Circulant Preconditioners for Domain Integral Equations in Electromagnetics

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Abstract — **In this paper, we present an optimal circulant preconditioner for domain integral equations in electromagnetics. The preconditioner is the best circulant fit to the discretized domain integral operator as measured by the Frobenius norm. We show that the discretized integral operators exhibit a Toeplitz-like structure for inhomogeneous objects and present an explicit expression for the elements of the optimal circulant. The circulant matrix can be used as an effective preconditioner in iterative solvers, since its action on a vector can be computed using the Fast Fourier Transform. Numerical experiments illustrate the performance of the preconditioner.**

1 INTRODUCTION

In this paper, we consider scattering of one-dimensional steady-state electromagnetic waves by penetrable inhomogeneous objects occupying a bounded domain in space. The objects are embedded in a homogeneous background medium and show dielectric contrast only. It is well known that the problem of finding the electromagnetic field in such a configuration can be formulated in terms of an integral equation for the total electric field strength inside the object. To approximately solve this integral equation, we discretize the configuration on a uniform grid and arrive at a system of equations in which the system matrix has a Toeplitz or Toeplitz-like structure. Matrix-vector multiplications with the system matrix can therefore be computed efficiently via the Fast Fourier Transform (FFT). Consequently, iterative solvers like GMRES and BiCGStab [1, 2] are often the solution methods of choice, since the system matrix is only required to form matrix-vector products in these methods.

To speed up the convergence rate of an iterative solver, the original system matrix is usually preconditioned and the resulting system is solved using an iterative solver. In this paper, we follow this approach and develop dedicated circulant preconditioners for electromagnetic scattering problems. Since our preconditioners are circulant by construction, we can again use the FFT to compute the action of a preconditioner on a vector. We show that the preconditioners can be very effective and significantly reduce the number of iterations to reach a certain prescribed error tolerance. Furthermore, we also show that our preconditioner coincides with the well known optimal circulant for Toeplitz systems if the object is homogeneous.

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2 BASIC EQUATIONS

Consider a one-dimensional configuration showing variation in the y-direction only. A penetrable slab has a width d and occupies the domain $\mathbb{D}^{\rm sc} = \{y \in \mathbb{R}; 0 < y < d\}$. The medium parameters of the slab are given by the position dependent conductivity $\sigma^{\rm sc}(y)$ and permittivity $\varepsilon^{\rm sc}(y)$ and the slab is embedded in a homogeneous background medium with constant medium parameters σ and ε. The slab shows no contrast in the permeability μ and we set $\eta = \sigma + {\rm j}\omega\varepsilon$ and $\zeta = {\rm j}\omega\mu$. In this 1D configuration, the electric field inside the slab satisfies the integral equation

$$E_z(y) - k_{\rm b}^2 \int_{y'=0}^{d} G(y-y')\chi(y')E_z(y')\,{\rm d}y' = E_z^{\rm inc}(y), \tag{1}$$

with $y \in \mathbb{D}^{\rm sc}$. In this equation, $E_z^{\rm inc}$ is the known incident electric field strength and $k_{\rm b}$ is the wave number of the background medium defined as

$$k_{\rm b} = (-\eta\zeta)^{1/2} \quad \text{with } {\rm Im}(k_{\rm b}) \le 0. \tag{2}$$

Furthermore, χ is the so-called contrast function given by

$$\chi(y) = \frac{\eta^{\rm sc}(y)}{\eta} - 1, \tag{3}$$

where $\eta^{\rm sc}(y) = \sigma^{\rm sc}(y) + {\rm j}\omega\varepsilon^{\rm sc}(y)$, and G is the Greens function of the homogeneous background medium given by

$$G(y) = \frac{\exp(-{\rm j}k_{\rm b}|y|)}{2{\rm j}k_{\rm b}} \quad y \in \mathbb{R}. \tag{4}$$

To discretize the object equation, we introduce the step size $\delta y = d/N$, with $N \in \mathbb{Z}_+$, and the grid node coordinates

$$y_n = \frac{\delta y}{2} + (n-1)\delta y, \quad \text{for } n = 1, 2, ..., N. \tag{5}$$

The grid nodes form the midpoints of the discretization cells

$$\mathbb{S}_k = \{y \in \mathbb{R}; (k-1)\delta y < y < k\delta y\} \tag{6}$$

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and the contrast function is taken to be constant within a discretization cell, that is, we have $\chi(y) = \chi_k$ if $y \in \mathbb{S}_k$, $k = 1, 2, ..., N$.

To arrive at a system of equations for the total electric field strength within the slab, we first require that Eq. (1) holds at the grid node coordinates. Taking $y = y_n$ in Eq. (1), we obtain

$$E_z(y_n) - k_\mathrm{b}^2 \int_{y'=0}^{d} G(y_n - y')\chi(y')E_z(y')\,\mathrm{d}y' = E_z^{\mathrm{inc}}(y_n), \tag{7}$$

for $n = 1, 2, ..., N$. Taking into account that the contrast is constant within a discretization cell, the above can be written as

$$E_z(y_n) - k_\mathrm{b}^2 \sum_{k=1}^{N} \chi_k \int_{y' \in \mathbb{S}_k} G(y_n - y')E_z(y')\,\mathrm{d}y' = E_z^{\mathrm{inc}}(y_n). \tag{8}$$

Applying the midpoint rule to the integral over the discretization cell results in

$$E_z(y_n) - k_\mathrm{b}^2 \delta y \sum_{k=1}^{N} \chi_k G(y_n - y_k)E_z(y_k) = E_z^{\mathrm{inc}}(y_n), \tag{9}$$

for $n = 1, 2, ..., N$. Finally, introducing the vectors

$$\mathbf{e}_z = [E_z(y_1), E_z(y_2), ..., E_z(y_N)]^T \tag{10}$$

and

$$\mathbf{e}_z^{\mathrm{inc}} = [E_z^{\mathrm{inc}}(y_1), E_z^{\mathrm{inc}}(y_2), ..., E_z^{\mathrm{inc}}(y_N)]^T, \tag{11}$$

we can write the discretized object equation as

$$\mathbf{K}\mathbf{e}_z = \mathbf{e}_z^{\mathrm{inc}} \quad \text{with} \quad \mathbf{K} = \mathbf{I} - \mathbf{G}\mathbf{X}. \tag{12}$$

Here, $\mathbf{I}$ is the identity matrix of order N, $\mathbf{X}$ is the contrast matrix given by

$$\mathbf{X} = \mathrm{diag}(\chi_1, \chi_2, ..., \chi_N), \tag{13}$$

and $\mathbf{G}$ is the spatial convolution matrix (Greens matrix) given by

$$\mathbf{G} = \begin{pmatrix} g_0 & g_{-1} & \cdots & g_{2-N} & g_{1-N} \\ g_1 & g_0 & \ddots & & g_{2-N} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ g_{N-2} & & & \ddots & g_{-1} \\ g_{N-1} & g_{N-2} & \cdots & g_1 & g_0 \end{pmatrix}, \tag{14}$$

with

$$g_n = \frac{k_\mathrm{b}\delta y}{2\mathrm{j}} \exp(-\mathrm{j}k_\mathrm{b}\delta y|n|) \tag{15}$$

for $n = 0, \pm1, \pm2, \pm(N-1)$. Notice that $\mathbf{G}$ is a complex symmetric Toeplitz matrix of order N. The action of this matrix on a vector can be computed with FFTs by embedding this matrix in a circulant of order $2N$. Furthermore, since $\mathbf{X}$ is diagonal, we conclude the action of the system matrix $\mathbf{K}$ on a vector can be computed at FFT speed as well. Also note that for an inhomogeneous slab the system matrix $\mathbf{K}$ is not Toeplitz, but for a homogeneous slab with a constant contrast χ we have $\mathbf{K} = \mathbf{I} - \chi\mathbf{G}$ and in this special case the system matrix is Toeplitz.

3 CONSTRUCTION OF THE PRECONDITIONER

As we have seen in the previous section, the system that we need to solve is $\mathbf{K}\mathbf{e}_z = \mathbf{e}_z^{\mathrm{inc}}$, where the system matrix is given by $\mathbf{K} = \mathbf{I} - \mathbf{G}\mathbf{X}$. We solve this system with an iterative solver since the action of matrix $\mathbf{K}$ on a vector can be computed via FFTs [2]. We prefer to maintain this FFT speed property when including a preconditioner for the above problem. To this end, we introduce a preconditioner whose action on a vector can also be computed at FFT speed.

In particular, let $\mathbf{M}$ be a nonsingular matrix and instead of solving Eq. (12), let us solve the preconditioned system

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{e}_z = \mathbf{M}^{-1}\mathbf{e}_z^{\mathrm{inc}}, \tag{16}$$

with $\mathbf{M}$ a preconditioner of the form $\mathbf{M} = \mathbf{I} - \mathbf{C}$, where $\mathbf{C}$ is a circulant matrix of order N given by

$$\mathbf{C} = \underset{\mathbf{Z}\ \text{circulant}}{\mathrm{argmin}} \|\mathbf{G}\mathbf{X} - \mathbf{Z}\|_\mathrm{F}, \tag{17}$$

and $\|\cdot\|_\mathrm{F}$ denotes the Frobenius norm. This circulant is known as the optimal circulant preconditioner and was introduced by T. Chan [3]. For Toeplitz matrices, the elements of the optimal circulant can be given explicitly in terms of the elements of the Toeplitz matrix. We now show that this property carries over to the volume scattering matrix $\mathbf{G}\mathbf{X}$. In particular, if the write the first column of matrix $\mathbf{C}$ as $\mathbf{c} = [c_0, c_1, ..., c_{N-1}]^T$ and introduce the cumulative contrast values

$$s_i = \sum_{j=1}^{N-i} \chi_j \quad \text{for } i = 0, 1, ..., N-1, \tag{18}$$

then the elements of the optimal circulant are given by

$$c_i = \frac{g_{N-i}(s_0 - s_i) + g_i s_i}{N} \quad \text{for } i = 0, 1, ..., N-1. \tag{19}$$

Notice that for $i=0$, we have

$$c_0 = g_0 \frac{s_0}{N} = g_0 \frac{1}{N} \sum_{j=1}^{N} \chi_j,$$

showing that the elements on the main diagonal of $\mathbf{C}$ are equal to the arithmetic average of the contrast values multiplied by the diagonal element of $\mathbf{G}$.

Formula (19) is obtained by essentially following the same steps as for Chan's preconditioner. We start by writing out $F = \|\mathbf{GX} - \mathbf{C}\|_{\mathrm{F}}^2$ in components. This gives

$$\begin{aligned} F = & \sum_{j=1}^{N} |g_0 \chi_j - c_0|^2 \\ & + \sum_{i=1}^{N-1} \sum_{j=N-i+1}^{N} |g_{N-i} \chi_j - c_i|^2 \\ & + \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} |g_i \chi_j - c_i|^2, \end{aligned} \tag{20}$$

where the first term on the right-hand side describes the mismatch on the diagonal and the second and third term describe the mismatch on the strictly upper and lower triangular parts of $\mathbf{GX}-\mathbf{C}$, respectively. Taking the derivative of F with respect to c_i and setting the result to zero, we obtain Eq. (19). Notice that the above optimal circulant reduces to T. Chan's preconditioner for Toeplitz matrices if the scatterer is homogeneous since then $\mathbf{X} = \chi \mathbf{I}$ for some constant contrast coefficient χ and matrix $\mathbf{GX}$ simplifies to the Toeplitz matrix $\chi \mathbf{G}$.

Having found the elements of the circulant, we now exploit the fact that a circulant matrix is diagonalized by the DFT matrix $\mathbf{F}$. In particular, we have $\mathbf{C} = \mathbf{F}^H \mathbf{D} \mathbf{F}$, where $\mathbf{D} = \sqrt{N} \mathrm{diag}(\mathbf{Fc})$ is a diagonal matrix with the eigenvalues of matrix $\mathbf{C}$ on its diagonal. With the help of this eigendecomposition, we can write

$$\mathbf{M}^{-1} = \mathbf{F}^H (\mathbf{I} - \mathbf{D})^{-1} \mathbf{F},$$

showing that the action of $\mathbf{M}^{-1}$ on a vector can be computed at FFT speed as well.

Finally, we mention that an alternate expression for the optimal circulant can be obtained from its definition. Using the eigendecomposition, we have

$$\|\mathbf{GX}-\mathbf{Z}\|_{\mathrm{F}} = \|\mathbf{GX}-\mathbf{F}^H\mathbf{DF}\|_{\mathrm{F}} = \|\mathbf{FGXF}^H-\mathbf{D}\|_{\mathrm{F}},$$

since the Frobenius norm is unitarily invariant. The last norm is minimized by taking $\mathbf{D}$ equal to the diagonal of $\mathbf{FGXF}^H$. In other words, the norm is minimized for $\mathbf{D} = \mathrm{d}\left(\mathbf{FGXF}^H\right)$, where $\mathrm{d}(\mathbf{A})$ is a diagonal matrix with the diagonal entries of matrix $\mathbf{A}$ on its diagonal. The optimal circulant for $\mathbf{GX}$ can now also be written as

$$\mathbf{C} = \mathbf{F}^H \mathrm{d}\left(\mathbf{FGXF}^H\right) \mathbf{F}. \tag{21}$$

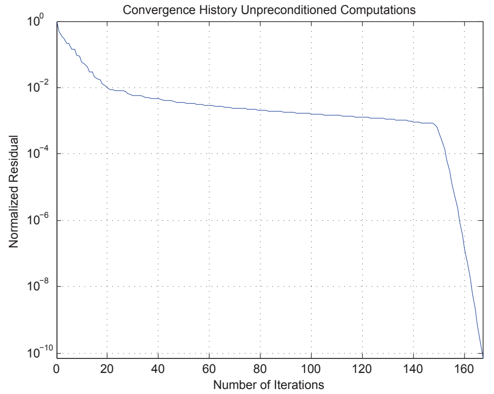

Figure 1: Convergence history of unpreconditioned GMRES for a homogeneous slab with $d = 4\lambda_0$ and a contrast $\chi = 128$ (dip = 0).

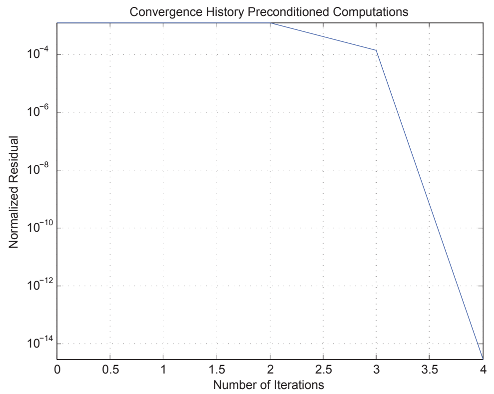

Figure 2: Convergence history of preconditioned GMRES for a homogeneous slab with $d = 4\lambda_0$ and a contrast $\chi = 128$ (dip = 0).

4 NUMERICAL RESULTS

In our numerical experiments, we consider a slab located in vacuum. The width of the slab is $d = 4\lambda_0$, where λ_0 is the wavelength in vacuum. The slab consists of three layers and all three layers have a width $d/3$. Furthermore, the outer layers have a contrast $\chi = 128$, while the contrast of the middle layer is given by

$$\chi_{\mathrm{mid}} = 128 - \mathrm{dip},$$

where $\mathrm{dip} \geq 0$ is the magnitude of the dip in the contrast profile.

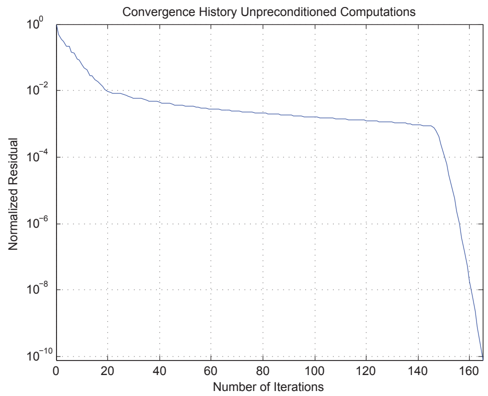

Figure 3: Convergence history of unpreconditioned GMRES for an inhomogeneous slab with $d = 4\lambda_0$ and dip $= 8$.

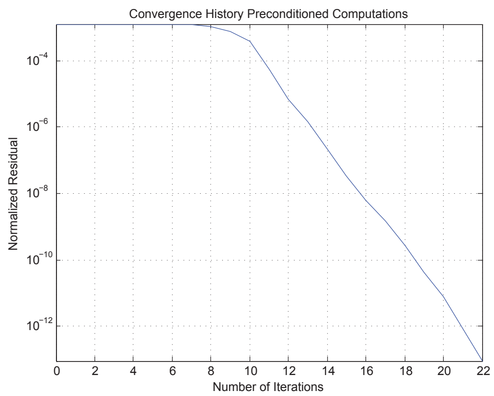

Figure 4: Convergence history of preconditioned GMRES for an inhomogeneous slab with $d = 4\lambda_0$ and dip $= 8$

As a first example, let us consider a homogeneous slab and take dip $= 0$. The integral equation is solved using the GMRES iterative solver with a vanishing total field as an initial guess. The iteration process is terminated as soon as the normalized residual falls below `1e-10`. Figure 1 shows the convergence history of unpreconditioned GMRES for this problem. We observe that it takes about 170 iterations to reduce the normalized residual to `1e-10`. If we now include our preconditioner (which coincides with T. Chan's preconditioner in this case), we obtain a dramatic improvement as illustrated in Figure 2. The normalized residual falls below `1e-10` after only four iterations of preconditioned GMRES in this case.

Subsequently, we perturb the Toeplitz structure of the object equation operator and take dip $= 8$. The convergence histories of unpreconditioned and preconditioned GMRES are shown in Figures 3 and 4, respectively. We observe that preconditioned GMRES again outperforms unpreconditioned GMRES, but the number of iterations has increased compared with the number of iterations required for the homogeneous slab. Further experimentation shows that the performance of the preconditioner worsens for increasing dip-values and improves again if the width of the middle layer is decreased. Based on these results, we conclude that the preconditioner is very efficient provided that contrast variations with respect to the average contrast of the object are electrically not "too large." Presently, we are trying to find a criterion that allows us to determine in a quantitative manner for what contrast variations the circulant preconditioner will be effective.

5 CONCLUSIONS

We have presented an explicit circulant preconditioner for Toeplitz-like matrices that result from a spatial discretization of the electric field domain integral equation. The preconditioner is very effective for inhomogeneous objects provided that the contrast variations of the object are not "too large." In the extreme case of a homogeneous object, the preconditioner dramatically improves the convergence rate of an iterative solver and coincides with the optimal circulant preconditioner of T. Chan for Toeplitz matrices. Future work focuses on extending the present preconditioning technique to scalar two- and three-dimensional scattering problems. The preconditioning technique may also be very effective for two- and three-dimensional vectorial problems. Such problems, however, are much more complex, since the electric field integral equation then contains a gradient-divergence operator and additional preconditioning techniques may be required.

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