RANGING ENERGY OPTIMIZATION FOR ROBUST SENSOR POSITIONING

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ABSTRACT

We address ranging energy optimization for an unsynchronized localization system, which features robust sensor positioning, in the sense that specific accuracy requirements are fulfilled within a prescribed service area. Optimization problems related to the ranging energy of a sensor and beacons are proposed, after which a practical algorithm based on semidefinite programming is presented. The effectiveness of the algorithm is illustrated by a numerical experiment.

Index Terms— Cramér-Rao bound, semidefinite programming, localization

1. INTRODUCTION

As an important component of a wireless sensor network, sensor localization has been attracting intensive research interest. To position an untethered sensor, algorithms based on ranging and fusion are commonly used [1, 2, 3]. More specifically, ranging is first performed to estimate the distances between sensors and beacons with known positions. Then, the sensor positions are estimated by fusion in a centralized or multi-hop manner. Thanks to the superior multipath resolution capability of ultra-wideband (UWB), time-of-arrival (TOA) based ranging using UWB pulses is practically preferred [2]. Particularly, the TOA-based two-way ranging (TWR) is supported by the IEEE 802.15.4a standard [1].

For the ranging-and-fusion type of algorithms, the positioning accuracy improves if the ranging energy of sensors and beacons is enhanced [2, 3]. In real scenarios, a beacon might have a reliable power supply so its ranging energy can be easily increased, but the ranging energy of an untethered sensor must be reduced in order to prolong the system lifetime. Therefore, the positioning accuracy is actually dominated by the sensor ranging energy, which should be small but sufficiently high to fulfill prescribed accuracy requirements. This motivates us to investigate the following ranging energy optimization problem: how to allocate the ranging energy to the sensors and beacons, so that the sensor ranging energy is minimized and specific accuracy requirements are fulfilled as well?

We will address this problem for an unsynchronized robust sensor positioning (RSP) system, which consists of power-supplied beacons connected to a central processing unit (CPU),

as well as sensors randomly deployed within a prescribed service area. The positioning is implemented by TOA-based TWR between the beacons and a sensor, followed by a fusion at the CPU to produce a position estimate. In particular, this system features RSP, in the sense that specific accuracy requirements are fulfilled within the service area. To reduce the implementation complexity, the ranging energy of both the sensor and beacons, is fixed and determined in the system design phase.

We assume that the most favorable TOA and position estimators are used. It is well known that the Cramér-Rao bound (CRB) sets a lower bound on the variance of any unbiased estimator, and the maximum likelihood estimator asymptotically attaining the CRB produces an unbiased estimator with Gaussian distribution [4]. Consequently, we will assume the considered system produces unbiased and Gaussian distributed TOA and position estimates achieving the CRB. The optimization result can be used as a benchmark to evaluate the energy efficiency of other localization algorithms.

The rest of this paper is organized as follows. In the next section, we will derive the positioning CRB, as well as a sufficient condition for RSP. Then, Section 3 will formulate the considered optimization problems and propose a practical algorithm based on semidefinite programming (SDP). In Section 4, we will illustrate the effectiveness of the proposed algorithm by a numerical experiment. Finally, we wrap up this paper by some conclusions in Section 5.

2. RSP SYSTEM SETUP AND PERFORMANCE

In this section, the RSP system is first described. Then, the TOA-based TWR procedure is introduced, and the positioning CRB is formulated. Finally, a sufficient condition for RSP is derived.

2.1. RSP system setup and parameters

We consider a 2D RSP system, with M beacons deployed and connected to a CPU through wired or radio links ($M \geq 3$). The m-th beacon is deployed at a known coordinate $\mathbf{p}_m = [x_m, y_m]^T$, ($m = 1, 2, \cdots, M$) and the sensor at an unknown coordinate $\mathbf{u} = [x, y]^T$ within a prescribed service area S. In addition, the clocks of the sensor and beacons are unsynchronized but run at the same pace. We assume the two-sided

power spectral density of the additive white Gaussian noise at the sensor and the beacons is respectively $N_s/2$ and $N_b/2$.

Assume the channel between the sensor and beacon m has a line of sight (LOS) path, which incurs a propagation delay $d_m(\mathbf{u})/c$ and an attenuation a_m. Here, c and $d_m(\mathbf{u})$ represent the signal propagation speed and the distance between the sensor and beacon m, respectively. In addition, we assume $a_m^2 = \alpha d_m(\mathbf{u})^{-\beta}$, where α and β refer to the path gain at 1m and the path-loss coefficient, respectively. During the ranging phase, both the beacons and the sensor will broadcast ranging signals, which consist of UWB pulses modulated by known data symbols and seperated sparsely, in order to eliminate multipath interference and improve the TOA estimation from the LOS signal component. More specifically, the sensor (or beacon m) broadcasts a pulse train of energy E_s (or E_m). The TOA estimation CRB at beacon m (or the sensor) can be expressed as $\sigma_{s \to m}^2 (\sigma_{m \to s}^2)$ [2]:

$$\sigma_{s \to m}^2 = \frac{N_b}{2a_m^2 \omega_s^2 E_s}, \ \sigma_{m \to s}^2 = \frac{N_s}{2a_m^2 \omega_b^2 E_m} \quad (1)$$

where $\omega_s = \sqrt{\frac{\int \omega^2 |S(\omega)|^2 d\omega}{\int |S(\omega)|^2 d\omega}}$ and $\omega_b = \sqrt{\frac{\int \omega^2 |B(\omega)|^2 d\omega}{\int |B(\omega)|^2 d\omega}}$ represent respectively the root-mean-square frequency associated with $|S(\omega)|^2$ and $|B(\omega)|^2$. Here, $S(\omega)$ and $B(\omega)$ are the spectrum of the UWB pulses used by the sensor and beacons, respectively.

2.2. TOA-based TWR and positioning CRB

During the TOA-based TWR, $\hat{d}_m(\mathbf{u})$ is produced as an estimate of $d_m(\mathbf{u})$ [1]. First, the CPU schedules the beacons to broadcast ranging signals sequentially, so that they are seperated when arriving at the sensor. Let's say beacon m broadcasts a ranging signal of energy E_m at time $T_{m,0}$, and the sensor estimates its TOA as $T_{m,1} + e_{m,1}$, where $T_{m,1}$ and $e_{m,1}$ represent the exact TOA and the estimation error, respectively. After the sensor has generated all the TOA's, it broadcasts back a ranging signal of energy E_s at time T_s to the beacons. The associated TOA at beacon m is estimated as $T_{m,2} + e_{m,2}$, where $T_{m,2}$ and $e_{m,2}$ represent the exact TOA and the estimation error, respectively. We assume the calibration can be perfectly accomplished so that $T_{m,0}$ and T_s are precisely known by beacon m and the sensor, respectively. Finally, both the processing delay $T_s - (T_{m,1} + e_{m,1})$ produced by the sensor, and the total delay $(T_{m,2} + e_{m,2}) - T_{m,0}$ generated by beacon m, are transmitted through data packets to the CPU, which evaluates $\hat{d}_m(\mathbf{u})$ as:

$$\hat{d}_m(\mathbf{u}) = \frac{c}{2}[(T_{m,2} + e_{m,2} - T_{m,0}) - (T_s - T_{m,1} - e_{m,1})]$$
$$= d_m(\mathbf{u}) + \frac{c(e_{m,2}+e_{m,1})}{2} \quad (2)$$

We assume the TOA estimators attain the CRB with an unbiased Gaussian distribution. This means that $e_{m,1}$ and $e_{m,2}$ are zero mean Gaussian random variables with variance $\sigma_{m \to s}^2$ and $\sigma_{s \to m}^2$, respectively. In addition, $e_{m,1}$ and $e_{m,2}$ are

independent since they are estimated using independent signals at the sensor and beacon m, respectively. Consequently, $\hat{d}_m(\mathbf{u})$ is Gaussian distributed with mean $d_m(\mathbf{u})$ and variance $\kappa_m^2(\mathbf{u})$ expressed as:

$$\kappa_m^2(\mathbf{u}) = \frac{c^2}{4}(\sigma_{s \to m}^2 + \sigma_{m \to s}^2)$$
$$= \rho a_m^{-2}(E_s^{-1} + \gamma E_m^{-1}) \quad (3)$$

where $\rho = \frac{c^2 N_b}{8\omega_s^2}$ and $\gamma = \frac{N_s}{N_b} \frac{\omega_s^2}{\omega_b^2}$. In fact, γ represents the TOA estimation accuracy at the sensor relative to that at beacon m when $E_s = E_m$ ($\gamma < 1$ means that the TOA estimation at the sensor is more accurate).

After TWR, we have a set of independent data $\{\hat{d}_m(\mathbf{u})\}_{m=1}^M$ for estimating $\mathbf{u}$. The associated CRB can be evaluated as the inverse of the corresponding Fisher information matrix $\mathbf{F}(\mathbf{x}, \mathbf{u})$, which is formulated as [3]:

$$\mathbf{F}(\mathbf{x}, \mathbf{u}) = \sum_{m=1}^M \frac{(\mathbf{u} - \mathbf{p}_m)(\mathbf{u} - \mathbf{p}_m)^T}{(\kappa_m(\mathbf{u})d_m(\mathbf{u}))^2} = \sum_{m=1}^M x_m \mathbf{F}_m(\mathbf{u})$$
$$(4)$$

where $\mathbf{F}_m(\mathbf{u}) = \alpha\rho^{-1} d_m(\mathbf{u})^{-\beta-2}(\mathbf{u} - \mathbf{p}_m)(\mathbf{u} - \mathbf{p}_m)^T$ and $\mathbf{x} = [x_1, \cdots, x_M]^T$. Here, $x_m = (E_s^{-1} + \gamma E_m^{-1})^{-1}$ can be regarded as the effective energy that blends the effects of E_s and E_m on the CRB. Notice that $x_m \le min\{E_s, \frac{E_m}{\gamma}\}$, and x_m is an increasing function of both E_s and E_m.

2.3. A sufficient condition for RSP

We assume the position estimate $\hat{\mathbf{u}}$ is Gaussian distributed with mean $\mathbf{u}$ and covariance $\mathbf{F}(\mathbf{x}, \mathbf{u})^{-1}$. The RSP requirement we consider is that, the estimation error $\mathbf{e} = \mathbf{u} - \hat{\mathbf{u}}$ should fall within a circle $C = \{\mathbf{e}|\mathbf{e}^T\mathbf{e} \le R_c^2\}$ with a probability higher than P_c for any $\mathbf{u} \in S$, where R_c is the radius specified by the regulation authority. It is shown in [5] that $\mathbf{e}$ falls within the ellipse $E = \{\mathbf{e}|\mathbf{e}^T\mathbf{F}(\mathbf{u})\mathbf{e} \le \phi = -2ln(1 - P_c)\}$ with probability P_c. This ellipse has a major principal axis of length $\sqrt{\phi/\lambda_{min}(\mathbf{x}, \mathbf{u})}$, where $\lambda_{min}(\mathbf{x}, \mathbf{u})$ denotes the minimal eigen-value of $\mathbf{F}(\mathbf{x}, \mathbf{u})$. Therefore, a sufficient condition for RSP is that, $\forall \mathbf{u} \in S$, $R_c \ge \sqrt{\phi/\lambda_{min}(\mathbf{x}, \mathbf{u})}$, or equivalently, $\forall \mathbf{u} \in S$, $\lambda_{min}(\mathbf{x}, \mathbf{u}) \ge \lambda_c = -2ln(1-P_c)/R_c^2$. We want the considered system to fulfill this RSP constraint.

3. RANGING ENERGY OPTIMIZATION

In this section, we will first formulate a few ranging energy optimization problems. Then, we will present a practical algorithm.

3.1. Ranging energy optimization problems

Since $\lambda_{min}(\mathbf{x}, \mathbf{u})$ is a non-decreasing function of x_m[1], the RSP constraint can be sastisfied by increasing entries of $\mathbf{x}$, which is in turn accomplished by enhancing E_s and E_m. One

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[1]This can be proved using Corollary 4.3.3 in [6].

of the optimization problems we consider is to find the threshold sensor energy E_{th}, above which RSP becomes feasible. Mathematically, this problem can be formulated as:

$$\begin{aligned}
&\min E_{\text{s}} \\
&s.t. \ E_{\text{s}} \geq 0, \ E_m \geq 0, \ m = 1, \cdots, M \\
&\quad \lambda_{min}(\mathbf{x}, \mathbf{u}) \geq \lambda_{\text{c}}, \ \forall \mathbf{u} \in S
\end{aligned} \tag{5}$$

To solve the above problem, consider an absolute value of E_{s} that is feasible. This means that there exists at least one $\mathbf{x}$ that fulfills both the RSP constraint and the constraint: $||\mathbf{x}||_\infty \leq E_{\text{s}}$, where $||\mathbf{x}||_\infty$ denotes the l_∞-norm, namely the maximal entry of $\mathbf{x}$. Then, we can further enhance each entry of $\mathbf{x}$ to its maximal possible value E_{s}, by increasing every E_m to be infinitely high. As a result, $\mathbf{x} = [E_{\text{s}}, \cdots, E_{\text{s}}]^T$ enables RSP as well, since $\lambda_{min}(\mathbf{x}, \mathbf{u})$ is non-decreasing with every x_m. Therefore, E_{th} can be evaluated as the optimal E_{s} for the following problem:

$$\begin{aligned}
&\min E_{\text{s}} \\
&s.t. \ x_m = E_{\text{s}} \geq 0, \ m = 1, \cdots, M \\
&\quad \lambda_{min}(\mathbf{x}, \mathbf{u}) \geq \lambda_{\text{c}}, \ \forall \mathbf{u} \in S
\end{aligned} \tag{6}$$

The optimal E_{s} for the above problem can be expressed as: $E_{\text{th}} = \frac{\lambda_{\text{c}}}{min_{\mathbf{u} \in S}\{\lambda_{min}(\mathbf{X}(\mathbf{u}))\}}$, where $\lambda_{min}(\mathbf{X}(\mathbf{u}))$ refers to the minimal eigen-value of $\mathbf{X}(\mathbf{u}) = \sum_{m=1}^{M} \mathbf{F}_m(\mathbf{u})$. It is important to notice that E_{th} is actually the minimal l_∞-norm of any $\mathbf{x}$ achieving RSP. This implies that at least one E_m has to be infinitely high, when E_{s} is reduced to the level E_{th}.

In real scenarios, the beacon energy E_m is usually constrained by E_{B} because of implementation difficulties, e.g., limited ranging duration and power due to the power-amplifier nonlinearity. Notice that E_{B} must be no less than γE_{th} to achieve RSP. In such cases, the optimization problem is to find the minimal E_{s} and associated E_m which makes RSP feasible. Mathematically, this problem can be formulated as:

$$\begin{aligned}
&\min E_{\text{s}} \\
&s.t. \ E_{\text{s}} \geq 0, \ E_{\text{B}} \geq E_m \geq 0, \ m = 1, \cdots, M \\
&\quad \lambda_{min}(\mathbf{x}, \mathbf{u}) \geq \lambda_{\text{c}}, \ \forall \mathbf{u} \in S
\end{aligned} \tag{7}$$

The solution to the above problem can be constructed by:

$$E_{\text{s}} = (E_{\text{th}}^{-1} - \gamma E_{\text{B}}^{-1})^{-1}, \ E_m = \gamma(x_m^{-1} - E_{\text{th}}^{-1} + \gamma E_{\text{B}}^{-1})^{-1} \tag{8}$$

where x_m is the m-th entry of $\mathbf{x}$, which belongs to the set $\Psi = \{\mathbf{x} : ||\mathbf{x}||_\infty = E_{\text{th}} \text{ and } \lambda_{min}(\mathbf{x}, \mathbf{u}) \geq \lambda_{\text{c}}, \forall \mathbf{u} \in S\}$. The above solution is justified as follows. First of all, it is feasible for (7). Second, $(E_{\text{th}}^{-1} - \gamma E_{\text{B}}^{-1})^{-1}$ is the optimal value of E_{s}, because for any smaller value of E_{s} RSP will not be possible, since the associated $||\mathbf{x}||_\infty$ is smaller than E_{th}. Third, when $E_{\text{s}} = (E_{\text{th}}^{-1} - \gamma E_{\text{B}}^{-1})^{-1}$, every x_m is no greater than E_{th} since $E_m \leq E_{\text{B}}$. To make RSP possible, $\mathbf{x}$ must belong to Ψ, therefore the optimal value of E_m is constructed as in (8).

It is interesting to notice that $\mathbf{x}_{\text{t}} = [E_{\text{th}}, \cdots, E_{\text{th}}]^T \in \Psi$ is the worst one for building a solution to (7), since every other

$\mathbf{x} \in \Psi$ results in a solution using less beacon energy. In order to build a solution with more efficient use of beacon energy, we can find a better $\mathbf{x} \in \Psi$ as the solution to the following problem:

$$\begin{aligned}
&\min \mathbf{w}^T \mathbf{x} \\
&s.t. \ E_{\text{th}} \geq x_m \geq 0, \ m = 1, \cdots, M \\
&\quad \lambda_{min}(\mathbf{x}, \mathbf{u}) \geq \lambda_{\text{c}}, \ \forall \mathbf{u} \in S
\end{aligned} \tag{9}$$

where $\mathbf{w} = [w_1, \cdots, w_M]^T$ is a weighting vector, and w_m denotes the priority assigned to E_m. Specifically, a greater w_m represents a higher priority to reducing E_m. One special case is to set all entries of $\mathbf{w}$ to zero except for $w_m = 1$. Using the associated optimal $\mathbf{x}$ for (9), the constructed solution to (7) reduces E_m to its minimal possible value.

In general, no closed-form solutions exist for (9). Nevertheless, $\lambda_{min}(\mathbf{x}, \mathbf{u}) \geq \lambda_{\text{c}}$ is equivalent to the linear matrix inequality $\mathbf{F}(\mathbf{x}, \mathbf{u}) \succeq \lambda_{\text{c}} \mathbf{I}$. Here, $\mathbf{I}$ denotes the 2×2 identity matrix, and $\mathbf{X} \succeq \mathbf{Y}$ means that $\mathbf{X} - \mathbf{Y}$ is positive semidefinite. Using this equivalence, (9) actually belongs to the class of SDP problems, which can be solved numerically with convex-optimization techniques [7].

3.2. A practical algorithm

To evaluate E_{th} and solve (9), the main difficulty lies in the fact that S is in general a continuous area. A practical method is to replace S with a discrete grid set $G = \{g_n\}_{n=1}^N$ within S. Then, we can easily evaluate E_{th} and solve (9) numerically. Let's denote Ω_{S} and Ω_{G} as the set of $\mathbf{x}$ that achieves RSP over S and G, respectively. Clearly, $\Omega_{\text{G}} \supseteq \Omega_{\text{S}}$. In order to keep the solutions to the earlier optimization problems unchanged, G should be carefully chosen such that $\Omega_{\text{G}} = \Omega_{\text{S}}$.

To find a way to generate G, consider the square cell $C_\triangle$ centered at $\mathbf{g}_{\text{c}}$ with lateral length $\triangle$ within S. In the appendix, we have shown that $\forall \mathbf{x}$, $\lambda_{min}(\mathbf{x}, \mathbf{u})$ is approximately a concave function of $\mathbf{u}$ within this cell, provided that $\triangle \ll d_{min}(\mathbf{g}_{\text{c}})$, where $d_{min}(\mathbf{g}_{\text{c}})$ refers to the distance of $\mathbf{g}_{\text{c}}$ to the closest beacon. As a result, if $\lambda_{min}(\mathbf{x}, \mathbf{u}) \geq \lambda_{\text{c}}$ is fulfilled at each corner point, RSP is achieved for the cell $C_\triangle$ as well.

Based on the above analysis, a method for producing G is introduced as follows. First, we divide S into L regions S_l, $l = 1, 2 \cdots, L$. Then, we sample S_l uniformly with a spacing $\triangle_l \ll min_{\mathbf{u} \in S_l} d_{min}(\mathbf{u})$ in both vertical and horizontal directions to generate a discrete set G_l. Finally, G is produced by combing all G_l as: $G = G_1 \cup \cdots \cup G_L$. According to the above analysis, for any $\mathbf{x}$ that achieves RSP over G, the RSP is fulfilled for S as well, since RSP is attained for each square cell formed by four adjacent points in G, and those cells cover S. This means that $\Omega_{\text{G}} \subseteq \Omega_{\text{S}}$. Since $\Omega_{\text{G}} \supseteq \Omega_{\text{S}}$ is always true, $\Omega_{\text{G}} = \Omega_{\text{S}}$ finally holds. This indicates that the solutions to the earlier problems remain unchanged after replacing S by the grid G generated by the aforementioned method.
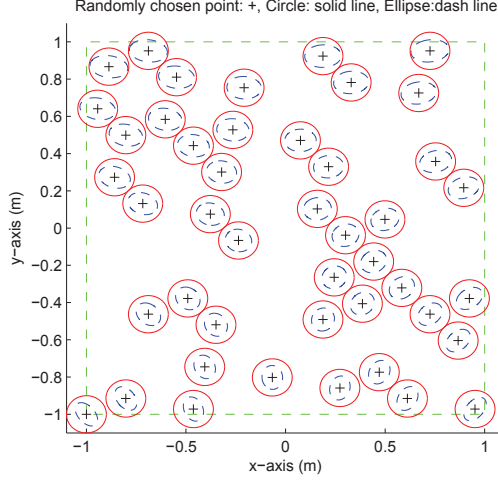
Randomly chosen point: +, Circle: solid line, Ellipse:dash line

Fig. 1. Ellipses and circles for randomly chosen points.

4. NUMERICAL EXPERIMENT

For illustration purposes, consider S as a square centered at $(0,0)$ with lateral length 2 m. There are three beacons located at $\mathbf{p}_1 = [0,8]^T$, $\mathbf{p}_2 = [-3,-3]^T$, and $\mathbf{p}_3 = [3,-3]^T$, respectively. The system parameters are set as: $\alpha = 1$, $\beta = 2$, $c = 3 \times 10^8$ m/s, $\frac{N_s}{2} = \frac{N_b}{2} = 0$ dBW/Hz, $\frac{\omega_s}{2\pi} = \frac{\omega_b}{2\pi} = 8$ GHz, $R_c = 10$ cm and $P_c = 0.8$.

To generate G, we sample the whole S uniformly with a spacing $\triangle = \delta \cdot d$, where $d = min_{\mathbf{u} \in S} d_{min}(\mathbf{u}) = 2.83$ m. During the evaluation, we find that when $\delta \leq 1\%$ the computed E_{th} remains essentially unchanged at -9.45dBJ, which implies that Ω_G is quite close to Ω_S beyond $\delta = 1\%$. Hence, we set $E_{\text{th}} = -9.45$dBJ, and replace S with G produced by $\delta = 1\%$ for the following evaluations.

Assume E_m is upper bounded by $E_B = 10$dBJ, and we would like to build a solution to (7) using (8). In addition, we hope this solution reduces E_1 to its minimal possible value. To this end, we assign $\mathbf{w} = [1,0,0]^T$ and compute the optimal $\mathbf{x}$ for (9) by Sedumi [8]. The result is $\mathbf{x} = [-10.1, -9.45, -9.45]^T$ in dBJ, and the constructed solution using (8) is $E_s = -9.4$dBJ, $E_1 = -1.8$dBJ, $E_2 = 10$dBJ, and $E_3 = 10$dBJ.

To show the effectiveness of the above results, we randomly select a set of points within S. Then, we plot the ellipse $\{\mathbf{e}|\mathbf{e}^T\mathbf{F}(\mathbf{u})\mathbf{e} = -2ln(1-P_c)\}$ and the circle $\{\mathbf{e}|\mathbf{e}^T\mathbf{e} = R_c^2\}$ for each point, using the $\mathbf{x}$ given above. It is shown in Figure 1 that each ellipse is enclosed by the associated circle, which indicates that the RSP is indeed accomplished for those randomly chosen points.

5. CONCLUSIONS

We have considered an RSP system, and proposed related ranging energy optimization problems, which aim to mini-

mize sensor ranging energy as well as fulfill an RSP constraint. A practical algorithm based on SDP has been presented, and we have demonstrated its effectiveness through a numerical experiment.

6. APPENDIX

When $\triangle \ll d_{min}(\mathbf{g}_c)$, $\forall \mathbf{u} \in C_{\triangle}$, $\lambda_{min}(\mathbf{x},\mathbf{u})$ can be approximated as follows:

$$
\begin{aligned}
\lambda_{min}(\mathbf{x},\mathbf{u}) &= \min_{\mathbf{v}:\mathbf{v}^T\mathbf{v}=1} \sum_{m=1}^{M} x_m \mathbf{v}^T \mathbf{F}_m(\mathbf{u}) \mathbf{v} \\
&\approx \min_{\mathbf{v}:\mathbf{v}^T\mathbf{v}=1} \sum_{m=1}^{M} x_m \mu_m \left| \mathbf{v}^T(\mathbf{u} - \mathbf{g}_c + \mathbf{g}_c - \mathbf{p}_m) \right|^2 \\
&\approx \min_{\mathbf{v}:\mathbf{v}^T\mathbf{v}=1} \sum_{m=1}^{M} x_m \mu_m \big(\mathbf{v}^T \mathbf{Z}_m \mathbf{v} + \\
&\qquad\qquad 2\mathbf{v}^T(\mathbf{g}_c - \mathbf{p}_m)\mathbf{v}^T(\mathbf{u} - \mathbf{g}_c) \big)
\end{aligned}
$$

where $\mu_m = \alpha\rho^{-1}d_m(\mathbf{g}_c)^{-\beta-2}$ and $\mathbf{Z}_m = (\mathbf{g}_c - \mathbf{p}_m)(\mathbf{g}_c - \mathbf{p}_m)^T$. Clearly, the expression inside the last bracket is an affine function of $\mathbf{u}$ parameterized by $\mathbf{v}$ and $\mathbf{x}$, so $\lambda_{min}(\mathbf{x},\mathbf{u})$ is approximately a pointwise minimum of this function over $\mathbf{v}$. Therefore, $\forall \mathbf{x}$, $\lambda_{min}(\mathbf{x},\mathbf{u})$ is a concave function of $\mathbf{u}$ [7].

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