Sparsity-aware Kalman Tracking of Target Signal Strengths on a Grid

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Abstract—Tracking multiple moving targets is known to be challenged by the nonlinearity present in the measurement equation, and by the computationally burdensome data association task. This paper introduces a grid-based model of target signal strengths leading to linear state and measurement equations, that can afford state estimation via sparsity-aware Kalman filtering (KF), and bypasses data association. Leveraging the sparsity inherent to the novel grid-based model, a sparsity-cognizant KF tracker is developed that effects sparsity through ℓ_1-norm regularization. The proposed tracker does not require knowledge of the number of targets or their signal strengths, and exhibits considerably lower complexity than the hidden Markov filter benchmark, especially as the number of targets increases. Numerical simulations demonstrate that the sparsity-cognizant tracker enjoys improved root mean-square error performance at reduced complexity when compared to its sparsity-agnostic counterparts.

Keywords: Multi-target tracking, Kalman filter, sparsity, compressed sensing.

I. Introduction

The major importance and continuously expanding interest in target tracking research and development are testified by the gamut of traditional and emerging applications, which include radar- and sonar-based systems, surveillance and habitat monitoring using distributed wireless sensors, collision avoidance modules envisioned for modern transportation systems, and mobile robot localization and navigation in the presence of static and dynamic obstacles, to name a few; see e.g., [3], [7], and references thereon.

At the core of long-standing research issues even for single-target tracking applications is the *nonlinear dependence* of the measurements on the desired state estimates, which challenges the performance of linearized Kalman filter (KF) trackers, including the extended (E)KF, the unscented (U)KF, and their iterative variants [3], [7]. This has motivated the development of particle filters (PF), which can cope with nonlinearities but tend to incur prohibitively high *complexity* in many critical applications. For multi-target tracking, *data association* constitutes another formidable challenge, especially when the ambient environment is cluttered, and the sensors deployed are unreliable. This challenge amounts to determining the target responsible for each measurement; see e.g., [7]. Once data association is established, targets can be tracked separately using the associated measurements, in conjunction with track fusion for improved accuracy.

Recent multi-target tracking schemes aim at bypassing data association at the price of tracking less informative estimates. Two such representatives are the probability hypothesis density (PHD) filter [19], and the Bayesian occupancy filter (BOF) [10]. The PHD filter tracks the so-termed target intensity, while the BOF adopts a grid-based model to describe target occupancy, and tracks the probability of a grid point being occupied by any target. PHD and BOF neither differentiate nor label individual targets, but rather determine the probabilistic presence of targets in space and thus bypass the more informative but also considerably complex task of data association.

The present paper develops a multi-target scheme for tracking target signal strengths. The latter are modeled using a state vector with entries representing the strength of signals emitted or reflected by targets located on (or close to) grid points of known positions. This grid-based state-space model bears resemblance to the BOF, and shares similar advantages in terms of avoiding data association. The main difference here is that state estimation becomes possible via KF applied to a linear state and measurement model at considerably reduced computational burden relative to the complexity incurred by the BOF. This is because the novel *grid-based* tracker exploits the *sparsity* present in the state vector, and leverages efficient solvers of (weighted) least-squares (LS) minimization problems regularized by the ℓ_1-norm of the desired state estimate.

Sparsity-aware estimators have been studied for variable selection in *static* linear regression problems, and have recently gained popularity in signal processing and various other fields in the context of compressive sampling; see e.g., [4], [8], [9], [14]. However, few results pertain to the *dynamic* scenario encountered with target tracking. When measurements arrive sequentially in time, a sparsity-aware recursive least-squares scheme was reported in [1], but its tracking capability is confined only to slow model variations; see also [2] for a sparsity-cognizant smoothing scheme which does not lend itself to filtering; and also [18], where a so-called KF-CS-residual scheme is reported for tracking slowly varying sparsity patterns. Different from existing alternatives, the present paper develops a sparsity-aware Kalman tracker along with its error covariance, without requiring knowledge

of the number of (possibly fast-moving) targets or their signal strengths. Alongside the sparsity-aware KF tracker proposed here, a sparsity-cognizant iterated extended KF (IEKF) tracker is developed in [12] which accommodates sparsity by viewing it as an extra measurement. The IEKF tracker also allows for the development of more accurate error covariance matrices.

The rest of the paper is organized as follows. Section II develops the novel grid-based sparse model, for which a sparsity-agnostic KF tracker is introduced in Section III. The sparsity-cognizant KF tracker is presented in Section IV for a single-target setup. The multi-target scenario is treated in Section V. Numerical results are presented in Section VI, followed by concluding remarks in Section VII.

II. GRID-BASED STATE SPACE MODEL

Consider the problem of tracking M moving targets using N active (e.g., radar) or passive (e.g., acoustic) sensors deployed to provide situational awareness over a geographical area. Associated with each target, say the mth one per time k, is its position vector $\mathbf{p}_k^{(m)}$, and the signal of strength $s_k^{(m)}$ that the target reflects or emits. Sensor n measures the superposition of received target signals, namely

$$y_{n,k} = \sum_{m=1}^{M} h(d_k^{(m,n)}) s_k^{(m)} + \nu_{n,k}, \quad n = 1, \ldots, N \quad (1)$$

where $h(\cdot)$ denotes the distance-dependent propagation function; $d_k^{(m,n)} := \|\mathbf{p}_k^{(m)} - \mathbf{q}_n\|_2$ is the distance between the known position $\mathbf{q}_n$ of sensor n and the unknown position vector $\mathbf{p}_k^{(m)}$ of target m; and, $\nu_{n,k}$ is zero-mean Gaussian noise at sensor n. Function $h(\cdot)$ satisfies $h(0) = 1$, is non-negative, decreasing, and is either assumed known from the physics of propagation or it is acquired through training [14].

At each time k, a centralized processor has available the measurement vector $\mathbf{y}_k := [y_{1,k}, y_{2,k}, \ldots, y_{N,k}]^T$, based on which the target positions $\{\mathbf{p}_k^{(m)}\}_{m=1}^M$ are to be tracked. For clarity in exposition, consider first the single-target scenario ($M = 1$), and drop the superscript (m) until Section V, where the multi-target extension is discussed.

The major challenge in tracking and localization problems is that the measurements in (1) are nonlinear functions of the wanted target position vectors. A neat approach to arrive at a linear measurement model is to adopt a set of G (possibly regularly spaced) grid points at known positions $\{\mathbf{g}_i\}_{i=1}^G$, where target(s) could be potentially located; see also e.g., [10], [9], and [4]. Using a sufficiently dense grid, it is possible to capture the target location at a prescribed spatial resolution using a $G \times 1$ vector $\mathbf{x}_k$ having all entries equal to zero except for the i_k-th entry, $x_k^{(i_k)}$, which is equal to the target signal strength at time k if and only if the target is located at the i_k-th grid point, that is $\mathbf{p}_k = \mathbf{g}_{i_k}$. Note that if the target is located exactly on a grid point i_k, then $x_k^{(i_k)} \equiv s_k \neq 0$ will be the only nonzero entry of $\mathbf{x}_k$. However, to account for target presence off the preselected grid points, it will be allowed for the unknown target signal strength s_k to "spill over" grid points around i_k and thus render nonzero a few

neighboring entries of $\mathbf{x}_k$. For $M = 1$, these considerations lead to a measurement equation given by [cf. (1)]

$$y_{n,k} = \sum_{i=1}^{G} h(d^{(i,n)}) x_k^{(i)} + v_{n,k} = \mathbf{h}_n^T \mathbf{x}_k + v_{n,k} \quad (2)$$

where $\mathbf{h}_n^T := [h(d^{(1,n)}), h(d^{(2,n)}), \ldots, h(d^{(G,n)})]^T$; $d^{(i,n)} := \|\mathbf{q}_n - \mathbf{g}_i\|_2$ now denotes the known time-invariant distance between the nth sensor and the ith grid point; and the noise $v_{n,k}$ replacing $\nu_{n,k}$ in (1) captures the unmodeled dynamics in the aforementioned "spill over" effect. Notwithstanding, the measurements are here linear functions of the unknown $\mathbf{x}_k$ whose nonzero entries reveal the grid point(s) where target signal strength is present at time k.

The next step is to model the evolution of $\mathbf{x}_k$ in time as the target moves across the grid. To this end, consider expressing each entry of $\mathbf{x}_k$ as $x_k^{(j)} = s \cdot p(x_k^{(j)} \neq 0)$, where $s \geq 0$ denotes a nonnegative proportionality constant, and $p(x_k^{(j)} \neq 0)$ stands for the probability of the target signal strength to be present on grid point j at time k. Invoking a total probability argument yields $p(x_k^{(j)} \neq 0) = \sum_{i=1}^{G} p(x_k^{(j)} \neq 0, x_{k-1}^{(i)} \neq 0)$, which after employing Bayes' rule leads to

$$p(x_k^{(j)} \neq 0) = \sum_{i=1}^{G} f_k^{(ij)} p(x_{k-1}^{(i)} \neq 0) \quad (3)$$

where $f_k^{(ij)} := p(x_k^{(j)} \neq 0 | x_{k-1}^{(i)} \neq 0)$ denotes the probability that the target moves from grid point i at time $k-1$ to grid point j at time k. Since each $\mathbf{x}_k$ entry is proportional to the probability of the target located at the corresponding grid point, (3) implies the following Gauss-Markov recursion

$$x_k^{(j)} = \sum_{i=1}^{G} f_k^{(ij)} x_{k-1}^{(i)}, \quad j = 1, \ldots, G. \quad (4)$$

Concatenating (4) for $j = 1, \ldots, G$, and (2) for $n = 1, \ldots, N$, one arrives at the *grid-based* model

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} \quad (5a)$$
$$\mathbf{y}_k = \mathbf{H} \mathbf{x}_k + \mathbf{v}_k \quad (5b)$$

where the $G \times G$ state transition matrix $\mathbf{F}_k$ has entry (i,j) given by $f_k^{(ji)}$; the measurement matrix is defined as $\mathbf{H} := [\mathbf{h}_1, \ldots, \mathbf{h}_n]^T$; and likewise for the noise vector $\mathbf{v}_k := [v_{1,k}, \ldots, v_{N,k}]$.

A distinct feature of model (5) is that the unknown $\mathbf{x}_k$ is *sparse* $\forall k$, since only a few out of the G entries are nonzero (in fact exactly one if the target is located on the grid). This sparsity attribute will prove to be instrumental for enhancing tracking performance. Given $\mathbf{y}_{1:k} := \{\mathbf{y}_1^T, \ldots, \mathbf{y}_k^T\}^T$, the goal of this paper is to track $\mathbf{x}_k$ using a *sparsity-aware* Kalman filter (KF).

Having available $\hat{x}_k^{(j)}$ estimates, and recalling that $x_k^{(j)} := s \cdot p(x_k^{(j)} \neq 0)$, one can estimate the constant capturing the target's signal strength at time k as [cf. (3)]

$$\hat{s}_k = \sum_{j=1}^{G} \hat{x}_k^{(j)}$$

and the target's position vector at time k as

$$\hat{\mathbf{p}}_k = (1/\hat{s}_k) \sum_{j=1}^{G} \mathbf{g}_j \hat{x}_k^{(j)}. \quad (6)$$

In addition to reduced complexity, an attractive feature of the present formulation relative to e.g., [10] is that even for finite G, there is no need to assume that the target is located on grid points since (6) allows for interpolating the target position vectors regardless. The following remark is useful to appreciate this point.

Remark 1. Given measurements $\mathbf{y}_{1:k}$ and supposing that the target signal strength s is known, the maximum a posteriori probability (MAP) optimal tracker of $p(\mathbf{x}_k|\mathbf{y}_{1:k})$ is a hidden Markov model (HMM) filter implementing the following recursions

$$p\left(x_k^{(j)} \neq 0|\mathbf{y}_{1:k-1}\right) = \sum_{i=1}^{G} f_k^{(ij)} \, p\left(x_{k-1}^{(i)} \neq 0|\mathbf{y}_{1:k-1}\right)$$

$$p\left(x_k^{(j)} \neq 0|\mathbf{y}_{1:k}\right) = \frac{p(\mathbf{y}_k|x_k^{(j)} \neq 0)p(x_k^{(j)} \neq 0|\mathbf{y}_{1:k-1})}{\sum_i p(\mathbf{y}_k|x_k^{(i)} \neq 0)p(x_k^{(i)} \neq 0|\mathbf{y}_{1:k-1})}$$

where $f_k^{(ij)}$ is the transition probability as in (3). Knowing $f_k^{(ij)}$ specifies a Markov chain (MC) with the position of the nonzero entry in $\mathbf{x}_k$ determining the current state of the MC. While complexity of these HMM recursions may be affordable for $M = 1$, it becomes prohibitive for multiple targets because the number of MC states, $G!/(G-M)!$, grows combinatorially in M. Even for a single target though, a large G increases grid density and hence spatial resolution at the expense of increasing complexity. In addition to these challenges, the BOF in [10] entails approximations to arrive at related HMM recursions, and requires s and M to be known beforehand.

One more remark is now in order.

Remark 2. Although the measurement vector in (5b) comprises scalar measurements from N geographically distributed sensors per time k, it is possible to form $\mathbf{y}_k$ with samples of the continuous-time signal received at a *single* (e.g., a radar or sonar) sensor by over-sampling at a rate faster than the rate $\mathbf{x}_k$ changes, so long as the state-space model (5) is guaranteed to be observable (and thus $\mathbf{x}_k$ is ensured to be identifiable).

III. KF FOR TRACKING TARGET SIGNAL STRENGTH

If the non-negativity constraints for $\mathbf{x}_k$ were absent, the optimal state estimator for (5) in the MAP, mean-square (MS), or least-squares (LS) error sense would be the KF. The same direction is pursued here in the presence of non-negativity constraints. Suppose that an estimate $\hat{\mathbf{x}}_{k-1|k-1}$ and error covariance matrix $\mathbf{P}_{k-1|k-1}$ are available from the previous time step. At time k, the KF state predictor and its error covariance are obtained as

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1} &= \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} \\ \mathbf{P}_{k|k-1} &= \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k \end{aligned} \quad (7)$$

where $\mathbf{Q}_k$ is a regularizing matrix added to ensure that $\mathbf{P}_{k|k-1}$ remains positive definite. Typically, $\mathbf{Q}_k = \epsilon \mathbf{I}_G$ where $\mathbf{I}_G$ is the identity matrix of size G and $\epsilon > 0$ is small.

For the KF corrector update, consider the LS formulation of the KF; see e.g., [16]. The predictor updates (7) can be derived based on LS too, and as will be seen later on, a regularized form of LS will be useful to effect sparsity-awareness. Viewing $\hat{\mathbf{x}}_{k|k-1}$ as a noisy measurement of $\mathbf{x}_k$, it follows readily from

(7) that $\hat{\mathbf{x}}_{k|k-1} = \mathbf{x}_k + \mathbf{e}_{k|k-1}$, where $\mathbf{e}_{k|k-1}$ has covariance matrix $\mathbf{P}_{k|k-1}$. Stacking $\hat{\mathbf{x}}_{k|k-1}$ and $\mathbf{y}_k$ to form an augmented measurement vector, the following linear regression arises

$$\begin{bmatrix} \hat{\mathbf{x}}_{k|k-1} \\ \mathbf{y}_k \end{bmatrix} = \begin{bmatrix} \mathbf{I}_G \\ \mathbf{H} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \mathbf{e}_{k|k-1} \\ \mathbf{v}_k \end{bmatrix}$$

where the augmented noise vector has block diagonal covariance matrix denoted as $\mathrm{diag}(\mathbf{P}_{k|k-1}, \mathbf{R}_k)$. The weighted (W)LS estimator for this linear regression problem is given by

$$\hat{\mathbf{x}}_{k|k} = \arg \min_{\mathbf{x}_k \geq \mathbf{0}} \|\hat{\mathbf{x}}_{k|k-1} - \mathbf{x}_k\|_{\mathbf{P}_{k|k-1}^{-1}}^2 + \|\mathbf{y}_k - \mathbf{H}\mathbf{x}_k\|_{\mathbf{R}_k^{-1}}^2 \quad (8)$$

where $\|\mathbf{x}\|_{\mathbf{A}}^2 := \mathbf{x}^T \mathbf{A}\mathbf{x}$. In the absence of non-negativity constraints, the optimal state corrector $\hat{\mathbf{x}}_{k|k}$ can be found in closed form as the cost is quadratic, and likewise its error covariance can be updated as

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{H}^T(\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + \mathbf{R}_k)^{-1}\mathbf{H}\mathbf{P}_{k|k-1}. \quad (9)$$

To solve (8) with non-negativity constraints, a gradient projection algorithm will be developed in Section IV. However, (9) will still be used bearing in mind that this update is approximate now. The KF tracker implemented by (7)-(9) is sparsity-agnostic, as it does not explicitly utilize the prior knowledge that $\mathbf{x}_k$ is sparse.

IV. SPARSITY-AWARE KF TRACKERS

Taking into account sparsity, this section develops sparsity-cognizant trackers. To this end, the degree of sparsity quantified by the number of nonzero entries of $\mathbf{x}_k$, namely the ℓ_0-norm $\|\mathbf{x}_k\|_0$, is used to regularize the LS cost of the previous section. Unfortunately, such a regularization results in a non-convex optimization problem that is NP-hard to solve, and motivates relaxing the ℓ_0-norm with its closest convex approximation, namely the ℓ_1-norm. Thus, the proposed sparsity-cognizant tracker is based on the state corrector minimizing the following ℓ_1-regularized WLS cost function

$$\hat{\mathbf{x}}_{k|k} = \arg \min_{\mathbf{x}_k \geq \mathbf{0}} J(\mathbf{x}_k) \quad (10)$$

$$J(\mathbf{x}_k) := \|\hat{\mathbf{x}}_{k|k-1} - \mathbf{x}_k\|_{\mathbf{P}_{k|k-1}^{-1}}^2 + \|\mathbf{y}_k - \mathbf{H}\mathbf{x}_k\|_{\mathbf{R}_k^{-1}}^2 + 2\lambda_k\|\mathbf{x}_k\|_1.$$

The state corrector minimizing (10), together with the covariance update[1] in (9) and the prediction step in (7), form the recursions of the sparsity-aware KF tracker. Relevant design choices and algorithms for minimizing (10) are discussed next.

A. Parameter Selection

The scalar parameter λ_k in (10) controls the sparsity-bias tradeoff [13]. The corrector $\hat{\mathbf{x}}_{k|k}$ becomes increasingly sparse as λ_k increases, and eventually vanishes, i.e., $\hat{\mathbf{x}}_{k|k} = \mathbf{0}$, when λ_k exceeds an upper bound λ_k^*. There are two systematic means of selecting λ_k. The first one popular for variable selection in linear regressions is cross-validation [13, pp. 241-249]. The second one is the so-termed absolute variance deviation (AVD) based selection that has been advocated in

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[1] A more accurate covariance update can be found in [12].

the context of outlier rejection setups [11]. Unfortunately, both methods require solving (10) many times for different trial values of λ_k. This can be acceptable for offline solvers of a linear regression problem or a fixed-interval smoothing scenario, but incurs large delays for real-time applications. For the tracking problem at hand, the simple rule advocated is to set $\lambda_k = \alpha \lambda_k^*$, where $\alpha \in (0, 1)$ is a fixed scaling value to avoid the trivial solution $\hat{\mathbf{x}}_{k|k} = \mathbf{0}$. The bound λ_k^* is derived below.

Proposition 1. *The solution to* (10) *reduces to* $\hat{\mathbf{x}}_{k|k} = \mathbf{0}$ *for any scalar* $\lambda_k \geq \lambda_k^*$, *where*

$$\lambda_k^* = \|\mathbf{P}_{k|k-1}^{-1}\hat{\mathbf{x}}_{k|k-1} + \mathbf{H}^T\mathbf{R}_k^{-1}\mathbf{y}_k\|_\infty. \quad (11)$$

Proof: See [12].

B. Gradient Projection Algorithms

As (10) is a convex problem, convex optimization software such as SeDuMi [15] can be utilized to solve it efficiently. In addition to these solvers, low-complexity iterative methods are developed here, by adopting the gradient projection (GP) algorithms in [6, pp. 212-217]. Note that these can be used to obtain the sparsity-agnostic tracker too, since the latter is obtained by minimizing a special case of (10) corresponding to $\lambda_k = 0$.

At each time k, the GP is initialized with $\hat{\mathbf{x}}_{k|k}(0) = \hat{\mathbf{x}}_{k|k-1}$ at iteration $l = 0$. Corrector iterates from l to $(l + 1)$ are obtained as follows

$$\hat{\mathbf{x}}_{k|k}(l + 1) = \left[\hat{\mathbf{x}}_{k|k}(l) - \gamma\nabla J\left(\hat{\mathbf{x}}_{k|k}(l)\right)\right]^+ \quad (12)$$

where $[x]^+$ denotes the projection onto the non-negative orthant, γ is the step size, and

$$\nabla J(\mathbf{x}) = 2\left(-\mathbf{P}_{k|k-1}^{-1}(\hat{\mathbf{x}}_{k|k-1} - \mathbf{x}) - \mathbf{H}^T\mathbf{R}_k^{-1}(\mathbf{y}_k - \mathbf{H}\mathbf{x}) + \lambda_k\mathbf{1}\right).$$

Here $J(\mathbf{x}_k)$ is differentiable because $\|\mathbf{x}_k\|_1 = \mathbf{x}_k^T\mathbf{1}$ when $\mathbf{x}_k \geq \mathbf{0}$.

While (12) is a Jacobi-type algorithm updating all the entries at once, one can also devise Gauss-Seidel variants, where entries are updated one at a time [6, pp. 218-219]. This is possible because the non-negative orthant is a constraint set expressible as the Cartesian product of one-dimensional sets, allowing entry-wise updates per iteration $(l + 1)$ as

$$\hat{x}_{k|k}^{(j)}(l + 1) = \max\left\{0, \ \hat{x}_{k|k}^{(j)}(l) - \gamma\nabla_j J\left(\tilde{\mathbf{x}}_{k|k}^{(j)}(l)\right)\right\} \quad (13)$$

where $\tilde{\mathbf{x}}_{k|k}^{(j)}(l) := \left\{\hat{\mathbf{x}}_{k|k}^{(1:j-1)}(l+1), \hat{\mathbf{x}}_{k|k}^{(j:G)}(l)\right\}$ has its first $j-1$ entries already updated in the $(l + 1)$st iteration. Convergence of the iterations in (13) to the optimum solution of (10) is guaranteed under mild conditions by the result in [6, p. 219]. Specifically, $J(\mathbf{x}_k)$ should be non-negative and its gradient should be Lipschitz continuous, both of which hold for the objective in (10).

Proposition 2. *Any limit point of the sequence generated by* (13), *with arbitrary initialization* $\hat{\mathbf{x}}_{k|k}^{(0)}$, *is an optimal solution of* (10) *provided that* γ *is chosen small enough.*

In practice, only a few gradient-projection iterations are run per time step k to allow for real-time sparsity-aware KF tracking.

V. EXTENSION TO MULTI-TARGET TRACKING

The target signal strength vector $\mathbf{x}_k$ in (2) can be adjusted to account for multiple targets. Suppose that the mth target moves to the $i_k^{(m)}$-th grid point at time k, and let $\mathbf{x}_k$ have $\bar{M} \geq M$ nonzero entries indexed by $\{i_k^{(m)}\}_{m=1}^{\bar{M}}$; that is, $x_k^{(i_k^{(m)})} = s_k^{(m)}$, and the rest $G - \bar{M}$ entries of $\mathbf{x}_k$ are zero. Clearly, the measurement model for $\mathbf{x}_k$ in (2) still holds, and so does the measurement equation in (5b). Using the same model for target state transitions, the state vector $\mathbf{x}_k$ has now entries given by $x_k^{(j)} := s^{(m)} \cdot p(x_k^{(j)} \neq 0)$. Assume now that all targets are *homogeneous*, in the sense that their transition probabilities are all equal to $\{f_k^{(ij)}\}_{i,j=1}^G$ for every target m. Accordingly, the state recursion in (3) holds, and so does the state equation (5a); see [12] for a detailed derivation.

In a nutshell, the grid-based state space model (5) holds for any $M \geq 1$ under the assumption of homogeneous target state transitions. Because $\mathbf{x}_k$ represents superimposed signal strengths rather than occupancy probabilities, the model applies even when targets are located not necessarily on grid points. The sparsity attribute of $\mathbf{x}_k$ is also present in the multi-target scenario because the combined spatial occupancy of all targets, measured by $\|\mathbf{x}_k\|_0$, is still much smaller than the grid size G. Therefore, the developed KF tracker applies directly for $M \geq 1$.

While $\mathbf{x}_k$ is tracked, two additional steps are needed for tracking multiple targets. First, the target positions per time k must be calculated along with M for each k. This can be accomplished via clustering techniques, such as the k-means algorithm [13]. Suppose that an estimate $\hat{M}$ is acquired and $\hat{\mathbf{x}}_{k|k}$ has been separated into $\hat{M}$ clusters. The mth cluster, represented by $G^{(m)}$, contains indices $i^{(m)} \in G^{(m)}$ with estimated entries $\hat{x}_{k|k}^{(i^{(m)})}$ corresponding to grid points located at position vectors $\mathbf{g}_{i^{(m)}}$. Similar to the single-target case in (6), the target positions can be estimated as

$$\hat{\mathbf{p}}_k^{(m)} = \frac{\sum_{i \in G^{(m)}} \mathbf{g}_i \hat{x}_{k|k}^{(i)}}{\sum_{i \in G^{(m)}} \hat{x}_{k|k}^{(i)}}, \quad m = 1, 2, \ldots, \hat{M}. \quad (14)$$

Second, given $\{\hat{\mathbf{p}}_k^{(m)}\}$, track-to-track association and labeling must be performed to produce target trajectories. This can be done using known methods such as the simple nearest-neighbor algorithm and optimization-based multiple hypotheses testing [3], [7]. (The omitted details along with a joint state-vector tracking and track association approach are provided in [12].)

Nevertheless, it is worth stressing that the developed KF trackers are immune to error propagation caused by position estimation errors, track association errors or even errors in estimating the number of targets, simply because these errors occur after $\hat{\mathbf{x}}_{k|k}$ vectors have been estimated.

Further elaboration on resolving track association issues, enhancing performance using multi-resolution grids, and relaxation of the target homogeneity assumption are interesting future research directions to be pursued.
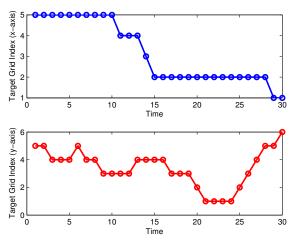
Figure 1. True target track on the grid.

Figure 2. RMSE for sparsity-agnostic and sparsity-aware KF trackers.

VI. NUMERICAL TESTS

Consider a 300×300 plane, in which $N = 10$ sensors are randomly placed. A 10×10 rectangular grid is adopted with equally-spaced grid points. Simulations are performed for both single- and multi-target scenarios.

A single target setup is considered first. The target starts at the center of the grid at time $k = 1$, and moves according to the following model: it stays on the current grid point with probability $1/3$, and moves to one of the four adjacent cells with probability $2/3$, which means moving up, down, right or left with equal probability of $1/6$ each. Whenever the target moves outside the boundaries, it is assumed remaining at its latest position during that time step. One random realization of this movement model is considered for $K_{\max} = 30$ time steps. In Fig. 1, the grid-point indices for the target positions are plotted along the x- and y-directions over time. The target signal strength is $s = 10$, and the measurement noise $v_k^{(n)}$ is zero-mean, independent identically distributed (i.i.d.) Gaussian with unit variance. The propagation function $h(x)$ in (1) is given by $h(x) = c/(c+x^2)$, where c is chosen so that $h(60) = 0.5$. Note that $h(x)$ decreases from $h(0) = 1$, as x increases.

The proposed sparsity-agnostic and sparsity-aware KF trackers in Sections III-IV are employed to estimate the target signal strengths and position vectors. The position estimation accuracy in terms of the average root mean square error (RMSE) is illustrated in Fig. 2, where the average RMSE = $\sqrt{\frac{1}{K_{\max}} \sum_{k=1}^{K_{\max}} \|\hat{\mathbf{p}}_k - \mathbf{p}_k\|_2^2}$. The position estimates are obtained by picking the center of the grid point corresponding to the peak of the estimated target signal strength profile. The x-axis in Fig. 2 is the weighting coefficient λ_k as a fraction of λ_k^* in (11). The sparsity-agnostic tracker corresponds to setting $\lambda_k = 0$ in (10), and results in constant RMSE. Matrix $\mathbf{Q}_k$ in (7) was set equal to $\mathbf{I}_G$. The sparsity-aware tracker performs much better than the sparsity-agnostic one for all $\lambda_k \neq 0$. The value of $\lambda_k = 0.1\lambda_k^*$ appears to be the best value, The optimal HMM filter exhibits the best performance, but requires
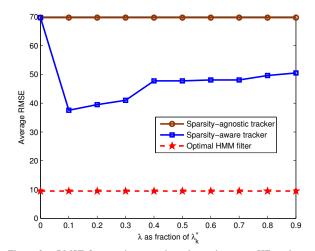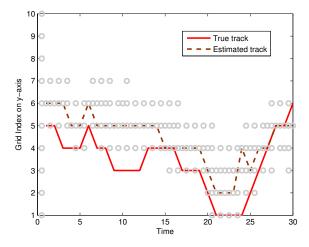

Figure 3. The nonzero support of the estimated target energy profiles over time, alone with true and estimated target trajectories (y-direction only).

knowledge of target signal strength.

Fig. 3 demonstrates the dynamic behavior of the sparsity-aware estimator in (10) with $\lambda_k = 0.9\lambda_k^*$. The estimated state vectors are depicted over time, with a circle representing a nonzero target signal strength at the corresponding grid point. The true target trajectory and the estimated one are plotted as well. For clarity, only the projection of the target track on the y-direction is depicted. It is seen that the "cloud" of nonzero target signal strengths follow the true track trajectory. The estimated target profile is spatially sparse. The size of the nonzero support indicates the uncertainty in target position estimates, which apparently does not grow over time, even when using a simple linear KF tracker to follow the state transition pattern.

In the multi-target setup, two targets are located on northwest and south-east positions on the grid at time $k = 1$. They start moving according to the same movement model used for the single-target case. In addition, if the model decides that the
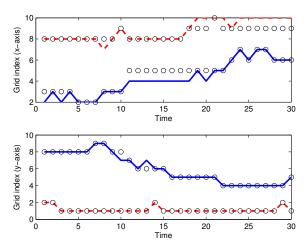
Figure 4. True target tracks and target position estimates ($N = 40$).

target should move into an occupied grid point, then the target does not move for that time step. This avoids target collisions, but still allows targets to get close, on adjacent grid points.

Simulations are performed on a sample trajectory obtained from these two moving targets. For the sparsity-aware KF tracker with $\lambda_k = 0.1\lambda_k^*$, its position estimation accuracy in the multi-target case is tested. The k-means algorithm is used to find clusters in the nonzero target strength profile estimates, and the peak of each cluster is selected to estimate the target position. These position estimates are plotted as circles in Fig. 4, for the x- and y-directions, respectively, along with the true target trajectories. The position estimation performance for the $N = 40$ sensors used here is more accurate than that with $N = 10$.

VII. CONCLUSIONS

The problem of tracking multiple targets on a plane has been investigated. A grid-based state space model was introduced to describe the dynamic behavior of target signal strengths. This model not only renders the nonlinear estimation problem linear, but also facilitates incorporation and exploitation of the sparsity present. A sparsity-aware Kalman tracker promoting sparsity of the state estimates through ℓ_1-norm minimization was developed. The proposed tracker does not require knowledge of the number of targets or their signal strengths, and incurs markedly lower complexity than the optimal hidden Markov model filter. It offers improved tracking performance at reduced sensing cost, especially when compared to sparsity-agnostic trackers.

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