SEPARATION OF ZERO/CONSTANT MODULUS SIGNALS

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Zero/constant modulus (ZCM) signals are complex signals for which every sample is either zero or has modulus 1. Such signals arise after imprecise carrier-to-baseband conversion of binary $\{0,1\}$-modulated signals, or with intermittent phase-modulated signals. We consider the separation of linear superpositions of such signals using analytic techniques. An application is the separation of multiple partly coinciding aircraft transponder signals (SSR reply signals).

1. INTRODUCTION

Aircraft transponder signals (secondary surveillance radar (SSR) mode-S reply signals [1,2]) are in essence binary PAM signals with alphabet $\{0,1\}$ and symbol period $0.5\mu s$, modulated on a carrier $f_c = 1090 \text{MHz} \pm 3 \text{MHz}$. A data burst consists of 56 or 112 bits, which are subsequently Manchester encoded ($0 \rightarrow 01$, $1 \rightarrow 10$). It does not contain training bits. Because transponders can be triggered by neighboring ground stations, it frequently occurs that a ground station receives a superposition of two such signals, partially overlapping in frequency and time. We are interested in the possibility of separating them using adaptive antenna arrays.

A simple and quite general data model is obtained by defining a zero/constant modulus (ZCM) signal as a complex signal for which every sample is either zero or has modulus 1. The received data consists of several unknown linear combinations of such signals. Indeed, the zero symbols in this model are either part of the message, or can account for the fact that the message has finite duration. Likewise, the wide tolerance on the carrier frequencies in the SSR application implies that, after conversion to baseband, a residual carrier is present so that we have a ZCM signal rather than a binary $\{0,1\}$-signal.

Several techniques have been developed to estimate and separate linear superpositions of signals. They can broadly be characterized as (1) those that use properties of the channel, such as a parametrized multipath model and a known or structured antenna array, and (2) those that use properties of the signals. Some of the properties used in the latter category are *training* (known data), *constant modulus*, *finite alphabet*, *cyclo-stationarity*, and *statistical independence*. For each of these, several methods are available to estimate the mixing matrix. Typical methods are based on cost-function optimization using gradient-search or iterative techniques. Such methods are very much dependent on accurate initial points.

Since data bursts are short, we are interested in algebraic algorithms in which the mixing matrix is found as the best-fitting solution to a set of algebraic equations. For constant-modulus signals, a successful algorithm is the Analytic CMA (ACMA) [3], which solves an overdetermined set of quadratic equations. The algorithm has been specialized to separate superpositions of binary $\{\pm1\}$ or $\{0,1\}$ signals [4], and has interesting but unexplored connections to algebraic techniques for stochastic source separation, cf. e.g., [5]. In the present paper, we explore ways to account for the present more general conditions on the signals: data blocks due to indepen-
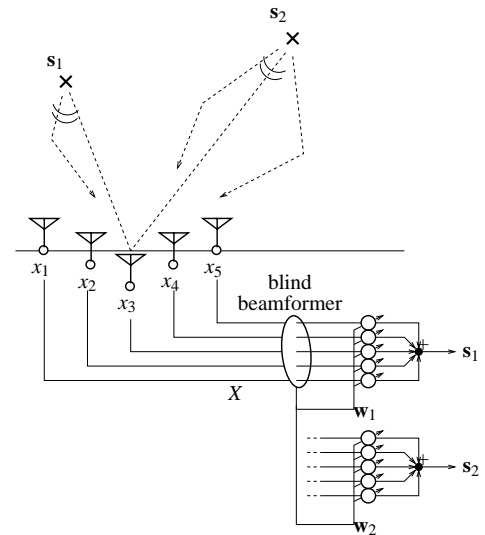

Figure 1. Beamforming scenario

dent sources are not necessarily synchronized in time, and carrier frequencies are not 100% identical.

2. SEPARATION BASED ON THE ZCM PROPERTY

2.1. Algorithm outline

We consider a scenario as depicted in figure 1. Suppose we have collected a data block $X = [\mathbf{x}_1 \cdots \mathbf{x}_N]$ of received signals, consisting of N complex-valued vector samples from M antennas. If d sources are present and the multipath delay spread is small relative to the sampling period, then the data matrix $X : M \times N$ is described by the standard model

$$X = AS = \mathbf{a}_1 \mathbf{s}_1 + \cdots + \mathbf{a}_d \mathbf{s}_d \,,$$

where all entries $(\mathbf{s}_i)_k \in$ ZCM. We assume that $d \leq M$, and that A and S have full rank d. The objective is to retrieve all (nontrivial) ZCM signals present in X, i.e. to find all complex beamforming vectors $\mathbf{w}$ that lead to linearly independent ZCM signals $\mathbf{s} = \mathbf{w}^* X$. To avoid nonuniqueness in case $d < M$, we first replace X by any full rank d matrix $V = [\mathbf{v}_1 \cdots \mathbf{v}_N]$ that has the same row span, for example as obtained from an SVD of X. Thus we look for all d linearly independent $\mathbf{w} \neq 0$ such that

$$\mathbf{w}^* V = \mathbf{s} \in \text{ZCM} \,.$$

Since A and S are assumed to have full rank d, this equation has at least d solutions $\{\mathbf{w}_1, \cdots, \mathbf{w}_d\}$. If sufficient samples are taken, then it is known for the CM case that generically there are precisely d solutions (unique up to arbitrary initial phase) [4], and by extension, this applies to the ZCM decomposition as well.

The ZCM property can be written as

$$s = 0 \quad \text{or} \quad |s| = 1 \quad \Leftrightarrow \quad s(s^*s - 1) = 0 \quad \Leftrightarrow \quad ss^*s = s \,.$$

Substitution of $\mathbf{w}^*\mathbf{v}_k = s_k$ gives

$$\mathbf{s} = \mathbf{w}^*V \in \text{ZCM} \quad \Leftrightarrow \quad \mathbf{w}^*\mathbf{v}_k\mathbf{v}_k^*\mathbf{w}\mathbf{w}^*\mathbf{v}_k = \mathbf{w}^*\mathbf{v}_k \quad \forall k.$$

The next step is to use a Kronecker product notation to separate the unknown $\mathbf{w}$ from the known $\mathbf{v}_k$'s. Note that the left-hand side contains only third-order terms of the entries of $\mathbf{w}$, whereas the right-hand side only has first-order terms. This imbalance is overcome by defining

$$\alpha = \|\mathbf{w}\|^2 = \mathbf{w}^*\mathbf{w}$$

(which is constant for each $\mathbf{w}$) and multiplying the right-hand side by $1 = \frac{1}{\alpha}\mathbf{w}^*\mathbf{w}$:

$$\begin{aligned}
&\mathbf{s} = \mathbf{w}^*V \in \text{ZCM} \\
\Leftrightarrow \quad & \mathbf{w}^*\mathbf{v}_k\mathbf{v}_k^*\mathbf{w}\mathbf{w}^*\mathbf{v}_k = \frac{1}{\alpha}\mathbf{w}^*\mathbf{w}\mathbf{w}^*\mathbf{v}_k \quad \forall k \\
\Leftrightarrow \quad & (\mathbf{v}_k \otimes \bar{\mathbf{v}}_k \otimes \mathbf{v}_k)^{\mathrm{T}}(\bar{\mathbf{w}} \otimes \mathbf{w} \otimes \bar{\mathbf{w}}) = \\
& = \frac{1}{\alpha}\text{vec}(I_d \otimes \mathbf{v}_k)^{\mathrm{T}}(\bar{\mathbf{w}} \otimes \mathbf{w} \otimes \bar{\mathbf{w}}) \quad \forall k.
\end{aligned}$$

Define matrices P_1, P_2 with rows $(\mathbf{v}_k \otimes \bar{\mathbf{v}}_k \otimes \mathbf{v}_k)^{\mathrm{T}}$ and $\text{vec}(I_d \otimes \mathbf{v}_k)^{\mathrm{T}}$, respectively. Then then ZCM separation problem is seen to be equivalent to finding all solutions $(\alpha, \mathbf{y})$, $\alpha \neq 0$ to

$$\alpha P_1\mathbf{y} = P_2\mathbf{y} \qquad \text{where } \mathbf{y} = \bar{\mathbf{w}} \otimes \mathbf{w} \otimes \bar{\mathbf{w}}. \qquad (1)$$

This is a (singular) matrix pencil problem of the form $A\mathbf{x} = \lambda B\mathbf{x}$, with "tall" A and B that are perhaps not full rank. If P_1 has full column rank (which requires at least $N > d^3$, and an additional processing step outlined in section 2.3 below), then it can be shown that the pair (P_1, P_2) has precisely d generalized eigenvalues (rank-reducing numbers), necessarily equal to $\alpha_i = \|\mathbf{w}_i\|^2$. If there are no repeated nonzero eigenvalues, then the corresponding eigenvectors are $\mathbf{y}_i \sim \bar{\mathbf{w}}_i \otimes \mathbf{w}_i \otimes \bar{\mathbf{w}}_i$, from which $\mathbf{w}_i$ is immediately obtained, up to scaling. The correct scaling of $\mathbf{w}_i$ follows from the corresponding eigenvalue α_i.

2.2. Repeated eigenvalues

If some of the nonzero eigenvalues in (1) are repeated, then the corresponding eigenvectors form an arbitrary basis of a subspace which contains the structured vectors we are looking for, and we need to find the correct linear combinations such that the Kronecker structure holds. In particular, if α is repeated δ times, with eigenvectors $\{\mathbf{y}_i\}_1^\delta$, then we need to find δ independent vectors $[\lambda_1 \cdots \lambda_\delta]$, each such that the resulting combination admits a factorization

$$\lambda_1\mathbf{y}_1 + \cdots + \lambda_\delta\mathbf{y}_\delta = \bar{\mathbf{w}} \otimes \mathbf{w} \otimes \bar{\mathbf{w}}.$$

This leads to a generalized "tensor-eigenvalue" problem that can be solved, much as in [4]. We omit the details of this in this paper. Note that, if V was taken to have orthonormal rows, then $\alpha_i = \mathbf{w}_i^*\mathbf{w}_i = \mathbf{s}_i^*\mathbf{s}_i$, so that there are repeated eigenvalues whenever two signals have an equal number of nonzero entries. The case of repeated eigenvalues can generically be avoided by not taking an orthonormal V.

In the simulations in section 4, we test both versions of the algorithm: AZCMA1 solves the generalized tensor-eigenvalue problem on the full collection of eigenvectors $\{\mathbf{y}_i\}_1^d$, whereas AZCMA2 simply assumes that all eigenvalues are distinct and omits the extra processing step.

2.3. Singular pencils

The above technique relied on the assumption that P_1 is full rank. However, this is never the case: the structure of the rows of P_1 implies that some of its columns are repeated. Similarly, $\mathbf{y} = \bar{\mathbf{w}} \otimes \mathbf{w} \otimes \bar{\mathbf{w}}$ has repeated entries, and we want our solutions to satisfy this property. Because it is known which entries are repeated, it is straightforward to enforce this: define a selection matrix J of size $d^3 \times \frac{1}{2}d^2(d+1)$, such that

$$\mathbf{y} = J\mathbf{y}',$$

where $\mathbf{y}'$ generically has no repeated entries. Set $P_1' := P_1J$, $P_2' := P_2J$, then generically P_1 has no repeated columns and has full rank. At this point, the pencil problem $\alpha P_1\mathbf{y} = P_2\mathbf{y}$ is replaced by

$$\alpha P_1'\mathbf{y}' = P_2'\mathbf{y}'.$$

Note that P_2' has only d nonzero columns. Hence, there are (at most) d nonzero eigenvalues α. If P_1' has full rank, then necessarily, these must be equal to the $\alpha_i = \|\mathbf{w}_i\|^2$. After finding solutions $\mathbf{y}_i'$, we set $\mathbf{y}_i = J\mathbf{y}_i'$ to add back the repeated entries, which brings us back to the case considered before.

2.4. Degenerate cases

Depending on the signals, there are other cases in which P_1' does not have full column rank. E.g., if two signals are purely CM, without zero entries, then one can show that P_1' is rank-deficient. In general, the columns of the signal matrix S should have "sufficiently many" different combinations with zero/nonzero entries. P_1' is also singular if two sources share exactly the same frequency.

Cases with singular P_1' are hard to solve. A more general Schur form is needed to reveal the eigenvalue structure [6], the eigenvalues are not equal to $\|\mathbf{w}_i\|^2$, and we end up with extra solution vectors $\mathbf{y}'$: a basis of the null space of P_1'. Hence, we cannot assume that the individual basis vectors $\mathbf{y}_i$ factor as $\bar{\mathbf{w}}_i \otimes \mathbf{w}_i \otimes \bar{\mathbf{w}}_i$, and we have to solve the generalized tensor-eigenvalue problem. A complication is that now the number of basis vectors is larger than the number of solutions. This case is not treated by the technique in [4] and remains a topic for future research.

2.5. Postprocessing

A further improvement of the estimate for $W = [\mathbf{w}_1 \cdots \mathbf{w}_d]$ is obtained by using it as initial point in an iterative algorithm to minimize the cost function $\min_{W, S \in \text{ZCM}} \|W^*X - S\|_F$. A simple algorithm based on alternating projections mimics the Gerschberg-Saxton constant-modulus algorithm GSA [7] (cf. [4]), and the ILSP algorithm for digital signal separation in [8]:

> **ZCMA**
> for $k = 1, 2, \cdots$
>   a. $S^{(i)} := \text{Proj}_{\text{ZCM}}\left[W^{(k)*}X\right]$
>   b. $W^{(k+1)*} := S^{(k)}X^\dagger$

where $\text{Proj}_{\text{ZCM}}(a) = \text{U}(|a| - \frac{1}{2}) \cdot \frac{a}{|a|}$, and $\text{U}(a)$ is the unit step function. Note that, unlike GSA and ILSP, this iterative algorithm relies on a proper initial scaling of W, or else the projection might map everything to 0. Initialized by one of the algebraic algorithms, the iterative algorithm converges in 2–3 iterations.

3. SEPARATION BASED ON FREQUENCY ESTIMATION

3.1. Algorithm outline

In section 2, our source data model was $s_k = 0 \vee |s_k| = 1$. However, for a binary $\{0,1\}$-source with some unknown residual carrier frequency f, more structure is available:

$$s_k = 0 \vee e^{i2\pi f k} \quad \forall k.$$

Call $\phi = e^{i2\pi f}$ ($|\phi| = 1$). The influence of k is removed by looking at $s_{k-1}^* s_k$, which is either 0 or ϕ. Altogether, we have

$$\left\{ \begin{array}{rcl} s_k^* s_k & = & 0 \vee 1 \\ s_{k-1}^* s_k & = & 0 \vee \phi \quad \forall k \end{array} \right.$$

$$\Leftrightarrow$$

$$\left\{ \begin{array}{rcl} s_k^* s_k (s_k^* s_k - 1) & = & 0 \\ s_{k-1}^* s_k (s_{k-1}^* s_k - \phi) & = & 0 \quad \forall k. \end{array} \right. \quad (2)$$

Similar as before, substitute $s_k = \mathbf{w}^* \mathbf{v}_k$. Define the Kronecker products $\mathbf{y} = \bar{\mathbf{w}} \otimes \mathbf{w}$, $\mathbf{p}_k^{(0)} = \mathbf{v}_k \otimes \bar{\mathbf{v}}_k$, $\mathbf{p}_k^{(1)} = \mathbf{v}_k \otimes \bar{\mathbf{v}}_{k-1}$. Then (2) becomes

$$\left\{ \begin{array}{rcl} [\mathbf{p}_k^{(0)} \otimes \mathbf{p}_k^{(0)}]^T (\mathbf{y} \otimes \mathbf{y}) & = & \mathbf{p}_k^{(0)T} \mathbf{y} \\ [\mathbf{p}_k^{(1)} \otimes \mathbf{p}_k^{(1)}]^T (\mathbf{y} \otimes \mathbf{y}) & = & \phi \mathbf{p}_k^{(1)T} \mathbf{y} \quad \forall k. \end{array} \right. \quad (3)$$

Collecting all equations into matrices as before, we obtain

$$\left[\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & \phi P_{22} \end{array} \right] \left[\begin{array}{c} \mathbf{z} \\ \mathbf{y} \end{array} \right] = 0 \quad (|\phi| = 1, \; \mathbf{y} = \bar{\mathbf{w}} \otimes \mathbf{w}, \; \mathbf{z} = \mathbf{y} \otimes \mathbf{y}), \quad (4)$$

and we have to find all d solution triples $(\phi, \mathbf{z}, \mathbf{y})$ of the indicated structure.

Again, the problem is essentially a generalized eigenvalue problem. This is immediately seen by writing (4) as

$$\left[\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & 0 \end{array} \right] \left[\begin{array}{c} \mathbf{z} \\ \mathbf{y} \end{array} \right] = -\phi \left[\begin{array}{cc} 0 & 0 \\ 0 & P_{22} \end{array} \right] \left[\begin{array}{c} \mathbf{z} \\ \mathbf{y} \end{array} \right]. \quad (5)$$

Hence, $-\phi$ is one of the generalized eigenvalues of the above matrix pair, and $[\mathbf{z}^T \; \mathbf{y}^T]^T$ is its corresponding eigenvector. In fact, there are d^2 finite eigenvalues, and we must choose the d eigenvalues that are on the unit circle (the others are located randomly throughout the plane). If the resulting $\{\phi_i\}$ are distinct (the residual frequencies modulo the sampling rate are not the same), then the second block of the corresponding eigenvectors $\{\mathbf{y}_i\}$ can directly be factored as $\mathbf{y}_i = \bar{\mathbf{w}}_i \otimes \mathbf{w}_i$, which gives the desired beamforming vectors. On the other hand, if frequencies are exactly coinciding, then again we need to solve a tensor-eigenvalue problem to find out which linear combinations of the corresponding eigenvectors lead to the desired Kronecker structure.

Because of the arbitrary normalization of eigenvectors, the factorization $\mathbf{y} = \bar{\mathbf{w}} \otimes \mathbf{w}$ determines the direction of $\mathbf{w}$, but not its scaling. The latter can be computed by looking at the norm of the corresponding $\mathbf{z}$: since $\mathbf{z} = \mathbf{y} \otimes \mathbf{y}$, we have $\|\mathbf{w}\|^2 = \|\mathbf{z}\| / \|\mathbf{y}\|$.

3.2. Real processing

Because of its symmetries, it is possible to transform $[\mathbf{z}^T \; \mathbf{y}^T]^T$ to a *real* vector without duplicates: there is a matrix J with a simple fixed structure, such that

$$J \left[\begin{array}{c} \mathbf{z}' \\ \mathbf{y}' \end{array} \right] = \left[\begin{array}{c} \mathbf{z} \\ \mathbf{y} \end{array} \right],$$

where z' and y' are real-valued without duplicate entries. We have to transform the P_{ij} to $P_{ij}' = P_{ij} J$ accordingly. After this transformation, $P_{11}', P_{12}', \mathbf{z}', \mathbf{y}'$ are real-valued, and since $|\phi| = 1$, we can come up with an additional relation $\bar{P}_{22}' \mathbf{y}' = -\phi \bar{P}_{21}' \mathbf{z}'$. Using this relation, the generalized eigenvalue problem (5) becomes

$$\left[\begin{array}{cc} P_{11}' & P_{12}' \\ P_{21}' & 0 \\ 0 & \bar{P}_{22}' \end{array} \right] \left[\begin{array}{c} \mathbf{z}' \\ \mathbf{y}' \end{array} \right] = -\phi \left[\begin{array}{cc} 0 & 0 \\ 0 & P_{22}' \\ \bar{P}_{21}' & 0 \end{array} \right] \left[\begin{array}{c} \mathbf{z}' \\ \mathbf{y}' \end{array} \right].$$

As it turns out, this data extension greatly improves the quality of the estimates. This algorithm is called AFZA in the simulations in section 4.

3.3. Alternative

Alternatively, we can try to combine both algorithms, by multiplying (3) by $1 = \frac{1}{\alpha} \mathbf{w}^* \mathbf{w}$. This then leads to a pair of coupled eigenvalue equations of the form

$$\begin{array}{rcl} (\alpha P_{11} + P_{12}) \mathbf{z} & = & 0 \\ (\beta P_{21} + P_{22}) \mathbf{z} & = & 0, \qquad \mathbf{z} = (\bar{\mathbf{w}} \otimes \mathbf{w}) \otimes (\bar{\mathbf{w}} \otimes \mathbf{w}), \end{array}$$

where $|\beta| = \alpha$. Unfortunately, it is not clear how such equations are optimally solved.

4. SIMULATION RESULTS

To get a crude understanding of the performance of the proposed algorithms, we simulated a scenario with $d = 2$ sources, arriving at an antenna array consisting of $M = 2$ antennas spaced at half-wavelength. The signals were Manchester coded binary signals (i.e. "0" is transmitted as "01", and "1" is transmitted as "10"). The data block covers 40 symbols (corresponding to $N = 80$ symbol periods of the coded signal), and signal 1 is present throughout. The sampling period T is equal to the coded (shorter) symbol period. The first signal has a randomly selected residual carrier frequency and arrives from $0°$ (broadside). The second signal has a different residual carrier frequency and direction and starts after a certain time separation, so that it is only partially present in the data block. It has the same power as the first signal.

The algorithms are tested for a range of SNRs, angle, time and frequency separations, and the results are averaged over 300 runs. The resulting worst residual signal-to-interference ratio (SIR) among the recovered signals is plotted in figure 2, labeled with '+' signs. The initial worst SIR before separation is 0 dB.

As discussed in section 2.5, the resulting estimates of W can also be used as initial points in the ZCMA iterative algorithm. This usually gives a significant improvement in the performance, as is shown by the 'o' curves. For comparison, we also plot the results of the iterative algorithm initialized with the true A, which indicates the 'best' (be it non-ML) type of performance that can be expected for this problem. It is seen that the algorithms after postprocessing usually reach the same performance.

Other observations are

- All three algorithms fail for precisely equal frequencies. For AFZA this is because the eigenvalues coincide, which is not taken into account in the implementation. For the two AZCMA algorithms, this is because the P_1 matrix becomes singular, for which there is yet no solution. The problem goes away already if the frequency difference is more than about 1 % of the symbol rate, or 1 cycle in the entire data batch.
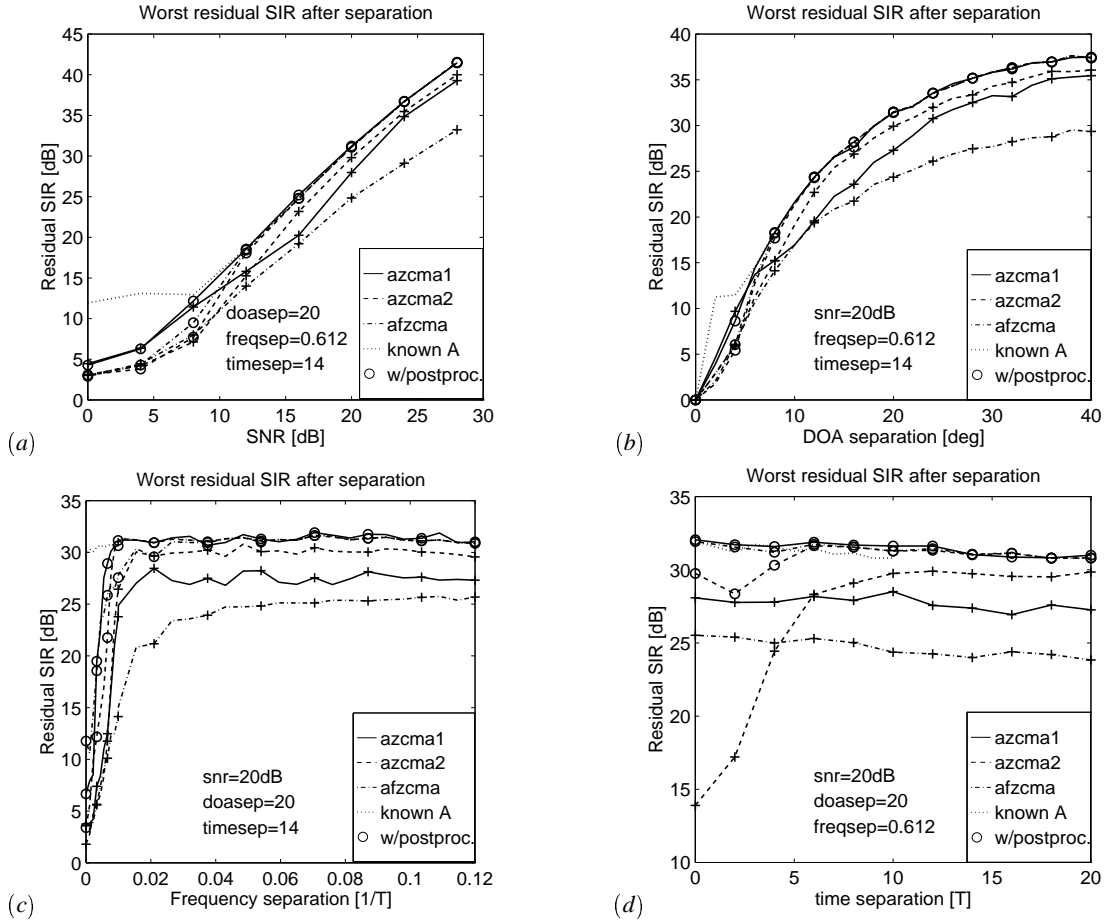
Figure 2. Average worst residual SIR among the recovered signals, as a function of (*a*) SNR, (*b*) DOA separation, (*c*) frequency separation, (*d*) time separation.

• The time separation is critical for AZCMA2. If it is small, then the number of 1's in both signals is approximately the same, so that the nonzero eigenvalues of P_1 coincide. This is not accounted for in AZCMA2, so that the algorithm fails in these cases.

• AFZA is usually less accurate than the other two methods. This is partly due to the fact that it is based on 4-th powers of the data rather than 3-rd. An important effect is also that the estimate of the frequencies is based on information on $s_{k-1}^* s_k$, which is nonzero only if both s_{k-1} and s_k are nonzero. This makes these estimates much more sensitive to noise.

The main conclusion to be drawn from the exploration in this paper is that it is certainly possible in theory to separate ZCM signals based on algebraic algorithms. Practical issues and efficient implementations have not been studied yet. The problem is rich in structure, and admits significant further improvements.

5. REFERENCES

[1] R.M. Trim, "Mode S: an introduction and overview," *Electronics & Communication Engineering Journal*, vol. 2, pp. 53–59, Apr. 1990.

[2] "International standards and recommended practices, aeronautical telecommunications: Annex 10," in *The convention on international civil aviation*, Int. Civil Aviation Org., 1985.

[3] A.J. van der Veen, "Analytical method for blind binary signal separation," *to appear IEEE Trans. Signal Processing*, 1997.

[4] A.J. van der Veen and A. Paulraj, "An analytical constant modulus algorithm," *IEEE Trans. Signal Processing*, vol. 44, pp. 1136–1155, May 1996.

[5] J.-F. Cardoso, "Super-symmetric decomposition of the fourth-order cumulant tensor. Blind identification of more sources than sensors," in *Proc. IEEE ICASSP*, (Toronto), pp. 3109–3112 vol.5, 1991.

[6] Th. Beelen and P. Van Dooren, "An improved algorithm for the computation of Kronecker's canonical form of a singular pencil," *Lin. Alg. Appl.*, vol. 105, pp. 9–65, 1988.

[7] R.W. Gerchberg and W.O. Saxton, "A practical algorithm for the determination of phase from image and diffraction plane pictures," *Optik*, vol. 35, pp. 237–246, 1972.

[8] S. Talwar, M. Viberg, and A. Paulraj, "Blind estimation of synchronous co-channel digital signals using an antenna array. Part I: Algorithms," *IEEE Trans. Signal Processing*, vol. 44, pp. 1184–1197, May 1996.