Finite Word Length Effects on Transmission Rate in Zero Forcing Linear Precoding for Multichannel DSL

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Abstract—Crosstalk interference is the limiting factor in transmission over copper lines. Crosstalk cancellation techniques show great potential for enabling the next leap in DSL transmission rates. An important issue when implementing crosstalk cancelation techniques in hardware is the effect of finite word length on performance. In this paper, we provide an analysis of the performance of linear zero-forcing precoders, used for crosstalk compensation, in the presence of finite word length errors. We quantify analytically the tradeoff between precoder word length and transmission rate degradation. More specifically, we prove a simple formula for the transmission-rate loss as a function of the number of bits used for precoding, the signal-to-noise ratio, and the standard line parameters. We demonstrate, through simulations on real lines, the accuracy of our estimates. Moreover, our results are stable in the presence of channel estimation errors. Lastly, we show how to use these estimates as a design tool for DSL linear crosstalk precoders. For example, we show that for standard VDSL2 precoded systems, 14 bit representation of the precoder entries results in capacity loss below 1% for lines over 300 m.

Index Terms—Capacity estimates, linear precoding, multichannel DSL, quantization, vectoring.

I. INTRODUCTION

DSL systems are capable of delivering high data rates over copper lines. A major problem of DSL technologies is the electromagnetic coupling between the twisted pairs within a binder group. Reference [1] and the recent experimental studies in [2] and [3] have demonstrated that *vectoring* and *crosstalk cancelation* allow a significant increase of the data rates of DSL systems. In particular, linear precoding has recently drawn considerable attention [4], [5] as a natural method for crosstalk precompensation as well as crosstalk

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cancelation in the receiver. In [2] and [3], it is shown that optimal cancellation achieves capacity boost ranging from $2\times$ to $4\times$ and also substantially reduces per-loop capacity spread and outage, which are very important metrics from an operator's perspective. References [5] and [6] advocate the use of a diagonalizing precompensator and demonstrate that, without modification of the customer premise equipment (CPE), one can obtain near optimal performance. Recent work in [7] and [8] has shown that a low-order truncated series approximation of the inverse channel matrix affords significant complexity reduction in the computation of the precoding matrix. Implementation complexity (i.e., the actual multiplication of the transmitted symbol vector by the precoding matrix) remains high, however, especially for multicarrier transmission, which requires one matrix-vector multiplication for each tone. Current advanced DSL systems use thousands of tones. In these conditions, using minimal word length in representing the precoder matrix is important. However, using coarse quantization will result in substantial rate loss. The number of quantization bits per matrix coefficient is an important parameter that affects the system's performance–complexity tradeoff, which we focus on in this paper. We provide closed-form sharp analytic bounds on the absolute and relative transmission-rate loss. We show that both absolute and relative transmission loss decay exponentially as a function of the number of quantizer bits and provide explicit bounds for the loss in each tone. Under analytic channel models as in [9] and [10], we provide refined and explicit bounds for the transmission loss across the band and compare these to simulation results. This explicit relationship between the number of quantizer bits and the transmission-rate loss due to quantization is a very useful tool in the design of practical systems.

The structure of this paper is as follows. In Section II, we present the signal model for a precoded discrete multichannel system and provide a model for the precoder errors we study. In Section III, a general formula for the transmission loss of a single user is derived. In Section IV, we focus on the case of full channel state information where the rate loss of a single user results from quantization errors only. Here we prove the main result of this paper in Theorem 4.1. We provide explicit bounds on the rate loss under an analytic model for the transfer function as in [9]. We also study a number of natural design criteria. In Section V, we provide simulation results on measured lines, which support our analysis. Moreover, we show through simulation that our results are valid in the presence of measurement errors. The appendixes provide full details of the mathematical claims used in the main text.

II. PROBLEM FORMULATION

A. Signal Model

In this section, we describe the signal model for a precoded discrete multitone (DMT) system. We assume that the transmission scheme is frequency-division duplexing, where the upstream and the downstream transmissions are performed at separate frequency bands. Moreover, we assume that all modems are synchronized. Hence, the echo signal is eliminated, as in [1], and the received signal model at frequency f is given by

$$\mathbf{x}(f) = \mathbf{H}(f)\mathbf{s}(f) + \mathbf{n}(f) \tag{1}$$

where $\mathbf{s}(f)$ is the vectored signal sent by the optical network unit (ONU), $\mathbf{H}(f)$ is a $p \times p$ matrix representing the channels, $\mathbf{n}(f)$ is additive Gaussian noise, and $\mathbf{x}(f)$ (conceptually) collects the signals received by the individual users. The users estimate rows of the channel matrix $\mathbf{H}(f)$, and the ONU uses this information to send $\mathbf{P}(f)\mathbf{s}(f)$ instead of $\mathbf{s}(f)$. This process is called *crosstalk precompensation*. In general, such a mechanism yields

$$\mathbf{x}(f) = \mathbf{H}(f)\mathbf{P}(f)\mathbf{s}(f) + \mathbf{n}(f). \tag{2}$$

Denote the diagonal of $\mathbf{H}(f)$ by $\mathbf{D}(f) = \text{diag}(\mathbf{H}(f))$ and let $\mathbf{P}(f) = \mathbf{H}(f)^{-1}\mathbf{D}(f)$, as suggested in [5]. With this, we have

$$\mathbf{x}(f) = \mathbf{D}(f)\mathbf{s}(f) + \mathbf{n}(f) \tag{3}$$

showing that the crosstalk is eliminated. Note that with $\mathbf{F}(f) = \mathbf{H}(f) - \mathbf{D}(f)$, we have the following formula for the matrix $\mathbf{P}(f)$:

$$\mathbf{P}(f) = \left(\mathbf{I} + \mathbf{D}^{-1}(f)\mathbf{F}(f)\right)^{-1}. \tag{4}$$

Following [5], we assume that the matrices $\mathbf{H}(f)$ are row-wise diagonally dominant, namely, that

$$\|h_{ii}\| \gg \|h_{ij}\|, \ \forall i \neq j. \tag{5}$$

In fact, motivated in part by Gersgorin's theorem [11], we propose the parameter $r(\mathbf{H})$

$$r(\mathbf{H}) = \max_{1 \leq i \leq N} \left(\frac{\sum_{j \neq i} |h_{ij}|}{|h_{ii}|} \right) \tag{6}$$

as a measure for the dominance. In most downstream scenarios, the parameter r is indeed much smaller than one. We emphasize that typical downstream VDSL channels are row-wise diagonally dominant even in mixed-length scenarios, as demonstrated in [8].

B. A Model for Precoder Errors

In practical implementations, the entries of the precoding matrix $\mathbf{P}$ will be quantized. The number of quantizer bits used is dictated by complexity and memory considerations. Indeed, relatively coarse quantization of the entries of the precoder $\mathbf{P}$ allows significant reduction of the time complexity and the amount of memory needed for the precoding process. The key problem is to determine the transmission-rate loss of an individual user caused by such quantization. Another closely

related problem is the issue of robustness of linear precoding with respect to errors in the estimation of the channel matrix. The mathematical setting for both is that of error analysis. Let

$$\mathbf{P} = (\mathbf{I} + \mathbf{D}^{-1}\mathbf{F} + \mathbf{E}_1)^{-1} + \mathbf{E}_2 \tag{7}$$

where:
- $\mathbf{E}_1$ models the *relative error in quantizing or measuring* the channel matrix $\mathbf{H}$;
- $\mathbf{E}_2$ models the *errors caused by quantizing* the precoder $\mathbf{P}$.

The problem is to determine the capacity of the system, and the capacity of each user, in terms of the system parameters and the statistical parameters of the errors. Note that (7) captures three types of errors: errors in the estimation of $\mathbf{H}$, quantization errors in the representation of $\mathbf{H}$, and quantization errors in the representation of the precoder $\mathbf{P}$.

Our focus will be on the study of the effect of quantization errors in the representation of the precoder on the capacity of an individual user. Nevertheless, the estimation errors resulting from measuring the channel cannot be ignored. We will show that the analysis of quantization errors and estimation errors can be dealt with separately (see Remark 3.2 after Lemma 3.1). This allows us to carry out analysis under the assumption of perfect channel information. Then, we show in simulations that when the estimation errors in channel measurements are reasonably small, our analytical bounds remain valid.

C. System Model

We now list our assumptions regarding the errors $\mathbf{E}_1, \mathbf{E}_2$, the power spectral density of the users, and the behavior of the channel matrices.

1) Perfect CSI: Perfect channel state information (CSI), namely

$$\mathbf{E}_1(f) = 0, \ \forall f. \tag{8}$$

2) Quant (2^{-d}): The quantization error of each matrix element of the precoder is at most 2^{-d}. Namely

$$|\mathbf{E}_2(f)_{i,j}| \leq 2^{-d}, \ \forall f, \ \forall i, j. \tag{9}$$

3) DD: The channel matrices are row-wise diagonally dominant (DD)

$$r(\mathbf{H}(f)) \leq 1, \ \forall f. \tag{10}$$

4) SPSD: The power spectral density (PSD) of all the users of the binder is the same. Namely, we assume that for some fixed unspecified function $P(f)$, we have

$$P_i(f) = P(f), \ \forall i. \tag{11}$$

The main result of this paper (Theorem 4.1) is based on assumptions (8)–(11). Assumption SPSD can be lifted, as shown in Appendix H. For the sake of clarity, we present only the simplified result in the body of this paper.

In order to obtain sharp analytic estimates on the transmission loss in actual DSL scenarios, we need to incorporate some of the properties of the channel matrices of DSL channels into our model. In particular, we will assume the following.

5) Werner Channel Model: The matrix elements of the channel matrices $\mathbf{H}(f)$ behave as in the model of [9]. Namely, following [9], we assume the following model for insertion loss:

$$|\mathbf{H}^{IL}(f,\ell)|^2 = e^{-2\alpha\ell\sqrt{f}} \qquad (12)$$

where ℓ is the DSL loop length (in meters), f is the frequency in hertz, and α is a parameter that depends on the cable type. Furthermore, crosstalk is modeled as

$$|\mathbf{H}^{\text{FEXT}}(f,\ell)|^2 = K(\ell)f^2|\mathbf{H}^{IL}(f,\ell)|^2. \qquad (13)$$

Here $K(\ell)$ is a random variable studied in [10]. The finding is that $K(\ell)$ is a log-normal distribution with expectation, denoted there $c_1(\ell)$, that increases linearly with ℓ.

An additional assumption that we will make concerns the behavior of the row dominance of the channel matrices $\mathbf{H}(f,\ell)$.

6) Sublinear Row Dominance:

$$r(\mathbf{H}(f,\ell)) \leq \gamma_1(\ell) + \gamma_2(\ell)f \qquad (14)$$

where $\gamma_2(\ell) = O(\sqrt{\ell})$.

Remark 2.1: Note that

$$\frac{|\mathbf{H}^{\text{FEXT}}(\ell,f)|}{|\mathbf{H}^{IL}(f,\ell)|} = \sqrt{K(\ell)}f.$$

The sublinearity in f follows by studying $r(\mathbf{H}(\ell,f))$ in terms of p^2 random variables behaving as $K(\ell)$.

D. Justification of the Assumptions

Perfect CSI is plausible due to the quasi-stationarity of DSL systems (long coherence time), which allows us to estimate the channel matrices at high precision.

Quant (2^{-d}) is a weak assumption on the type of the quantization process. Informally, it is equivalent to an assumption on the number of bits used to quantize an entry in the channel matrix. In particular, our analysis of the capacity loss will be independent of the specific quantization method, and our results are valid for any technique that quantizes matrix elements with bounded errors.

Assumption DD reflects the diagonal dominance of DSL channels. While linear precoding may result in power fluctuations, the diagonal dominance property of DSL channel matrices makes these fluctuations negligible within the 3.5 dB fluctuation allowed by the PSD template (G993.2). For example, if the row dominance is up to 0.1, the effect of precoding on the transmit powers and spectra will be at most 1 dB.

Assumption SPSD [see (11)] is justified in a system with ideal full-binder precoding, where each user will use the entire PSD mask allowed by regulation. Note that in [3], it is shown that DSM3 provides significant capacity gains only when almost all pairs in a binder are coordinated. Thus the equal transmit spectra assumption is reasonable in these systems. However, we also provide in Appendix H a generalization of the main result to a setting in which this assumption is not satisfied.

Assumption Werner Channel Model does not need justification, whereas our last assumption, Sublinear Row Dominance, was verified on measured lines [3] and can also be deduced analytically from Werner's model. In practice, the type of fitting required to obtain $\gamma_1(\ell), \gamma_2(\ell)$ from measured data is simple and can be done efficiently. Moreover, the line parameters tabulated in standard (e.g., R,L,C,G parameters of the two-port model), together with the 99% worst case power-sum model used in standards [12], provide another way of computing the constants $\gamma_1(\ell), \gamma_2(\ell)$.

III. A GENERAL FORMULA FOR TRANSMISSION LOSS

The purpose of this section is to provide a general formula for the transmission-rate loss of a single user, resulting from errors in the estimated channel matrix as well as errors in the precoder matrix. First, we develop a useful expression for the equivalent channel in the presence of errors. This is given in (17). Next, a formula for the transmission loss is obtained in (30). The formula compares the achievable rate of a communication system using an ideal ZF precoder as in (4) versus that of a communication system whose precoder is given by (7). This formula is the key to this whole paper. Note that we use a gap analysis as in [13] and [14]. A useful corollary in the form of (34) is derived. This will be used in the next section to obtain bounds on capacity loss due to quantization.

Let $\mathbf{H}(f) = \mathbf{D}(f) + \mathbf{F}(f)$ be a decomposition of the channel matrix at a given frequency to diagonal and nondiagonal terms. Thus $\mathbf{D}(f)$ is a diagonal matrix whose diagonal is identical to that of $\mathbf{H}(f)$. Also, we let $\text{SNR}_i(f)$ be the signal-to-noise ratio of the ith receiver at frequency f

$$\text{SNR}_i(f) = \frac{P_i(f)|d_{i,i}(f)|^2}{E|n_i(f)|^2}. \qquad (15)$$

In this formula, $P_i(f)$ is the PSD of the ith user at frequency f and $n_i(f)$ is the associated noise term. We denote

$$\sigma_{n_i}^2(f) = E|n_i(f)|^2. \qquad (16)$$

A. A Formula for the Equivalent Channel in the Presence of Errors

We first derive a general formula for the equivalent signal model. The next lemma provides a useful reformulation of the signal model in (2).

Lemma 3.1: The precoded channel (2) with precoder as in (7) is given by

$$\mathbf{x}(f) = \mathbf{D}(f)\mathbf{s}(f) + \mathbf{D}(f)\boldsymbol{\Delta}(f)\mathbf{s}(f) + \mathbf{n}(f) \qquad (17)$$

with

$$\boldsymbol{\Delta}(f) = (\mathbf{I} + \mathbf{D}^{-1}(f)\mathbf{F}(f))\mathbf{E}_2(f)$$
$$- \mathbf{E}_1(f)(\mathbf{I} + \mathbf{D}^{-1}(f)\mathbf{F}(f) + \mathbf{E}_1(f))^{-1}. \qquad (18)$$

The proof is deferred to Appendix A.

Remark 3.2: For our analysis, we will assume that $\mathbf{E}_1(f) = 0$, in which case the formula for the matrix $\boldsymbol{\Delta}$ simplifies to

$$\boldsymbol{\Delta}(f) = (\mathbf{I} + \mathbf{D}^{-1}(f)\mathbf{F}(f))\mathbf{E}_2(f). \tag{19}$$

The relevance of (18) for the experimental part of the paper (where $\mathbf{E}_1(f)$ is not assumed to be zero) is explained in the next remark.

Remark 3.3: In (30), we show that the impact of the errors $\mathbf{E}_1(f)$ and $\mathbf{E}_2(f)$ on the transmission loss of a user can be computed from the matrix $\boldsymbol{\Delta}$. Thus, an important consequence of the lemma is that the effect on transmission loss due to estimation errors (encoded in the matrix $\mathbf{E}_1(f)$) and due to quantization errors (encoded in the matrix $\mathbf{E}_2(f)$) can be studied separately, as they contribute to different terms in the above expression for $\boldsymbol{\Delta}$.

B. Transmission Loss of a Single User

Consider a communication system as defined in (3) and denote by B the frequency band of the system. We let $\mathrm{SNR}_i(f)$ be as in (15) and let Γ be the Shannon gap comprising modulation loss, coding gain, and noise margin. Let R_i be the achievable transmission rate of the ith user in the system defined in (3). Recall that in such a system, the crosstalk is completely removed, and therefore

$$R_i = \int_{f \in B} \log_2(1 + \Gamma^{-1}\mathrm{SNR}_i(f))df. \tag{20}$$

Let

$$R_i(f) = \log_2(1 + \Gamma^{-1}\mathrm{SNR}_i(f)) \tag{21}$$

be the transmission rate at frequency f (formally, it is just the density of that rate). Let $\tilde{R}_i(f)$ be the transmission rate at frequency f of the ith user, when the precoder in (7) is used. We note that while $R_i(f)$ is a number, the quantity $\tilde{R}_i(f)$ depends on the random variables $\mathbf{E}_1, \mathbf{E}_2$ and hence is itself a random variable. Let $\tilde{R}_i$ be the transmission rate of the ith user for the equivalent system in (17). Thus

$$\tilde{R}_i = \int_{f \in B} \tilde{R}_i(f)df. \tag{22}$$

By (17), the ith user receives

$$x_i(f) = d_{i,i}(f)s_i(f) + d_{i,i}\sum_{j=1}^{p}\Delta_{i,j}(f)s_j(f) + n_i(f)$$
$$= d_{i,i}(f)(1 + \Delta_{i,i}(f))s_i(f) + N_i(f) \tag{23}$$

where $N_i(f) = d_{i,i}(f)\sum_{j \neq i}^{p}\Delta_{i,j}(f)s_j(f) + n_i(f)$. Assuming Gaussian signaling, i.e., that all $s_i(f)$ are Gaussian, we conclude that $N_i(f)$ is Gaussian. A similar conclusion is valid in the case of a large number of users, due to the central limit theorem. In practice, the Gaussian assumption is a good approximation even for a modest number of (e.g., eight) users. Recall also that Gaussian signaling is the optimal strategy in the case of exact channel knowledge. Therefore, we can use the capacity formula

for the Gaussian channel, even under precoder quantization errors.

Definition 3.1: The transmission loss $L_i(f)$ of the ith user at frequency f is given by

$$L_i(f) = R_i(f) - \tilde{R}_i(f). \tag{24}$$

The *total loss* of the ith user is

$$L_i = \int_{f \in B} L_i(f)df. \tag{25}$$

We are ready to deduce a formula for the rate loss of the ith user as a result of the nonideal precoder in (17). Our result will be given in terms of the matrix $\boldsymbol{\Delta}$. Recall that $\boldsymbol{\Delta}$ generally depends on both precoder quantization errors $\mathbf{E}_2$ and estimation errors $\mathbf{E}_1$.

Denote by $\boldsymbol{\Delta}_{i,j}$ the (i,j)th element of the matrix $\boldsymbol{\Delta}$ and let

$$\delta_i(f) = \Gamma \sum_{j \neq i}\frac{P_j(f)}{P_i(f)}|\Delta_{i,j}(f)|^2. \tag{26}$$

Let

$$a_i(f) = \delta_i(f)\Gamma^{-1}\mathrm{SNR}_i(f)$$
$$= \sum_{j \neq i}\frac{P_j(f)}{P_i(f)}|\Delta_{i,j}(f)|^2\mathrm{SNR}_i(f) \tag{27}$$

$$q_i(\boldsymbol{\Delta}, f) = \frac{|1 + \Delta_{i,i}(f)|^2}{a_i(f) + 1} \tag{28}$$

and

$$k_i(f) = \frac{\Gamma^{-1}\mathrm{SNR}_i(f)}{\Gamma^{-1}\mathrm{SNR}_i(f) + 1}. \tag{29}$$

Note that $a_i(f)$ and hence $q_i(\boldsymbol{\Delta}, f)$ are independent of the Shannon gap Γ. The next lemma provides a formula for the exact transmission-rate loss due to the errors modeled by the matrices $\mathbf{E}_1$ and $\mathbf{E}_2$. The result is stated in terms of quantities $q(\boldsymbol{\Delta}, f)$ and the effective SNR $\Gamma^{-1}\mathrm{SNR}_i(f)$.

Lemma 3.4: Let $\mathbf{H}(f)$ be the channel matrix at frequency f and let $\mathbf{E}_1, \mathbf{E}_2$ be the estimation and quantization errors, respectively, as in (7). Let $L_i(f)$ be the loss in transmission rate of the ith user defined in (24). Then

$$L_i(\boldsymbol{\Delta}, f) = -\log_2\left(1 - k_i(f)(1 - q_i(\boldsymbol{\Delta}, f))\right) \tag{30}$$

where $q_i(\boldsymbol{\Delta}, f)$ is given in (28) and $k_i(f)$ is given in (29).

In particular, if $\Delta_{i,i}(f) = -1$, the transmission loss is $\log_2(1 + \Gamma^{-1}\mathrm{SNR}_i(f))$, where $\mathrm{SNR}_i(f)$ is defined in (15). Lastly, if $\Delta_{i,i}(f) \neq -1$, we have

$$L_i(\boldsymbol{\Delta}, f) \leq Max\left(0, \log_2\left(\frac{1}{q_i(\boldsymbol{\Delta}, f)}\right)\right). \tag{31}$$

The proof of this lemma is deferred to Appendix B.

To formulate a useful corollary, we introduce the quantities

$$M_i(f) = \max_{j \neq i}\frac{P_j(f)}{P_i(f)} \tag{32}$$

$$t_i(f) = \max_{1 \leq j \leq n}|\boldsymbol{\Delta}_{i,j}|. \tag{33}$$

Corollary 3.5: Let $\mathbf{H}(f)$ be the $p \times p$ channel matrix at frequency f and let $\mathbf{E}_1(f)$, $\mathbf{E}_2(f)$ be the estimation and quantization errors, respectively, as in (7). Let $L_i(f)$ be the transmission-rate loss of the ith user defined in (24). Assume that $t_i(f) < 1$. Then

$$L_i(\mathbf{\Delta}, f) \leq \log_2 \left(\frac{1 + (p-1)M_i(f)t_i^2(f)\text{SNR}_i(f)}{(1 - t_i(f))^2} \right). \quad (34)$$

Proof: By (27), we have

$$a_i(f) = \sum_{j \neq i} \frac{P_j(f)}{P_i(f)} |\mathbf{\Delta}_{i,j}(f)|^2 \text{SNR}_i(f)$$

$$\leq M_i(f)t_i(f)^2(p-1)\text{SNR}_i(f) \quad (35)$$

$$1 + a_i(f) \leq 1 + (p-1)M_i(f)t_i(f)^2\text{SNR}_i(f). \quad (36)$$

Since $|\mathbf{\Delta}_{i,i}(f)| \leq t_i(f)$, we get

$$|1 + \mathbf{\Delta}_{i,i}(f)|^2 \geq (1 - t_i(f))^2. \quad (37)$$

Thus by (28), we have

$$\frac{1}{q_i(\mathbf{\Delta}, f)} = \frac{a_i(f) + 1}{|1 + \mathbf{\Delta}_{i,i}(f)|^2}$$

$$\leq \frac{1 + (p-1)M_i(f)t_i^2(f)\text{SNR}_i(f)}{(1 - t_i(f))^2}. \quad (38)$$

Notice that the right-hand side is larger than one and, using (31) of the previous lemma, the proof is complete.

Remark 3.6: We note that under simplifying assumptions, such as assumption SPSD [see (11)], the above formula reduces to

$$L_i(\mathbf{\Delta}, f) \leq \log_2(1 + (p-1)t_i^2(f)\text{SNR}_i(f))$$

$$-2\log_2(1 - t_i(f)). \quad (39)$$

Under the assumption Perfect CSI, we have $\mathbf{\Delta}(f) = (\mathbf{I} + \mathbf{D}^{-1}(f)\mathbf{F}(f))\mathbf{E}_2(f)$, and since we further assumed that the channel matrices $\mathbf{H}(f)$ are row-wise diagonally dominant, we see that $\mathbf{\Delta}(f) \approx \mathbf{E}_2(f)$. Thus, $t_i(f) \approx 2^{-d}$ and we obtain a bound of the form

$$L_i(\mathbf{\Delta}, f) \leq \log_2(1 + (p-1)\text{SNR}_i(f)2^{-2d})$$

$$-2\log_2(1 - 2^{-d}). \quad (40)$$

For a statement of a bound of this form, see (41) of Theorem 4.1.

IV. TRANSMISSION-RATE LOSS RESULTING FROM QUANTIZATION ERRORS IN THE PRECODER

In the ZF precoder studied earlier, we can assume without loss of generality that the entries are of absolute value less than one. Each of these values is now represented using $2d$ bits (d bits for the real part and d bits for the imaginary part, not including the sign bit). We first consider an ideal situation in which we have perfect channel estimation.

A. Transmission Loss With Perfect Channel Knowledge

Consider the case where $\mathbf{E}_1 = \mathbf{0}$ and the quantization error is given by an arbitrary matrix $\mathbf{E}_2$ with the property that each entry

is a complex number with real and imaginary parts bounded in absolute value by 2^{-d}. We will not make any further assumptions about the particular quantization method employed and will provide upper bounds for the capacity loss. We do not assume any specific random model for the values of $\mathbf{E}_2$ because we are interested in obtaining absolute upper bounds on capacity loss.

The following theorem describes the transmission-rate loss resulting from quantization of the precoder.

1) Main Theorem 4.1: Let $\mathbf{H}(f)$ be the channel matrix of p twisted pairs at frequency f and $r(f) = r(\mathbf{H}(f))$ as in (6). Assume Perfect CSI (8), Quant (2^{-d}) (9), SPSD (11), and that the precoder $\mathbf{P}(f)$ is quantized using $d \geq 1/2 + \log_2(1 + r(f))$ bits. The transmission-rate loss of the ith user at frequency f due to quantization is bounded by

$$L_i(d, f) \leq \log_2(1 + \gamma(d, f)\text{SNR}_i(f))$$

$$-2\log_2(1 - v(f)2^{-d}) \quad (41)$$

where

$$\gamma(d, f) = 2(p-1)(1 + r(f))^2 2^{-2d} \quad (42)$$

and

$$v(f) = \sqrt{2}(1 + r(f)). \quad (43)$$

Furthermore, suppose $d \geq 1/2 + \log_2(1 + r_{\max})$ with

$$r_{\max} = \max_{f \in B}(r(\mathbf{H}(f))). \quad (44)$$

Then the transmission loss in the band B is at most

$$\int_{f \in B} \log_2(1 + \gamma(d)\text{SNR}_i(f))df - 2|B|\tau(d) \quad (45)$$

where $|B|$ is the total bandwidth

$$\gamma(d) = 2(1 + r_{\max})^2(p-1)2^{-2d} \quad (46)$$

and

$$\tau(d) = \log_2(1 - (1 + r_{\max})2^{-d+0.5}). \quad (47)$$

The proof of the theorem is deferred to Appendix C.

We now record some useful corollaries of the theorem illustrating its value.

Corollary 4.2: The transmission-rate loss $L_i(\mathbf{\Delta}, f)$ due to quantization of the precoding matrix by d bits is bounded by

$$L_i(\mathbf{\Delta}, f) \leq \log_2(1 + \gamma(d, f)\text{SNR}_i(f))$$

$$-2\log_2(1 - v(f)2^{-d}) \quad (48)$$

where $\gamma(d, f) = 2(p-1)(1 + r(f))^2 2^{-2d}$ and $v(f) = \sqrt{2}(1 + r(f))$. If $r(f) \leq 1$, a simplified looser bound is given by

$$L_i(\mathbf{\Delta}, f) \leq 2^{-d+3.5} + \log_2(1 + 8(p-1)\text{SNR}_i(f)2^{-2d}). \quad (49)$$

For the derivation of the first inequality, see (98) in Appendix C. The simplified bound is based on the estimate $-\log_2(1 - z) \leq 2z$ valid for $0 \leq z \leq 0.5$.

The next result is of theoretical value. It describes the asymptotic behavior of $L_i(d)$ for very large d.

Corollary 4.3: Under the assumptions of the theorem and assuming that $r_{\max} \leq 1$

$$L_i(d) = O(2^{-d}).$$

More precisely, we have

$$L_i(d) = \theta\left(\frac{\sqrt{32}}{\ln(2)}2^{-d}B\right).$$

Remark 4.4: By definition, $f(n) = \theta(g(n))$ if and only if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$

We note that for many practical values of the parameters (e.g., $\text{SNR}(f) = 80$ dB, $d \leq 20$, $p \leq 100$), the first term in (45), involving 2^{-2d}, is dominant. Since we are interested in results that have relevance to existing systems, we will develop in the next section, and under some further assumptions (e.g., (12) and (13)), a bound for $L_i(d)$ of the form $a_1 2^{-2d} + a_2 2^{-d}$ where the coefficients a_1, a_2 are expressible using the system parameters. This is proposition 4.8.

2) Ensuring Bounded Transmission Loss in Each Frequency Bin: We now turn to study the natural design requirement that the transmission loss caused due to quantization of precoders should be bounded by a certain fixed quantity, say 0.1 bit/s/Hz/ user, *on a per-tone basis*. Such a design criterion is examined in the next corollary.

Corollary 4.5: Let $t > 0$ and let d be an integer with

$$d \geq d(t). \tag{50}$$

With

$$d(t) = \begin{cases} d_l(t), & \text{if } 2^t - 1 \leq \frac{B^2}{4A} \\ d_u(t), & \text{otherwise.} \end{cases}$$

Here

$$d_l(t) = \log_2\left(\frac{1.25v(f)2^{t+1}}{t\ln(2)}\right) \tag{51}$$

and

$$d_u(t) = 0.5\log_2\left(\frac{5(p-1)(1+r)^2\text{SNR}_i(f)}{t\ln(2)}\right). \tag{52}$$

Then the transmission loss at tone f due to quantization with d bits is at most t bps/Hz.

Proof: By Theorem 4.1, the loss at a tone f is bounded by $\log_2\left(1 + 2^{-2d}u(f)\right) - \log_2((1 - v(f)2^{-d})^2)$, where $u(f) = 2(p-1)(1+r)^2\text{SNR}_i(f)$ and $v(f) = \sqrt{2}(1 + r(f))$.

Using $1 - 2t \leq (1-t)^2$, we get

$$L_i(d,f) \leq \log_2\left(1 + 2^{-2d}u(f)\right) - \log_2(1 - 2v(f)2^{-d}).$$

We will show that the inequality

$$\log_2\left(\frac{1 + 2^{-2d}u(f)}{1 - 2v(f)2^{-d}}\right) \leq t \tag{53}$$

is satisfied for any $d \geq d(t)$ as in (50) Let $z = 2^{-d}$ so that (53) is

$$\frac{1 + z^2 u(f)}{1 - 2v(f)z} \leq 2^t. \tag{54}$$

This yields a quadratic inequality of the form

$$Az^2 + Bz \leq T \tag{55}$$

with $A = u(f)$, $B = 2^{t+1}v(f)$, and $T = 2^t - 1$. Using Lemma 10.1 (see Appendix D), we see that if $d \geq d_0(t)$, where

$$d_0(t) = \begin{cases} d_{\text{low}}(t), & \text{if } 2^t - 1 \leq \frac{B^2}{4A} \\ d_{up}(t), & \text{otherwise.} \end{cases}$$

Here

$$d_{\text{low}}(t) = \log_2\left(\frac{1.25v(f)2^{t+1}}{(2^t - 1)}\right) \tag{56}$$

and

$$d_{up}(t) = 0.5\log_2\left(\frac{5(p-1)(1+r)^2\text{SNR}_i(f)}{(2^t - 1)}\right). \tag{57}$$

Then $L_i(d,f) \leq t$. But $d_0(t) \leq d(t)$ because $2^t - 1 \geq \ln(2)t$ and the result follows.

Remark 4.6: The qualitative behavior is $d(t) \approx a_1 - \log_2(t)$ for very small values of t, whereas $d(t) \approx a_2 - 0.5\log_2(t)$ for larger values of t.

B. Applications of the Main Theorem

We now apply Theorem 4.1 to analyze the required quantization level for DSM level 3 precoders under several design criteria. To that end, let R_i be the transmission rate of the ith user (20) and let L_i be the transmission loss of the ith user as in (24). The *relative transmission loss* is defined by

$$\eta_i = \frac{L_i}{R_i} = \frac{\int_{f \in B} L_i(f)df}{\int_{f \in B} R_i(f)df}. \tag{58}$$

The design criteria are as follows.
- Absolute/relative transmission loss across the band is bounded.
- Absolute/relative transmission loss for each tone is bounded.

1) Bound on Absolute Transmission Loss: From now on, we will assume that the transfer function obeys a parametric model as in [9]. Thus we assume (12) and (13).

To bound the absolute transmission loss, we estimate the integral in (45) of Theorem 4.1.

Using the model (12), one can easily see that

$$\text{SNR}_i(f) = \frac{P_i(f)}{\sigma_{n_i}^2(f)}e^{-2\alpha\ell\sqrt{f}}.$$

Moreover, under the assumption (14), we have a linear bound on the quantity $r(\mathbf{H}(f, \ell))$, that is

$$r(\mathbf{H}(f, \ell)) \leq \gamma_1(\ell) + \gamma_2(\ell)f$$

where $\gamma_2(\ell) = O(\sqrt{\ell})$. Putting these together, we can estimate the integral occurring in the bound (45), and the final conclusion is described in Theorem 4.8.

The parameters $\gamma_1(\ell), \gamma_2(\ell)$ enter our bounds through the following quantity:

$$\rho_\ell = \gamma_0(\ell)^2 + 12\gamma_0(\ell)\frac{\gamma_2(\ell)}{(\alpha\ell)^2} + 240\left(\frac{\gamma_2(\ell)}{(\alpha\ell)^2}\right)^2 \quad (59)$$

with $\gamma_0(\ell) = 1 + \gamma_1(\ell)$.

Remark 4.7: The quantity ρ_ℓ behaves as $1 + C\ell^{-3/2}$ and is close to one for $\ell = 300$ m.

We are now ready to formulate one of the main results of this paper:

Theorem 4.8: Under assumptions Perfect CSI, Quant (2^{-d}), SPSD, Werner Channel Model, and Sublinear Row Dominance [see (8), (9), and (11)–(14)], we have

$$\frac{L_i(d)}{B} \leq \xi_\ell 2^{-2d} + 2^{-d+3.5} \quad (60)$$

where

$$\xi_\ell = \frac{4}{\ln(2)}(p-1)\frac{P}{\sigma_n^2}\frac{1}{\alpha^2 B}\frac{1}{\ell^2}\rho_\ell. \quad (61)$$

We provide a proof of this result in Appendix E.

2) Bound on Relative Transmission Loss: The most natural design criterion is to ensure that the *relative capacity loss* is below a predetermined threshold. We will keep our assumption that the insertion loss behaves as in the model (12) and (13).

Let $\text{SNR}_i = P_i/\sigma_{n_i}^2$ and $\text{SNR}_i' = P_i/\sigma_{n_i}^2 e^{-\alpha\sqrt{B}}$ be the SNRs of the ith user at the lowest and highest frequencies, respectively. We also denote by $\widetilde{\text{SNR}} = \text{SNR}_i/\Gamma$ and by $\widetilde{\text{SNR}}_i' = \text{SNR}_i'/\Gamma$. Finally, we denote

$$c_i = \frac{1}{3}\log_2(\widetilde{\text{SNR}}_i) + \frac{2}{3}\log_2(\widetilde{\text{SNR}}_i'). \quad (62)$$

The next proposition shows that c_i provides a lower bound on the spectral efficiency of the ith user.

Proposition 4.1: Assume that the attenuation transfer characteristic of the channel is given by (12). Then the spectral efficiency is bounded below by

$$\frac{1}{B}R_i \geq c_i. \quad (63)$$

The proof is deferred to Section IV-A (Appendix F).

Corollary 4.9: Let $\eta_i(d)$ be the relative transmission rate loss of the ith user as in (58). Assume that the transfer function satisfies (12) and (13).

Then

$$\eta_i(d) \leq \zeta_\ell 2^{-2d} + \frac{1}{c_i}2^{-d+3.5} \quad (64)$$

where

$$\zeta_\ell = \frac{\xi_\ell}{c_i} = \frac{4}{\ln(2)}(p-1)\frac{P}{\sigma_n^2}\frac{1}{\alpha^2 B}\frac{1}{\ell^2}\frac{1}{c_i}\rho_\ell. \quad (65)$$

Proof: This is an immediate consequence of the upper bound on the average loss L_i/B and the lower bound on $1/BR_i$.

3) Ensuring Bounded Relative Transmission Loss in the Whole Band: The next corollary yields an upper bound for the number of quantized bits required to ensure that the relative loss is below a given threshold.

Corollary 4.10: Let $0 \leq \tau \leq 1$ and let $d \geq d(\tau)$, where

$$d(\tau) = \begin{cases} \log_2(\frac{12\sqrt{2}}{c_i\tau}), & \text{if } \tau \leq \frac{32}{\zeta_\ell c^2} \\ 0.5\log_2(\frac{2.5}{\zeta_\ell\tau}), & \text{otherwise.} \end{cases}$$

Then the relative transmission loss caused by quantization with d bits is at most τ.

The proof is a simple application of the previous bound on the relative transmission loss and Lemma 10.1 Appendix D.

V. SIMULATION RESULTS

To check the quality of the bounds in Theorem 4.1 and its corollaries, we compared the bounds with simulation results, based on measured channels. We have used the results of the measurement campaign conducted by France Telecom R&D as described in [10]. All experiments used the band 0–30 MHz.

A. Full Band

For each experiment, we generated 1000 random precoder quantization error matrices $\mathbf{E}_2(f)$, with independent identically distributed (i.i.d.) elements and independent real and imaginary parts, each uniformly distributed in the interval $[-2^{-d}, 2^d]$. We add the error matrix to the precoder matrix to generate the quantized precoder matrix. Repeating this in each frequency, we produced a simulation of the quantized precoded system and computed the resulting channel capacity of each of the ten users. Then we computed the relative and absolute capacity loss of each of the users. In each bin, we picked the worst case out of 1000 quantization trials and obtained a quantity we called *maximal loss*. The quantity maximal loss is a random variable depending on the number of bits used to quantize the precoder matrices. Each value of this random variable provides a lower bound for the actual worst case that can occur when the channel matrices are quantized. We compare this lower bound with our upper bounds of Theorem 4.1. We have checked our bounds in the following scenario: each user has flat PSD of -60 dBm/Hz and the noise has flat PSD of -140 dBm/Hz. The Shannon gap is assumed to be 10.7 dB. As can be seen in Fig. 1, the bound given by (45) is sharp. We also checked the more explicit bound (64), which is based on the model (12) and (13). We validated the linear behavior of the row dominance $r(\mathbf{H}(f))$ as a function of the tone f as predicted by (14). Next we used (12) to fit the parameter α of the cable via the measured insertion losses. The process of fitting is described in detail in [10]. Its value, which was used in the bound (64), was $\alpha = 0.0019$. The parameters $\gamma_1 = 0.1596$ and $\gamma_2 = 3.1729 \ 10^{-8}$ were estimated from the measured channel matrices by simple line fit. The results are depicted in Fig. 1.
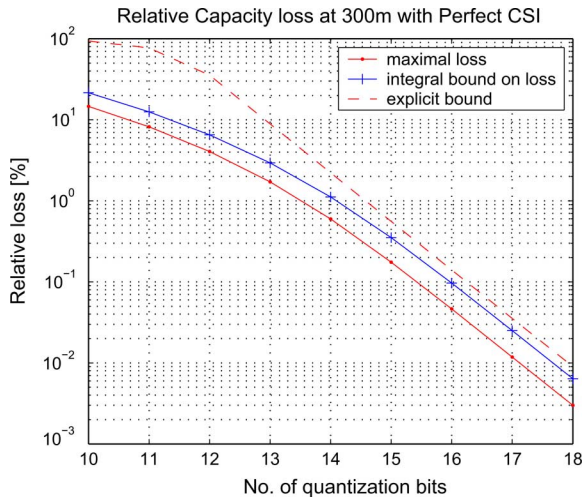
Fig. 1.   Relative capacity loss versus number of quantizer bits in perfect CSI in a system of ten users. Integral bound on loss is obtained via (45); explicit bound is obtained via (64), (65), and (62).
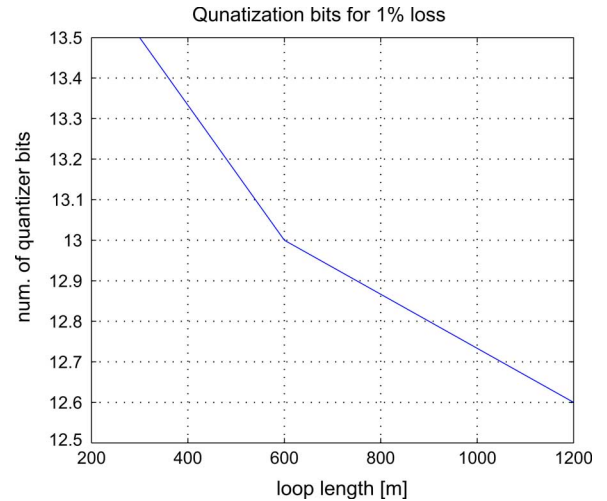

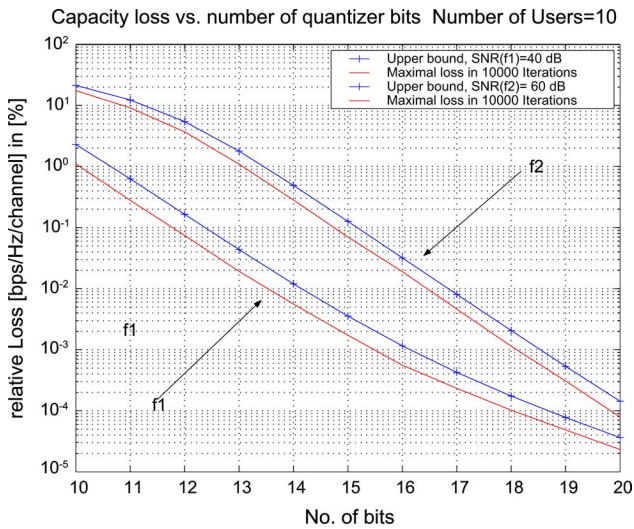Fig. 3.   Number of quantization bits required versus loop length.

Fig. 2.   Capacity loss versus quantizer bits. Perfect CSI in system of ten users.

Fig. 4.   Capacity loss versus quantizer bits. Imperfect CSI in system of ten users. CSI based on 1000 measurements.

B. Single Frequency

The bounds provided for the entire band are results of bounds on each frequency bin. To show that our bounds are sharp even without averaging over the frequency band, we studied the capacity loss in specific frequency bins. We concentrated on the same scenario as before (i.e., with ten users), the noise is -140 dBm/Hz, and the power of the users is -60 dBm/Hz. We picked measured matrices $\mathbf{H}(f_1)$, $\mathbf{H}(f_2)$ so that $\mathrm{SNR}(f_1)$ is 40 dBm and $\mathrm{SNR}(f_2)$ is 60 dBm. As before, we systematically generated an error matrix $\mathbf{E}_2$ by choosing its entries to be i.i.d., uniformly distributed with maximal absolute value $2^{-d+0.5}$. Next, we computed the transmission-rate loss using (30). By repeating this process $N = 10\,000$ times and choosing the worst event of transmission-rate loss, we obtained a lower bound estimate of worst case transmission-rate loss. This was compared to the bounds of Corollary 4.2. The results are depicted in Fig. 2. Fig. 2 uses (48). In particular, we see that for $\mathrm{SNR} = 60$ dB and transmission-rate loss of 1%, simulation indicates quantization with
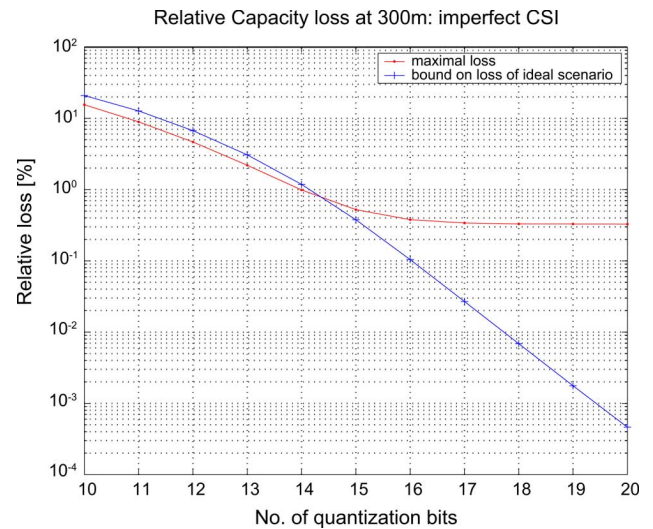
13 bits. The analytic formula indicates 14 bits. Similarly, when $\mathrm{SNR} = 40$ dB, and again allowing the same transmission-rate loss of 1%, simulation suggests using 10 bits for quantization. The simple analytic estimate requires 11 bits.

C. The Number of Quantizer Bits Needed to Assure 99% of Capacity

In the next experiment, we have studied the number of bits required to obtain a given transmission loss as a function of the loop length. Fig. 3 depicts the number of bits required to ensure transmission-rate loss below 1% as a function of loop length. We see that 14 bits are sufficient for loop lengths up to 1200 m. Fewer bits are required for longer loops.

D. Stability of the Results

In the next experiment, we validated that the analytic results proven for perfect CSI are valid even when CSI is imperfect as long as channel measurement errors are not the dominating

cause for capacity loss. To model the measurement errors of the channel matrix $\mathbf{H}(f)$, we used matrices with Gaussian entries with variance proportional to $\mathrm{SNR}(f)$. More precisely, we assumed that the estimation error of the matrix $\mathbf{H}(f)$ is a Gaussian with zero mean and with variance $\sigma^2_{\mathbf{H}(f)} = 1/N\mathrm{SNR}_i(f)$, where N is the number of samples used to estimate the channel matrix $\mathbf{H}(f)$. For $N = 1000$, we estimated the loss in a frequency bin as the worst case out of 500 realizations of quantization noise combined with measurement noise. Fig. 3 shows that as long as the quantization noise is dominant, we can safely use our bounds for the transmission loss. We comment that the stationarity of DSL channels allows accurate channel estimation.

VI. CONCLUSION

In this paper, we analyzed effects of finite word length on the achievable rate of vector DSL systems with zero-forcing precoding. The results of this paper provide simple analytic expressions for the loss due to finite word length. These expressions allow simple optimization of linearly precoded DSM level 3 systems.

We validated our results using measured channels. Moreover, we showed that our bounds can be adapted to study the effect of measurement errors on the transmission loss. In practice, for loop lengths between 300 and 1200 m, one needs 14 bits to represent the precoder elements in order to lose no more than 1% of the capacity.

APPENDIX A
PROOF OF LEMMA 3.1

In this section, we prove Lemma 3.1.

Proof: For simplicity, we will omit the explicit dependency of the matrices $\mathbf{H}(f), \mathbf{D}(f), \mathbf{F}(f), \mathbf{P}(f)$ on the frequency f. We show that

$$\mathbf{H}\mathbf{P} = \mathbf{D} + \mathbf{D}\boldsymbol{\Delta} \tag{66}$$

with $\boldsymbol{\Delta}$ as above. Indeed, $\mathbf{H} = \mathbf{D}(\mathbf{I} + \mathbf{D}^{-1}\mathbf{F})$, and thus

$$\mathbf{H}\mathbf{P} = \mathbf{D}(\mathbf{I} + \mathbf{D}^{-1}\mathbf{F})((\mathbf{I} + \mathbf{D}^{-1}\mathbf{F} + \mathbf{E}_1)^{-1} + \mathbf{E}_2). \tag{67}$$

Hence

$$\mathbf{H}\mathbf{P} = \mathbf{D}(\mathbf{I} + \mathbf{D}^{-1}\mathbf{F} + \mathbf{E}_1 - \mathbf{E}_1)$$
$$\times (\mathbf{I} + \mathbf{D}^{-1}\mathbf{F} + \mathbf{E}_1)^{-1} + \mathbf{D}(\mathbf{I} + \mathbf{D}^{-1}\mathbf{F})\mathbf{E}_2. \tag{68}$$

Thus

$$\mathbf{H}\mathbf{P} = \mathbf{D} - \mathbf{D}\mathbf{E}_1(\mathbf{I} + \mathbf{D}^{-1}\mathbf{F} + \mathbf{E}_1)^{-1} + \mathbf{D}(\mathbf{I} + \mathbf{D}^{-1}\mathbf{F})\mathbf{E}_2 \tag{69}$$

which proves the lemma.

APPENDIX B
PROOF OF LEMMA 3.4

In this Appendix, we prove Lemma 3.4.

Proof: By (17), the ith user receives

$$x_i(f) = d_{i,i}(f)s_i(f) + d_{i,i}\sum_{j=1}^{p}\Delta_{i,j}(f)s_j(f) + n_i(f)$$
$$= d_{i,i}(f)(1 + \Delta_{i,i}(f))s_i(f) + N_i(f) \tag{70}$$

with $N_i(f) = d_{i,i}(f)\sum_{j\neq i}^{p}\Delta_{i,j}(f)s_j(f) + n_i(f)$. For a large number of users, we may assume that $N_i(f)$ is again a Gaussian noise and the *transmission rate* at frequency f of the system described by (70) will be

$$R_i(\Delta, f) = \log_2\left(1 + \mathrm{SINR}_i(\Delta, f)\right) \tag{71}$$

where

$$\mathrm{SINR}_i(\Delta, f) = \frac{P_i(f)|d_{i,i}(f)|^2|(1 + \Delta_{i,i}(f))|^2}{\Gamma(\sum_{j\neq i} P_j(f)|d_{i,i}(f)|^2|\Delta_{i,j}(f)|^2 + |n_i(f)|^2)}.$$

Note that this quantity appeared in the main body of the paper just after (22), where it was denoted $\tilde{R}_i(f)$. Dividing both the numerator and the denominator by $P_i(f)|d_{i,i}(f)|^2$, we get

$$\mathrm{SINR}_i(\Delta, f) = \frac{|(1 + \Delta_{i,i}(f))|^2}{\Gamma\sum_{j\neq i}\frac{P_j(f)}{P_i(f)}|\Delta_{i,j}(f)|^2 + \frac{\Gamma|n_i(f)|^2}{P_i(f)|d_{i,i}(f)|^2}}. \tag{72}$$

Thus

$$R_i(\Delta, f) = \log_2\left(1 + \frac{|(1 + \Delta_{i,i}(f))|^2}{\delta_i(f) + \frac{1}{e\mathrm{SNR}_i(f)}}\right) \tag{73}$$

where we have defined

$$e\mathrm{SNR}_i(f) = \frac{\mathrm{SNR}_i(f)}{\Gamma} = \frac{P_i(f)|d_{i,i}(f)|^2}{\Gamma|n_i(f)|^2} \tag{74}$$

and

$$\delta_i(f) = \Gamma\sum_{j\neq i}\frac{P_j(f)}{P_i(f)}|\Delta_{i,j}(f)|^2. \tag{75}$$

To get the transmission-rate loss, we denote

$$e\mathrm{SNR}_i(\Delta, f) = \frac{|(1 + \Delta_{i,i}(f))|^2}{\delta_i(f) + \frac{1}{e\mathrm{SNR}_i(f)}}. \tag{76}$$

Notice that

$$e\mathrm{SNR}_i(f) = e\mathrm{SNR}_i(\mathbf{0}, f).$$

By (24), we have

$$L_i(\boldsymbol{\Delta}, f) = R_i(f) - R_i(\Delta, f)$$
$$= \log_2(1 + e\mathrm{SNR}_i(f))$$
$$- \log_2(1 + e\mathrm{SNR}_i(\Delta, f)). \tag{77}$$

We then have

$$
\begin{aligned}
& L_i(\boldsymbol{\Delta}, f) \\
&= -\log_2\left(\frac{1 + e\mathrm{SNR}_i(\Delta, f)}{1 + e\mathrm{SNR}_i(f)}\right) \\
&= -\log_2\left(1 - \frac{e\mathrm{SNR}_i(f) - e\mathrm{SNR}_i(\Delta, f)}{1 + e\mathrm{SNR}_i(f)}\right). \quad (78)
\end{aligned}
$$

To keep the equations short, we introduce a local notation

$$
\delta_i^{SNR}(f, \Delta) = e\mathrm{SNR}_i(f) - e\mathrm{SNR}_i(\Delta, f). \quad (79)
$$

We have

$$
\delta_i^{\mathrm{SNR}}(\Delta, f) = e\mathrm{SNR}_i(f) - \frac{|(1 + \Delta_{i,i}(f))|^2}{\delta_i + \frac{1}{e\mathrm{SNR}_i(f)}} \quad (80)
$$

so

$$
\delta_i^{\mathrm{SNR}}(\Delta, f) = \frac{e\mathrm{SNR}_i(f)\delta_i(f) + 1 - |(1 + \Delta_{i,i}(f))|^2}{\delta_i + \frac{1}{e\mathrm{SNR}_i(f)}} \quad (81)
$$

and, finally

$$
\begin{aligned}
\delta_i^{\mathrm{SNR}}(\Delta, f) &= e\mathrm{SNR}_i(f) \\
&\times \frac{e\mathrm{SNR}_i(f)\delta_i(f) + 1 - |(1 + \Delta_{i,i}(f))|^2}{\delta_i(f)e\mathrm{SNR}_i(f) + 1}. \quad (82)
\end{aligned}
$$

Hence

$$
L_i(\boldsymbol{\Delta}, f) = -\log_2\left(1 - c_i(f)\frac{a_i(f) + 1 - |1 + \Delta_{i,i}|^2}{a_i(f) + 1}\right) \quad (83)
$$

where

$$
a_i(f) = \delta_i(f)e\mathrm{SNR}_i(f) \quad (84)
$$

$$
c_i(f) = \frac{e\mathrm{SNR}_i(f)}{e\mathrm{SNR}_i(f) + 1} \quad (85)
$$

and $\delta_i(f)$ is given in (75). With the notations (28) and (29), we get

$$
L_i(\boldsymbol{\Delta}, f) = -\log_2\left(1 - k_i(f)(1 - q_i(\boldsymbol{\Delta}, f))\right). \quad (86)
$$

To prove the bound, we consider two cases. When $q(\boldsymbol{\Delta}, f) > 1$, we see from (86) that $L_i(\boldsymbol{\Delta}, f) \leq 0$. This clearly indicates transmission gain, and the stated inequality is valid. On the other hand, if $q_i(\boldsymbol{\Delta}, f) \leq 1$, we get

$$
\frac{e\mathrm{SNR}_i}{e\mathrm{SNR}_i + 1}(1 - q_i(\boldsymbol{\Delta}, f)) \leq 1 - q_i(\boldsymbol{\Delta}, f) \quad (87)
$$

and using the monotonicity of $-\log_2(1 - u)$ (increasing) in the interval (0,1), we get

$$
\begin{aligned}
L_i(\boldsymbol{\Delta}, f) &\leq -\log_2\left(1 - (1 - q_i(\boldsymbol{\Delta}, f))\right) \\
&= \log_2\left(\frac{1}{q_i(\boldsymbol{\Delta}, f)}\right) \quad (88)
\end{aligned}
$$

and the lemma is proved.

APPENDIX C
PROOF OF THEOREM 4.1

For the proof of the theorem, we need a simple lemma.

Lemma 9.1: Let $\mathbf{A}$ be a complex $p \times p$ matrix and define $\mathbf{D}$ to be the diagonal matrix with $\mathbf{D}_{i,i} = \mathbf{A}_{i,i}$ for $i = 1, \ldots, p$. Let $\mathbf{E}$ be a $p \times p$ matrix whose entries are complex numbers with real and imaginary parts bounded by 2^{-d}. Lastly, let $\mathbf{B} = \mathbf{D}^{-1}\mathbf{A}\mathbf{E}$. Then $|\mathbf{B}_{i,j}| \leq 2^{-d+1/2}(1 + r(A))$.

Proof: Let $\mathbf{Q} = \mathbf{D}^{-1}\mathbf{A} = \mathbf{I} + \mathbf{D}^{-1}(\mathbf{A} - \mathbf{D})$. Then we have

$$
\sum_{k=1}^{p} |\mathbf{Q}_{ik}| \leq 1 + r(\mathbf{A}) \quad (89)
$$

for all $i = 1, \ldots, p$. Therefore

$$
\begin{aligned}
|\mathbf{B}_{i,j}| &= \left|\sum_{k=1}^{p} \mathbf{Q}_{ik}\mathbf{E}_{kj}\right| \\
&\leq 2^{-d+1/2}\sum_{k=1}^{p} |\mathbf{Q}_{ik}| \\
&\leq 2^{-d+1/2}(1 + r). \quad (90)
\end{aligned}
$$

Proof of the Main Theorem: We first bound the loss $L_i(f)$ in a particular tone f. By Lemma 3.4, we have

$$
L_i(\boldsymbol{\Delta}, f) \leq Max\left(0, \log_2\left(\frac{1}{q_i(\boldsymbol{\Delta}, f)}\right)\right) \quad (91)
$$

where

$$
q_i(\boldsymbol{\Delta}, f) = \frac{|1 + \Delta_{i,i}(f)|^2}{a_i(f) + 1}. \quad (92)
$$

Here $\boldsymbol{\Delta}(f) = (\mathbf{I} + \mathbf{D}(f)^{-1}\mathbf{F}(f))\mathbf{E}_2(f)$, where $\mathbf{H}(f) = \mathbf{D}(f) + \mathbf{F}(f)$ is the channel matrix at frequency f and $\mathbf{E}_2(f)$ is a matrix whose entries are complex numbers with real and imaginary parts bounded by 2^{-d}. Applying Lemma 9.1 to the matrix $\mathbf{H}(f)$, we see that the entries $\boldsymbol{\Delta}_{i,j}(f)$ are all in a disk of radius $v(f)2^{-d}$ around zero. Using $r(f) \leq 5$, we obtain $v(f) = \sqrt{2}(1 + r(f)) \leq 6\sqrt{2}$. Using $d \geq 4$, we obtain $1 - 2^{-d}v(f) \geq 1 - 6\sqrt{2}/16 > 0$.

Thus

$$
|1 + \boldsymbol{\Delta}_{i,i}(f)|^2 \geq (1 - v2^{-d})^2. \quad (93)
$$

Using the assumption on the PSD of the different users, we obtain

$$
\begin{aligned}
a_i(f) &= \sum_{j \neq i} \frac{P_j(f)}{P_i(f)}|\Delta_{i,j}(f)|^2\mathrm{SNR}_i(f) \\
&= \sum_{j \neq i} |\Delta_{i,j}(f)|^2\mathrm{SNR}_i(f). \quad (94)
\end{aligned}
$$

Using Lemma 9.1, we have

$$
\sum_{j \neq i} |\Delta_{i,j}(f)|^2 \leq (p - 1)2^{-2d+1}(1 + r(f))^2. \quad (95)
$$

Thus

$$1 + a_i(f) \le 1 + (p-1)2^{-2d+1}(1+r(f))^2 \mathrm{SNR}_i(f)$$
$$\le 1 + \gamma(d,f)\mathrm{SNR}_i(f). \tag{96}$$

Combining (92), (93), and (96), we obtain

$$\frac{1}{q_i(\mathbf{\Delta},f)} \le \frac{1+\gamma(d,f)\mathrm{SNR}_i(f)}{(1-v(f)2^{-d})^2}. \tag{97}$$

Note that the right-hand side of the above inequality is positive and greater than one. Combining (31) and (97), we obtain

$$L_i(\mathbf{\Delta},f) \le \log_2\left(\frac{1+\gamma(d,f)\mathrm{SNR}_i(f)}{(1-v(f)2^{-d})^2}\right)$$
$$= \log_2(1+\gamma(d,f)\mathrm{SNR}_i(f))$$
$$- 2\log_2(1-v(f)2^{-d}). \tag{98}$$

Since $\gamma(d,f) \le 2(1+r_{\max})^2(p-1)2^{-2d}$ and $v(f) = \sqrt{2}(1+r(f)) \le \sqrt{2}(1+r_{\max})$, integrating this inequality over $f \in B$, we obtain (45) and the theorem is proved.

APPENDIX D
PROOFS OF COROLLARIES 4.8 AND 4.9

A Quadratic Inequality: In the proof of Corollaries 4.8 and 4.9, we use the following lemma.

Lemma 10.1: Let A, B, T be positive real numbers and let

$$d(T) = \begin{cases} \log_2\left(\frac{1.25B}{T}\right), & \text{if } T \le \frac{B^2}{4A} \\ 0.5\log_2\left(\frac{2.5A}{T}\right), & \text{otherwise.} \end{cases}$$

Then for $d \ge d(T)$, we have

$$A2^{-2d} + B2^{-d} \le T. \tag{99}$$

Proof: We let $x = 2^{-d}$ and observe that $f(x) = Ax^2 + Bx$ is monotone in $x > 0$ with one root of $f(x) = T$ exactly at $x_0 = \sqrt{B^2+4AT} - B/2A$. Thus for any $d > d_0(T) = \log_2(2A/\sqrt{B^2+4AT} - B)$, we have $A2^{-2d} + B2^{-d} = f(2^{-d}) \le f(2^{-d_0}) = f(x_0) = T$. To complete the proof, we will show that $d_0(T) \le d(T)$. Indeed

$$d_0(T) = \log_2\left(\frac{2A}{\sqrt{B^2+4AT} - B}\right)$$
$$= \log_2\left(\frac{2A(\sqrt{B^2+4AT}+B)}{4AT}\right). \tag{100}$$

Thus

$$d_0(T) = \log_2\left(\frac{B}{2T}\left(\sqrt{1+\frac{4AT}{B}}+1\right)\right). \tag{101}$$

If we let $\rho = 4AT/B^2$, then for $\rho < 1$, we have $\sqrt{1+\rho}+1 \le 2.5$, and this yields the bound

$$d_0(T) \le \log_2\left(\frac{1.25B}{T}\right) \tag{102}$$

for $T \le B^2/4A$. On the other hand, if $\rho > 1$, it is easy to see that $1 + \sqrt{1+\rho} \le 2.5\sqrt{\rho}$. Thus

$$d_0(T) \le \log_2\left(\frac{B}{2T}\left(2.5\sqrt{\frac{4AT}{B^2}}\right)\right) = \log_2\left(2.5\sqrt{\frac{A}{T}}\right). \tag{103}$$

Remark 10.2: Note that as T decreases to zero, the value of $d(T)$ increases and behaves as $\log_2(1/T)$.

APPENDIX E
PROOF OF THEOREM 4.8

Proof: Using Theorem 4.1, the capacity loss of the ith user $L_i(d)$ is bounded by

$$L_i(d) \le \int_{f\in B} \log_2\left(1+\gamma(d,f)\frac{P}{\sigma_n^2}e^{-\alpha_\ell\sqrt{f}}\right) df$$
$$- 2|B|\log_2(1-2^{-d+1.5}). \tag{104}$$

By assumption, $\gamma(d,f) \le 2(p-1)2^{-2d}(1+\gamma_1+\gamma_2 f)^2$. To bound the first term, we state here a simple lemma (for the proof, see Appendix G).

Lemma 11.1: Let $f(x) = P/\sigma_n^2 e^{-\alpha\sqrt{x}}$ and define

$$J_{a,b}(\mu) = \frac{1}{B}\int_0^B \log_2(1+\mu(a+bx)^2 f(x))dx. \tag{105}$$

We have

$$J(\mu) \le \frac{e^{\alpha\sqrt{B}}}{\alpha^2 B}\psi(a,b,\alpha)\log_2\left(1+\mu f(B)\right) \tag{106}$$

where

$$\psi(a,b,\alpha) = 2a^2 + 24\frac{ab}{\alpha^2} + 240\left(\frac{a}{\alpha^2}\right)^2.$$

We can now finish the proof of the theorem.

Let $a = 1+\gamma_1$, $b = \gamma_2$, and $\mu = 2(p-1)2^{-2d}$; and let $J = J_{a,b}$ as in the lemma above. From (104), we get

$$\frac{1}{B}L_i(d) \le J(2(p-1)2^{-2d}) - 2\log_2(1-2^{-d+1.5}). \tag{107}$$

Using the inequality $-\log_2(1-z) \le 2z$, for $z \le 1/2$, and the inequality provided by the lemma for $J(\mu)$, we obtain

$$\frac{1}{B}L_i(d) \le \frac{e^{\alpha_\ell\sqrt{B}}}{\alpha^2 B}\chi(\ell,\alpha)$$
$$\log_2(1+2(p-1)2^{-2d}f(B)) + 2^{-d+3.5} \tag{108}$$

with

$$\chi(\ell,\alpha) = 2(1+\gamma_1(\ell))^2$$
$$+24(1+\gamma_1(\ell))\frac{\gamma_2(\ell)}{(\alpha\ell)^2} + 240\left(\frac{\gamma_2(\ell)}{(\alpha\ell)^2}\right)^2.$$

Using $\log_2(1+t) \le ln(2)t$, the fact that $f(B) = P/\sigma_n^2 e^{-\alpha\sqrt{B}}$, and the definition of ρ_ℓ in (59), we obtain

$$\frac{1}{B}L_i(d) \le \frac{4}{ln(2)}(p-1)\frac{P}{\sigma_n^2}\frac{1}{(\alpha\ell)^2 B}\rho_\ell 2^{-2d} + 2^{-d+3.5}. \tag{109}$$

APPENDIX F
PROOF OF PROPOSITION 4.1

Proof: We begin with a bound on the transmission rate of the users. By (20) and the model (12), we obtain

$$R_i = \int_{f \in B} \log_2(1 + \Gamma^{-1}\mathrm{SNR}e^{-\alpha_\ell\sqrt{f}})df$$

$$\geq \log_2(e) \int_{f \in B} \ln(\Gamma^{-1}\mathrm{SNR}e^{-\alpha_\ell\sqrt{f}})df. \quad (110)$$

Thus

$$R_i \geq B\log_2(\Gamma^{-1}\mathrm{SNR}) - \log_2(e)\int_0^B \alpha_\ell\sqrt{f}df$$

$$\geq B\log_2(\Gamma^{-1}\mathrm{SNR}) - \frac{2}{3}\log_2(e)\alpha_\ell B\sqrt{B}. \quad (111)$$

We notice that this, with $\mathrm{SNR}' = \mathrm{SNR}e^{-\alpha_\ell\sqrt{B}}$, implies

$$\frac{1}{B}R_i \geq -\frac{2}{3}\log_2(e)(\ln(\mathrm{SNR}) - \ln(\mathrm{SNR}'))$$

$$+ \log_2(\Gamma^{-1}\mathrm{SNR})$$

$$= \frac{1}{3}\log_2(\mathrm{SNR}) + \frac{2}{3}\log_2(\mathrm{SNR}') - \log_2(\Gamma). \quad (112)$$

The proof is complete.

Remark 12.1: In practice, the estimation of α_ℓ is more reliable than the measurement of the transfer function at the edge of the frequency band. Thus, the equivalent form

$$\frac{1}{B}R_i \geq \log_2(\widetilde{\mathrm{SNR}}) - \frac{2}{3}\alpha_\ell\sqrt{B} \quad (113)$$

is more reliable.

APPENDIX G
PROOF OF LEMMA 11.1

In this section, we prove Lemma 11.1. Recall

$$J(\mu) = \frac{1}{B}\int_0^B \log_2(1 + \mu(a + bx)^2 f(x))dx \quad (114)$$

where $f(x) = P/\sigma_n^2 e^{-\alpha\sqrt{x}}$.

Lemma: Let $a \geq 1$ and $b \geq 0$.

Let M be the maximal value of $(a + bx)^2 f(x)$ in the interval $[0, B]$. We have

$$J(\mu) \leq \min\left(c\log_2\left(1 + \mu f(B)\right), \log_2\left(1 + M\mu\right)\right) \quad (115)$$

where

$$c = \frac{e^{\alpha\sqrt{B}}}{\alpha^2 B}\psi(a, b, \alpha).$$

In particular, we have

$$J(\mu) \leq \frac{2P}{ln(2)\alpha^2 B\sigma_n^2}\left(a^2 + 12\frac{ab}{\alpha^2} + 120\left(\frac{b}{\alpha^2}\right)^2\right)\mu \quad (116)$$

which is sharp for small values of μ.

Proof: The inequality $J(\mu) \leq \log_2(1+M\mu)$ is evident. To get the second bound, we compute the derivative with respect to μ

$$J'(\mu) = \frac{1}{Bln(2)}\int_0^B \frac{(a + bx)^2 f(x)}{1 + \mu(a + bx)^2 f(x)}dx. \quad (117)$$

Using the lower bound $1 + \mu(a + bx)^2 f(x) \geq 1 + \mu f(x) \geq 1 + f(B)\mu$, we obtain

$$J'(\mu) \leq \frac{1}{Bln(2)}\int_0^B \frac{(a + bx)^2 f(x)}{1 + \mu f(B)}dx. \quad (118)$$

We get

$$J'(\mu) \leq \frac{P}{\sigma_n^2}\frac{1}{Bln(2)}\frac{1}{(1 + \mu f(B))}\int_0^B h(x)dx \quad (119)$$

where

$$h(x) = (a^2 + 2abx + b^2x^2)e^{-\alpha\sqrt{x}}.$$

But $\int_0^\infty x^n e^{-\sqrt{x}}dx = 2\int_0^\infty t^{2n+1}e^{-t}dt = 2(2n+1)!$, and hence

$$\int_0^B e^{-\alpha\sqrt{x}}dx \leq \frac{1}{\alpha^2}\int_0^\infty e^{-\sqrt{x}}dx = \frac{2}{\alpha^2} \quad (120)$$

$$\int_0^B xe^{-\alpha\sqrt{x}}dx \leq \frac{1}{\alpha^4}\int_0^\infty xe^{-\sqrt{x}}dx = \frac{12}{\alpha^4} \quad (121)$$

$$\int_0^B x^2e^{-\alpha\sqrt{x}}dx \leq \frac{1}{\alpha^6}\int_0^\infty x^2e^{-\sqrt{x}}dx = \frac{240}{\alpha^6}. \quad (122)$$

Thus we get

$$J'(\mu) \leq \frac{P}{\sigma_n^2}\frac{1}{Bln(2)}\frac{1}{(1 + \mu f(B))}\left[\frac{2a^2}{\alpha^2} + \frac{24ab}{\alpha^4} + \frac{240b^2}{\alpha^6}\right]. \quad (123)$$

Integrating this inequality from $\mu = 0$ to t, we obtain

$$\int_0^t J'(\mu) \leq \frac{2P}{ln(2)\alpha^2 B\sigma_n^2}\frac{\ln(1 + tf(B))}{f(B)}$$

$$\times \left[a^2 + \frac{12ab}{\alpha^2} + \frac{120b^2}{\alpha^4}\right]. \quad (124)$$

Using the fact that $J(0) = 0$, we obtain the desired result.

Remark 13.1: We emphasize that M can be computed analytically. In fact, it is a routine exercise to write the maxima M of $f(x)$ in terms of a, b, α. Indeed

$$f'(x) = \frac{P}{\sigma_n^2} \left(2b(a+bx)e^{-\alpha\sqrt{x}} - \frac{(a+bx)^2}{2\alpha\sqrt{x}}e^{-\alpha\sqrt{x}} \right).$$

Thus $f'(x) = 0$ is equivalent to a quadratic equation and can be solved analytically. Since the function $f(x)$ may have at most two critical points, say, $x_1, x_2 \in [0, \infty)$, we find that

$$M = \max(f(0), f(x_1), f(x_1), f(B)).$$

APPENDIX H
LIFTING THE ASSUMPTION OF EQUAL PSD FROM THE MAIN THEOREM

In this Appendix, we prove a slight generalization of the main result, showing that the assumption of equal PSD in the binder is not necessary. The resulting bound is similar to that of the main Theorem 4.1.

To formulate the bound on the transmission loss, we introduce the quantities

$$P_{\max}(f) = \max_i(P_i(f)) \tag{125}$$

$$P_{\min}(f) = \min_{i: P_i(f) \neq 0}(P_i(f)). \tag{126}$$

We let $\rho(f) = P_{\max}(f)/P_i(f)$. We will say that the PSD satisfies the assumption SPSD(ρ) (or has dynamic range of width ρ) if we have

$$P_{\max}(f) \leq \rho P_{\min}(f).$$

We emphasize that this means that for each f such that $P_i(f) \neq 0$, we have

$$P_{\max}(f) \leq \rho P_i(f).$$

Remark 14.1: In realistic scenarios, the number ρ is limited by the maximal power backoff parameter of the modems in the system.

Theorem 14.2: Assume assumptions Perfect CSI, Quant (2^{-d}), and SPSD(ρ). Assume that the precoder $\mathbf{P}(f)$ is quantized using $d \geq 1/2 + \log_2(1 + r_{\max}))$ bits. Let $\mathbf{H}(f)$ be the channel matrix of p twisted pairs at frequency f. Let $r(f) = r(\mathbf{H}(f))$ as in (6). The transmission-rate loss of the ith user at frequency f due to quantization is bounded by

$$L_i(d, f) \leq \log_2(1 + \gamma(d, f)\text{SNR}_i(f)) - 2\log_2(1 - v(f)2^{-d}) \tag{127}$$

where

$$\gamma(d, f) = 2\rho(f)(p-1)(1 + r(f))^2 2^{-2d} \tag{128}$$

and

$$v(f) = \sqrt{2}(1 + r(f)). \tag{129}$$

Furthermore, the transmission loss in the band B is at most

$$\int_{f \in B} \log_2(1 + \gamma(d)\text{SNR}_i(f))df - 2|B|\tau(d) \tag{130}$$

where $|B|$ is the total bandwidth

$$\gamma(d) = 2\rho(1 + r_{\max})^2(p-1)2^{-2d} \tag{131}$$

and

$$\tau(d) = \log_2(1 - (1 + r_{\max})2^{-d+0.5}). \tag{132}$$

Proof: Only few changes in the proof of Theorem 4.1 are needed in order to derive the above theorem. In the proof of the main theorem, instead of (97), we have

$$a_i(f) = \sum_{j \neq i} \frac{P_j(f)}{P_i(f)} |\Delta_{i,j}(f)|^2 \text{SNR}_i(f)$$

$$\leq \sum_{j \neq i} \rho(f)|\Delta_{i,j}(f)|^2 \text{SNR}_i(f). \tag{133}$$

The bound on $\Delta_{i,j}(f)$ obtained in (95) is valid because our assumptions on the quantization are the same as in Theorem 4.1. Following the same line of reasoning as in (96) and (97) yields the bound (127). This, together with the assumption SPSD(ρ), easily yields (130). ∎

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