text{vec}\left[H_0^{(1)}(k_{\rm b}|\mathbf{x}_{ij} - \mathbf{x}_{\rm s}|)\right],$$

where $\gamma_{\rm s} = {\rm i}\omega\mu_{\rm b}f(\omega)/4$. Substituting this incident field in the discretized object equation and solving for the total field $\mathbf{u}$, we obtain

$$\mathbf{u} = \gamma_{\rm s}(\mathbf{I} - \zeta\mathbf{G}\mathbf{P})^{-1}\mathbf{s}$$

and the discretized data equation becomes

$$u^{\rm sc} = \gamma\zeta\mathbf{r}^T\mathbf{P}(\mathbf{I} - \zeta\mathbf{G}\mathbf{P})^{-1}\mathbf{s}, \tag{6}$$

where $\gamma = \gamma_{\rm r}\gamma_{\rm s}$. Introducing the vectors $\mathbf{x} = \mathbf{r}+\mathbf{s}$ and $\mathbf{y} = \mathbf{r}-\mathbf{s}$, the data equation can also be written in terms of two monostatic source/receiver setups as

$$u^{\rm sc} = \frac{\gamma\zeta}{4}\left[\mathbf{x}^T\mathbf{P}(\mathbf{I} - \zeta\mathbf{G}\mathbf{P})^{-1}\mathbf{x} - \mathbf{y}^T\mathbf{P}(\mathbf{I} - \zeta\mathbf{G}\mathbf{P})^{-1}\mathbf{y}\right]. \tag{7}$$

3 THE REDUCED-ORDER OBJECTIVE FUNCTION

Let $E_z^{\rm sc;m}$ denote the scattered electric field strength measured at the receiver location. We can try to find the medium parameters of the object by minimizing the objective function

$$F = \frac{|E_z^{\rm sc;m} - u^{\rm sc}|^2}{|E_z^{\rm sc;m}|^2}. \tag{8}$$

In this paper, however, we follow a more efficient approach and first construct an accurate reduced-order model for the modeled scattered field. We then minimize a reduced-order objective function following a modified Newton approach.

The reduced-order models are computed by exploiting the $\mathbf{P}$-symmetry of matrix $\mathbf{GP}$ in a Lanczos-type algorithm. These models are given b by [1]

$$u_k^{\rm sc} = \frac{\gamma\zeta}{4}\mathbf{e}_1^T\left[a_x(\mathbf{I}_k - \zeta\mathbf{T}_{{\rm x};k})^{-1} - a_y(\mathbf{I}_k - \zeta\mathbf{T}_{{\rm y};k})^{-1}\right]\mathbf{e}_1, \tag{9}$$

where $\mathbf{e}_1$ is the first column of the k-by-k identity matrix $\mathbf{I}_k$, $a_x = \mathbf{x}^T\mathbf{P}\mathbf{x}$, $a_y = \mathbf{y}^T\mathbf{P}\mathbf{y}$, and $\mathbf{T}_{{\rm x};k}$ and $\mathbf{T}_{{\rm y};k}$ are tridiagonal matrices of order k obtained after k iterations of the Lanczos algorithm with vectors $\mathbf{x}$ and $\mathbf{y}$ as starting vectors. The order k of the model is much smaller than the order of the original system and it can be shown that $u_k^{\rm sc}$ is actually a $[k-1/k]$ Padé approximant of $u^{\rm sc}$ around $\zeta = 0$.

To determine the order of the model, we first write the contrast coefficient as

$$\zeta = \varepsilon_{\rm r} - 1 + {\rm i}\frac{\sigma}{\omega\varepsilon_{\rm b}}, \tag{10}$$

and we assume that lower and upper bounds for the permittivity and conductivity can be given such that the permittivity $\varepsilon_{\rm r}$ and conductivity σ of the object satisfy

$$\varepsilon_{\rm r;min} \le \varepsilon_{\rm r} \le \varepsilon_{\rm r;max} \quad \text{and} \quad \sigma_{\min} \le \sigma \le \sigma_{\max}.$$

These ranges for the permittivity and conductivity, together with Eq. (10), specify our domain of interest $\mathbb{A}$ in the complex ζ-plane. We now require that the order of the model is such that $u_k^{\rm sc} \approx u^{\rm sc}$ for all $\zeta \in \mathbb{A}$. How to select k for a given domain of interest is discussed in [4].

Having the reduced-order model for the scattered field available, we introduce the discrepancy

$$d_k^{\rm sc} = E_z^{\rm sc;m} - u_k^{\rm sc}$$

and we try to find the medium parameters of the object by minimizing the reduced-order objective function

$$F_k = \frac{|d_k^{\rm sc}|^2}{|E_z^{\rm sc;m}|^2} \tag{11}$$

on $\mathbb{A}$, our domain of interest.

4 A MODIFIED NEWTON MINIMIZATION APPROACH

To arrive at Newton-type updating schemes, we start with a Taylor expansion of the reduced-order objective function. With $\zeta = \zeta_{\rm r} + {\rm i}\zeta_{\rm i}$, we have

$$\begin{aligned} F_k(\zeta_{\rm r} + \delta\zeta_{\rm r}, \zeta_{\rm i} + \delta\zeta_{\rm i}) = F_k(\zeta_{\rm r}, \zeta_{\rm i}) \\ + \mathbf{g}_\zeta^T\delta\mathbf{z} \\ + \frac{1}{2}\delta\mathbf{z}^T\mathbf{Z}_\zeta\delta\mathbf{z} \\ + \text{higher order terms,} \end{aligned} \tag{12}$$

where $\delta\mathbf{z} = [\delta\zeta_{\rm r}, \delta\zeta_{\rm i}]^T$. Explicit expressions for the gradient $\mathbf{g}$ and Hessian $\mathbf{Z}_\zeta$ are given in [1].

It is well known, of course, that to obtain the update, we need to solve Newton's equation

$$\mathbf{Z}_\zeta \delta\mathbf{z} = -\mathbf{g}_\zeta.$$

Before doing so, however, we first take into account that we are looking for contrast values ζ belonging to the domain $\mathbb{A}$ only. To this end, we transform our constraint minimization problem into an unconstrained one by using the nonlinear transformation [2]

$$\varepsilon_{\rm r}(\eta) = \varepsilon_{\rm r;min} + (\varepsilon_{\rm r;max} - \varepsilon_{\rm r;min})\frac{\eta^2}{\eta^2+1},$$

for $-\infty < \eta < \infty$, and

$$\sigma(\xi) = \sigma_{\rm min} + (\sigma_{\rm max} - \sigma_{\rm min})\frac{\xi^2}{\xi^2+1},$$

with $-\infty < \xi < \infty$. Considering now the reduced-order objective function as a function of η and ξ and setting $\mathbf{m} = [\eta, \xi]^T$, we have

$$F_k(\mathbf{m}+\delta\mathbf{m}) = F_k(\mathbf{m}) + \mathbf{g}^T\delta\mathbf{m} + \frac{1}{2}\delta\mathbf{m}^T\mathbf{Z}\delta\mathbf{m} \\ + \text{higher order terms}, \tag{13}$$

where $\delta\mathbf{m} = [\delta\eta, \delta\xi]^T$. The gradient and Hessian are given by

$$\mathbf{g} = \mathbf{D}\mathbf{g}_\zeta \quad \text{and} \quad \mathbf{Z} = \mathbf{D}\mathbf{Z}_\zeta\mathbf{D},$$

respectively, and in the above equations

$$\mathbf{D} = \\ 2\begin{pmatrix} (\varepsilon_{\rm r;max} - \varepsilon_{\rm r;min})\frac{\eta}{(\eta^2+1)^2} & 0 \\ 0 & \frac{\sigma_{\rm max}-\sigma_{\rm min}}{\omega\varepsilon_{\rm b}}\frac{\xi}{(\xi^2+1)^2} \end{pmatrix}.$$

Newton's equation now becomes

$$\mathbf{Z}_\zeta\mathbf{D}\delta\mathbf{m} = -\mathbf{g}_\zeta,$$

provided we are not in a false minimum introduced by the nonlinear transformation.

Now if $\mathbf{Z}_\zeta$ is positive definite, the solution of Newton's equation provides us with a descent direction. However, if the Hessian is not positive definite, the update direction is not necessarily a descent direction. To remedy this situation, we first compute the eigendecomposition of the Hessian and obtain

$$\mathbf{Z}_\zeta = s\mathbf{X}\Lambda\mathbf{X}^T$$

with $\mathbf{X}^T\mathbf{X} = \mathbf{X}\mathbf{X}^T = \mathbf{I}_2$ and $\Lambda = \text{diag}(\lambda_+, \lambda_-)$. It can be shown that λ_+ is always positive, but λ_- may become negative (see [1]). We "cure" this situation by replacing the Hessian in Newton's equation by the modified Hessian

$$\mathbf{Z}_\zeta^{\rm mn} = s\mathbf{X}\Lambda^{\rm mn}\mathbf{X}^T,$$

where $\Lambda^{\rm mn} = \text{diag}(\lambda_+, |\lambda_-|)$. The modified Newton update direction is given by

$$\delta\tilde{\mathbf{m}} = \mathbf{D}^{-1}\mathbf{X}(\Lambda^{\rm mn})^{-1}\mathbf{X}^T\mathbf{p}$$

and substitution of this direction in the Taylor expansion of Eq. (13) shows that the above update vector leads to a reduction of the objective function provided that $\delta\tilde{\mathbf{m}}$ is sufficiently small. Notice that at points where $\lambda_- > 0$, we have $\mathbf{Z}_\zeta^{\rm mn} = \mathbf{Z}_\zeta$ and our modified Hessian reduces to the original Hessian $\mathbf{Z}_\zeta$.

4.1 Backtracking

To obtain a sufficient decrease of the objective function, we introduce a step length ν_n in the direction $\delta\mathbf{m}$. In other words, the update equation is given by

$$\mathbf{m}_n = \mathbf{m}_{n-1} + \nu_n\delta\tilde{\mathbf{m}}_n.$$

Sufficient decrease of the objective function is described by the so-called Armijo condition [3]

$$F_k(\mathbf{m}+\nu\delta\mathbf{m}) \le F_k(\mathbf{m}) + c\nu\mathbf{g}^T\delta\mathbf{m},$$

where c is a small positive number (usually one sets $c = 10^{-4}$). Since Newton methods have a natural unit step length, we start the backtracking procedure with a step length $\nu_n^{(1)} = 1$. Next, we test if the Armijo condition is satisfied. If so, we terminate the backtracking procedure. Otherwise, we construct a quadratic interpolation polynomial in ν on the interval $[0, \nu_n^{(1)} = 1]$. Using all available information, namely, $F_k(\mathbf{m}_{n-1})$, $\mathbf{g}^T\delta\mathbf{m}$, and $F_k(\mathbf{m}_{n-1}+\delta\mathbf{m})$, the interpolation polynomial is given by

$$\left[F_k(\mathbf{m}_{n-1}+\delta\mathbf{m}) - F_k(\mathbf{m}_{n-1}) - \mathbf{g}^T\delta\mathbf{m}\right]\nu^2 \\ + \mathbf{g}^T\delta\mathbf{m}\,\nu + F_k(\mathbf{m}_{n-1})$$

and the new step length $\nu_n^{(2)}$ is obtained as the minimizer of this polynomial, that is,

$$\nu_n^{(2)} = -\frac{\mathbf{g}^T\delta\mathbf{m}}{2[F_k(\mathbf{m}_{n-1}+\delta\mathbf{m}) - F_k(\mathbf{m}_{n-1}) - \mathbf{g}^T\delta\mathbf{m}]}.$$

If this new step length satisfies the Armijo condition, we terminate backtracking. If the Armijo condition is not satisfied, we construct a cubic interpolation polynomial in ν using $F_k(\mathbf{m}_{n-1})$, $\mathbf{g}^T\delta\mathbf{m}$,

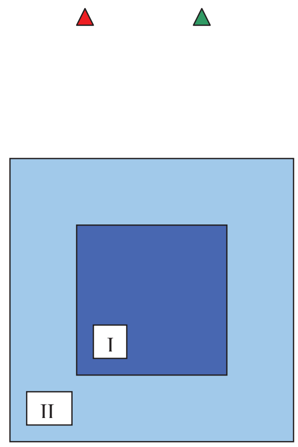

Figure 1: A source (triangle left) and receiver (triangle right) symmetrically located above an inhomogeneous block. The distance between the upper boundary of the block and the source/receiver unit is $\lambda_0/2$.

$F_k(\mathbf{m}_{n-1} + \delta\mathbf{m})$, and $F_k(\mathbf{m}_{n-1} + \nu_n^{(2)}\delta\mathbf{m})$ as interpolation data. The new step length $\nu_n^{(3)}$ is the minimizer of this polynomial. If this step size still does not satisfy the Armijo condition, we start iterating using a cubic interpolation polynomial based on $F_k(\mathbf{m}_{n-1})$, $\mathbf{g}^T\delta\mathbf{m}$, and the objective function values for the latest two step lengths.

5 NUMERICAL RESULTS

We apply our reduced-order Newton scheme to find effective conductivity and permittivity values of an inhomogeneous block. The configuration is shown in Figure 1. An inhomogeneous block is embedded in a vacuum domain and consists of two homogeneous subdomains. A source/receiver unit is symmetrically located above the block and the distance between the upper boundary of the block and the source/receiver unit is $\lambda_0/2$, where λ_0 is the wavelength in the background medium. The side length of the block is either $\lambda_0/3$ (small block) or λ_0 (large block). The true medium parameters in each subdomain are given in Table 1 and we look for effective permittivity and conductivity values in the range $1 \leq \varepsilon_r \leq 9$ and $0 \leq \sigma \leq 10$ mS/m. Table 1 shows the effective medium parameters obtained with our modified Newton scheme. Effective medium parameters for both the small and the large block have been determined starting from an initial guess that is obtained by inspecting the reduced-order objective function (see [4]) on a coarse grid. We observe that in this case the method produces reasonable effective permittivity values, but effective conductivity values may be less reliable. Using multiple frequency data or carrying out additional measurements using multiple sources and receivers may improve the effective inversion results. In addition, the effective medium parameters should be source/receiver independent. Future work focuses on finding conditions under which reliable effective medium parameters can be found.

Conductivity Permittivity	Eff. med. par. small block	Eff. med. par. large block
$\varepsilon_{r;I} = 6$ $\varepsilon_{r;II} = 4$ $\sigma_I = 8 \cdot 10^{-3}$ $\sigma_{II} = 5 \cdot 10^{-3}$	$\varepsilon_{r;ef} =$ 4.1 $\sigma_{ef} =$ 7.3e-03	$\varepsilon_{r;ef} =$ 5.10 $\sigma_{ef} =$ 1.0e-02
$\varepsilon_{r;I} = 4.75$ $\varepsilon_{r;II} = 4$ $\sigma_I = 5.5 \cdot 10^{-3}$ $\sigma_I = 5 \cdot 10^{-3}$	$\varepsilon_{r;ef} =$ 4.0 $\sigma_{ef} =$ 5.7e-03	$\varepsilon_{r;ef} =$ 4.5 $\sigma_{ef} =$ 9.6e-03
$\varepsilon_{r;I} = 6$ $\varepsilon_{r;II} = 4$ $\sigma_I = 0$ $\sigma_I = 0$	$\varepsilon_{r;ef} =$ 4.1 $\sigma_{ef} =$ 1.7e-03	$\varepsilon_{r;ef} =$ 5.2 $\sigma_{ef} =$ 1.7e-03

Table 1: Results effective inversion inhomogeneous block

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