The Bounded Real Lemma for Discrete Time-Varying Systems with application to Robust Output Feedback

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Abstract

In this paper we develop a solution to the discrete-time robust output feedback control problem for Linear Time-Varying (LTV) systems. The solution is developed along the strategy set up in [1] and the main ingredient in its derivation is the extension of the well-known bounded real lemma in a (discrete) time-varying context, developed in [2]. This approach contributes to the conceptual simplicity, and hence to the accessibility, of the solution.

Apart from that, we treat the ∞-horizon case for LTV system of non-uniform state dimension, and varying input and output dimension. Both situations can easily occur in practice, e.g. in multirate sampled data control systems.

1. Introduction.

In this paper, we analyze the topic of robust control of LTV systems. In a time-invariant context this topic is indicated by H_∞ control and in the past decade a burst of research activity has taken place in this field. Without giving a detailed overview of the contributions in this field, we mention two main strategies to solve the "standard" four block H_∞ problem. One is the approach indicated by the so-called "1984 approach" in [1] and is based on various standard factorizations, such as spectral and inner-outer factorization, of transfer functions. This approach is well documented in [4]. The other is the "Riccati state space approach" presented in [1], which establishes a striking parallel between state space solutions to LQG and H_∞ control problems.

Most of the research activities in this area are for continuous time systems, however solutions exists in a discrete time context, such as [5], [6], [7] and [8].

For LTV system a restricted number of solutions have been published. The earliest contribution to this

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topic is the paper [9], where the so-called "1984 approach" has been formulated into an operator theoretic framework covering discrete LTV systems. In that paper, it was however remarked that "at present, computation of uniformly optimal controllers for LTV systems is not feasible". With the algorithms that have recently been developed [22] to calculate an inner-outer, spectral factorization and to solve the Nehari problem, we are now in a position to map the solution of [9] into a computational scheme. However, as in the time-invariant case such a solution will give rise to controllers of large system order. A particular situation that needs to be avoided in practice.

Within the class of solutions following the "1984 approach", we have solved a prototype robust control problem, namely the (weighted) sensitivity minimization problem, for discrete LTV systems [10]. As in the time-invariant case, this problem has been formulated as a Nevanlinna-Pick interpolation problem based on the inner-outer factorization of the given causal plant. Related contributions for periodic time-varying systems are [11] and [12].

In the wake of the pioneering paper [1] a number of extensions have been published treating LTV systems. In the context of differential games we mention the contributions of [13], [14] and in the context of the maximum principle we mention [15]. Apart from the work in [14] which also treats the discrete time case, all these solutions are for the finite horizon case and for continuous time systems. The particularly more difficult infinite horizon case has only been treated in [15] and [16] for continuous time systems.

In this paper, we treat the infinite horizon case for LTV discrete time systems. Apart from this, the merits of the paper are: (1) the simplicity of the solution only based on the discrete time Bounded Real Lemma for LTV systems [2], (2) the treatment of varying state dimensions (and input-output dimensions). It has been observed that the latter situations can easily occur in practice. E.g. the change of the input/output dimension occurs in multirate sampled data systems.

The solution presented follows the strategy developed in [1] and continuous on the contributions made in [8] and [17], discussing related problems for LTI systems. As in [1], the three different stages along which we develop a solution are: (1) Solving the robust static state feedback control problem and its dual variant of robust state reconstruction, (2) Formulating the plant to be controlled as a linear fractional transformation of an "inner" operator and (3) Combining the first two stages in providing a solution to the robust output feedback problem.

The present paper is organized as follows. In section 2, we give a brief overview of the notation and the representation of a state space model of LTV systems used throughout the paper. The variants of the bounded real lemma necessary to tackle the problems in the first stage are presented in section 3 and applied to the robust static state feedback problem in section 4 and the robust state reconstruction problem in section 5. The equivalent representation of the given plant as a linear fractional transformation of an "inner" operator and the solution to the robust output feedback problem are treated in section 6.

2. Preliminaries.

In this section, we introduce the notation used in representing Linear Time-Varying (LTV) systems.

A state space realization of the LTV system P to be controlled, is denoted on a local time scale as:

$$
\begin{aligned}
x_{k+1} &= x_k A_k + u_k B_k \\
y_k &= x_k C_k + u_k D_k
\end{aligned}
\tag{1}
$$

where x_k, u_k and y_k are (finite dimensional) row vectors in respectively $\mathbb{C}^{N_k}$, $\mathbb{C}^{M_k}$ and $\mathbb{C}^{L_k}$ and the matrices $\{A_k, B_k, C_k, D_k\}$ are bounded matrices of appropriate dimensions. Remark that this notation is compatible with the earlier work on LTV systems as reported in [3] and [22].

To denote the state space representation more compactly, we introduce as done in the paper [3] and [22], the dimension space sequences $\mathcal{B}$,

$$\mathcal{B} = \cdots \times \boxed{\mathcal{B}_0} \times \mathcal{B}_1 \times \cdots$$

where $\mathcal{B}_k = \mathbb{C}^{N_k}$ and the square box identifies the space of the 0–th entry. In a similar way, we introduce the dimension space sequence $\mathcal{M}$ and $\mathcal{N}$ from the integer sequences $\{M_k\}$ and $\{L_k\}$. It is allowed that some integers in these sequences are zero. The space of sequences in $\mathcal{B}$ with finite 2-norm will be denoted by $\ell_2^{\mathcal{B}}$. Next we stack the sequence of state vectors x_k, input vectors u_k and output vectors y_k into ∞-dimensional row vectors x, u and y; denoted explicitly for the state vector sequence as,

$$x = \begin{bmatrix} \cdots & x_{-1} & \boxed{x_0} & x_1 & \cdots \end{bmatrix}$$

where the square identifies the position of the 0-th entry. Let $\mathcal{B}^{(-1)}$ denote the shifted dimension space sequence of $\mathcal{B}$, i.e.

$$\mathcal{B}^{(-1)} = \cdots \times \boxed{\mathcal{B}_1} \times \mathcal{B}_2 \times \cdots$$

and let $\mathcal{D}(\mathcal{M}, \mathcal{N})$ denote the Hilbert space of bounded diagonal operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$, then we can stack the system operators A_k, B_k, C_k and D_k into the diagonal operators A, B, C and D, as (denoted only explicitly for A):

$$A = diag \begin{bmatrix} \cdots & A_{-1} & \boxed{A_0} & A_1 & \cdots \end{bmatrix} \in \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)}),$$

$$C \in \mathcal{D}(\mathcal{B}, \mathcal{N}), \quad B \in \mathcal{D}(\mathcal{M}, \mathcal{B}^{(-1)}), \quad D \in \mathcal{D}(\mathcal{M}, \mathcal{N}).$$

Let the causal bilateral shift operator on sequences be denoted by Z, such that,

$$\begin{bmatrix} \cdots & x_{-1} & \boxed{x_0} & x_1 & \cdots \end{bmatrix} Z = \begin{bmatrix} \cdots & x_{-2} & \boxed{x_{-1}} & x_0 & \cdots \end{bmatrix}$$

then a compact notation on a global time scale of the state space representation (1) is:

$$\begin{matrix} xZ^{-1} & = & xA + uB \\ y & = & xC + uD \end{matrix} \quad \text{also denoted as} \quad \mathbf{P} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{2}$$

With this notation it is possible to represent a LTV system as an operator. Let the transition operator $\Phi(j, k)$ of the system with state space representation (2) be defined as,

$$\Phi(j, k) = \begin{cases} A_k A_{k+1} \cdots A_{j-1} & j > k \\ I & j = k \\ \text{undefined} & j < k \end{cases}$$

and let $\lim_{j \to \infty} \Phi(j, k) = 0 \ \forall k < \infty$, then the inverse of the operator $(I - AZ)$ exists and is in $\mathcal{U}$ and the operator representation of the (*asymptotically stable* (a.s)) LTV system P becomes:

$$P = D + BZ(I - AZ)^{-1}C \tag{3}$$

This transfer operator is *upper* triangular and in general the Hilbert space of bounded upper operators

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acting from $\ell_2^{\mathcal{M}}$ to $\ell_2^{\mathcal{N}}$ is denoted by $\mathcal{U}(\mathcal{M}, \mathcal{N})$ or denoted in short by $\mathcal{U}$. When the dimension N_k of the state vector is finite for all k then the operator represented as in Eq. (3) is *locally finite*. In the same way as $\mathcal{U}$, we denote the space of bounded operators by $\mathcal{X}(\mathcal{M}, \mathcal{N})$ and the space of bounded *lower* triangular operators by $\mathcal{L}(\mathcal{M}, \mathcal{N})$.

Finally, operators representing input-output maps are sometimes indexed. In this way, the input-output map T_{wz} relates the input sequence w to the output sequence z.

3. The Bounded Real Lemma and its Extension.

In this section, we consider a causal system T with state realization $\mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ such that A, B, C, D have the dimensions as indicated in the Preliminary section.

We only consider the bounded real lemma related to the spectral factorization of the operator $\Gamma_o I - T^* T$. Let us recall Theorem 11 and Proposition 14 of [2].

Theorem 1.   **(Theorem 11, [2])** *Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a locally finite operator with an observable state realization $\{A, B, C, D\}$ such that A is a.s. Let $\Gamma_o \in \mathcal{D}(\mathcal{N}, \mathcal{N})$ such that $\Gamma_o \gg 0$. Then $\Gamma_o I - T^* T \gg 0$ if and only if there exists a solution $M_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of*

$$
\begin{aligned}
M_o^{(-1)} \;=\;& A^* M_o A \;+\; [A^* M_o C + B^* D] (\Gamma_o I - D^* D - C^* M_o C)^{-1} \\
& [D^* B + C^* M_o A] + B^* B
\end{aligned}
\tag{4}
$$

such that $\Gamma_o I - D^* D - C^* M_o C \gg 0$ and $M_o \geq 0$. This M_o is unique. If in addition the realization of T is [uniformly] controllable, then M_o is [uniformly] positive.

If $\Gamma_o I - T^* T \gg 0$, let $W \in \mathcal{U}(\mathcal{N}, \mathcal{N})$ be a factor of $\Gamma_o I - T^* T = W^* W$. A realization $\{A, B_W, C, D_W\}$ for W such that W is outer is then given by the solution M_o of the above equation, and solutions D_W, B_W of

$$
\left\{
\begin{aligned}
D_W^* D_W \;&=\; \Gamma_o I - D^* D - C^* M_o C \\
B_W \;&=\; -D_W^{-*} [D^* B + C^* M_o A] \;.
\end{aligned}
\right.
\tag{5}
$$

Proposition 2.   **(Proposition 14, [2])** *Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{M})$ be an outer invertible operator, with state realization $\mathbf{T} = \{A, B, C, D\}$. Then $S = T^{-1} \in \mathcal{U}(\mathcal{M}, \mathcal{M})$ has a state realization given by*

$$
\mathbf{S} = \begin{bmatrix} A - CD^{-1}B & -CD^{-1} \\ D^{-1}B & D^{-1} \end{bmatrix}
\tag{6}
$$

Moreover, $\mathbf{T}$ is [uniformly] controllable if and only if $\mathbf{S}$ is [uniformly] controllable, $\mathbf{T}$ is [uniformly] observable if and only if $\mathbf{S}$ is [uniformly] observable. Let $A^\times = A - CD^{-1}B$. If A is a.s. and $\mathbf{T}$ is controllable or observable, then $A^\times$ is a.s.

Based on this proposition, we have the following Corollary to Theorem 1.

Corollary 3.   *Let the conditions of Theorem 1 hold, and let the same quantities as in this Theorem be defined. Then the operator $A_o^\times$, defined as:*

$$
A_o^\times = A + C(\Gamma_o I - D^* D - C^* M_o C)^{-1} [D^* B + C^* M_o A]
\tag{7}
$$

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is a.s.

In order to address the robust control problems of this paper, we need the following extension of the version of the bounded real lemma in Theorem 1. In the proof of this extension, we make use of the following definition and Lemma.

Definition 4. *The pair (A, B) is uniformly stabilizable if and only if there exists a bounded operator $F \in \mathcal{D}(\mathcal{B}, \mathcal{M})$ such that $A + FB$ is a.s.*

Lemma 5. **(Extended Lyapunov lemma, [19])** *Suppose the pair (A, B) is uniformly stabilizable. Then if there exists a solution $X \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ and $X \geq 0$ of:*

$$X^{(-1)} = A^*XA + B^*B \tag{8}$$

then A is a.s. Conversely, if A is a.s., then there exists a unique bounded solution $X \geq 0$ of Eq. (8).

Theorem 6. *Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a locally finite operator with realization $\{A, B, C, D\}$. Let $\Gamma_o \in \mathcal{D}(\mathcal{N}, \mathcal{N})$ such that $\Gamma_o \gg 0$. Then $\Gamma_o I - T^*T \gg 0$ and A is a.s. if and only if there exists a unique solution $M_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of Eq. (4) such that $\Gamma_o I - D^*D - C^*M_oC \gg 0$ and $M_o \geq 0$. This operator M_o defines the operator $A_o^\times$ as in Eq. (7) such that $A_o^\times$ is a.s.*

Remark 7. *It is easy to state the dual (controllable) variants of Theorem 1, Definition 4, Lemma 5 and Theorem 6. This is done in an extended version of this paper.*

4. Robust Static State Feedback.

Using the controllable dual of Theorem 6, we are now in a position to generalize the solution presented in [8] to the static H_∞ state feedback control problem for time-invariant systems to the time-varying case.

Consider the time-variant system T with state space realization:

$$
\begin{aligned}
xZ^{-1} &= xA &+ wB_1 &+ uB_2 \\
z &= xC_1 &+ wD_{11} &+ uD_{21} \\
y &= x
\end{aligned}
\tag{9}
$$

Note that we do not assume $T \in \mathcal{U}(\mathcal{M}_1 + \mathcal{M}_2, \mathcal{N}_1 + \mathcal{B})$ since we allow the A-operator of (9) to be unstable.

We make the following standard assumptions:

Assumptions 8. *(1) The pair (A, B_2) is uniformly stabilizable, and (2) The operator $D_{21}D_{21}^*$ is uniformly positive.*

The robust static state feedback control problem can be stated as follows (Fig. 1): For a given level of disturbance attenuation $\Gamma_c \gg 0$, $\Gamma_c \in \mathcal{D}(\mathcal{M}_1, \mathcal{M}_1)$, find (if it exists) a bounded static state feedback control law $u = yF = xF$, with $F \in \mathcal{D}(\mathcal{B}, \mathcal{M}_2)$, such that:

1. The A-operator $A + FB_2$ of the closed-loop system in Figure 1 is a.s. and

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2. The closed-loop operator T_{wz} between w and z with realization $\{A+FB_2, B_1, C_1+FD_{21}, D_{11}\}$ satisfies:

$$\Gamma_c I - T_{wz} T_{wz}^* \gg 0$$

Figure 1. Block-schematic representation of the robust static state feedback problem.

A solution to the robust static state feedback problem is provided in the next theorem.

Theorem 9. *Let T be a locally finite operator with state space realization (9) and satisfying the Assumptions 8. Furthermore, let $\Gamma_c \in \mathcal{D}(\mathcal{M}_1, \mathcal{M}_1)$ be a prescribed level of disturbance attenuation, such that $\Gamma_c \gg 0$. Then an operator $F \in \mathcal{D}(\mathcal{B}, \mathcal{M}_2)$ solves that the robust static state feedback control problem if and only if there exists a solution $M_c \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of,*

$$M_c = AM_c^{(-1)}A^* + \left[C_1 D_E^* + AM_c^{(-1)}B_E^*\right] \left(\begin{bmatrix} \Gamma_c I & 0 \\ 0 & 0 \end{bmatrix} - D_E D_E^* - B_E M_c^{(-1)} B_E^*\right)^{-1}$$

$$\left[D_E C_1^* + B_E M_c^{(-1)} A^*\right] + C_1 C_1^* \tag{10}$$

with $D_E = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$ and $B_E = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, such that

$$\Gamma_c I - D_{11} D_{11}^* - B_1 M_c^{(-1)} B_1^* \gg 0 \quad M_c \text{ is unique and } \geq 0$$

and the operator $A_c^\times$, defined as:

$$A_c^\times = A + \left[C_1 D_E^* + AM_c^{(-1)}B_E^*\right] \left(\begin{bmatrix} \Gamma_c I & 0 \\ 0 & 0 \end{bmatrix} - D_E D_E^* - B_E M_c^{(-1)} B_E^*\right)^{-1} B_E$$

is a.s. With this solution M_c of Eq. (10), the static state feedback law is given as,

$$F = \left[C_1 D_E^* + AM_c^{(-1)}B_E^*\right] \left(\begin{bmatrix} \Gamma_c I & 0 \\ 0 & 0 \end{bmatrix} - D_E D_E^* - B_E M_c^{(-1)} B_E^*\right)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{11}$$

Remark 10. *A dual Theorem can be formulated to solve the robust state reconstruction problem. This is done in an extended version of the paper, but see also [18].*

5. Robust Output Feedback.

Let the time-variant system T be given with state space realization:

$$
\begin{aligned}
xZ^{-1} &= xA &+ wB_1 &+ uB_2 \\
z &= xC_1 & &+ uD_{21} \\
y &= xC_2 &+ wD_{12} &+ uD_{22}
\end{aligned}
\tag{12}
$$

Consider the time-variant controller K with state space realization:

$$
\begin{aligned}
\xi Z^{-1} &= \xi\Phi + y\Psi_1 \\
u &= \xi\Psi_2 + y\Psi_3
\end{aligned}
\tag{13}
$$

where Φ, Ψ_1, Ψ_2 and Ψ_3 are bounded diagonal operators and where the state dimensions still has to be determined. Both systems are connected as displayed in Figure 2. Then under the following assumptions:

Figure 2. Block-schematic representation of the robust output feedback problem.

Assumptions 11. *(1) The pair $(A\ ,\ B_2)$ is uniformly stabilizable, the operator $D_{21}D_{21}^* \gg 0$ and $\Gamma_c = \gamma I_{\mathcal{M}_1}$ with $\gamma > 0$ is chosen such that a solution exists to the robust static state feedback problem treated in section 4 and solved in Theorem 9. (2) The pair $(\overline{A}\ ,\ \overline{C}_2) = (A + \tilde{B}_1 U_1^{-1}B_1\ ,\ C_2 + \tilde{B}_1 U_1^{-1}D_{12})$, with the quantities $\tilde{B}_1$ and U_1 defined in Eq. (15), is uniformly detectable and (3) The operator $D_{12}^*D_{12}$ is uniformly positive.*

We can state the robust feedback problem as follows (Figure 2): For a given level of disturbance attenuation $\Gamma_c = \gamma I_{\mathcal{M}_1}$ with $\gamma > 0$, find a state space realization $\{\Phi, \Psi_1, \Psi_2, \Psi_3\}$ of the controller K in Eq. (13), such that:

1. The A-operator of the closed-loop system in Figure 2, which has the following form:

$$
A_{\mathrm{cl}} = \begin{bmatrix} A + C_2\Psi_3(I - D_{22}\Psi_3)^{-1}B_2 & C_2\Psi_1 + C_2\Psi_3(I - D_{22}\Psi_3)^{-1}D_{22} \\ \Psi_2(I - D_{22}\Psi_3)^{-1}B_2 & \Phi + \Psi_2(I - D_{22}\Psi_3)^{-1}D_{22}\Psi_1 \end{bmatrix},
$$

is a.s. When this is the case, the closed-loop system depicted in Figure 2 is internally stable, as defined in Definition 14.

2. The operator T_{wz} between w and z in Figure 2 satisfies $\Gamma_c I - T_{wz}T_{wz}^* \gg 0$.

As outlined in the introduction, a solution to this problem will be developed in three different stages. The first stage is the solution to the robust static state feedback control problem discussed in section 4. Theorem 9 provides the static feedback gain operator F that solves this problem. Continuing with this solution we will now subsequently treat the next two stages, namely:

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Stage 2 Using the solution of the first stage we formulate and solve an intermediate problem that falls within the class of robust state reconstruction problems. The quantity that will be reconstructed in this intermediate problem is xF.

Stage 3 Relate the solution derived in Stage 2 to the original robust output feedback problem.

5.1. Equivalent representation of the given LTV plant T.

As for the solution given to the time-invariant H_∞ output feedback problem in [1], [17] we derive based on the solution to the robust static state feedback problem two LTV systems from the given plant T making use of the following identity:

$$xZ^{-1}M_c^{(-1)}Zx^* - xM_cx^* = 0$$

Making use of the state space representation in Eq. (12) and the expression for M_c in Theorem 9, rewritten as,

$$M_c = AM_c^{(-1)}A^* + \tilde{B}_1U_1^{-1}\tilde{B}_1^* + C_1C_1^* - U_3U_2^{-1}U_3^* \tag{14}$$

with,

$$
\begin{array}{llll}
U_1 &=& \Gamma_cI - B_1M_c^{(-1)}B_1^* & \tilde{B}_1 &=& AM_c^{(-1)}B_1^* \\
\tilde{B}_2 &=& C_1D_{21}^* + AM_c^{(-1)}B_2^* & \tilde{B}_3 &=& +B_2M_c^{(-1)}B_1^* \\
U_2 &=& B_2M_c^{(-1)}B_2^* + \tilde{B}_3U_1^{-1}\tilde{B}_3^* + D_{21}D_{21} & U_3 &=& \tilde{B}_2 + \tilde{B}_1U_1^{-1}\tilde{B}_3^*
\end{array}
\tag{15}
$$

the above identity can be written as,

$$
\begin{aligned}
0 &= x\left[U_3U_2^{-1}U_3^* - \tilde{B}_1U_1^{-1}\tilde{B}_1^*\right]x^* - xC_1C_1^*x^* \\
&\quad + wB_1M_c^{(-1)}B_1^*w^* + uB_2M_c^{(-1)}B_2^*u^* \\
&\quad + 2xAM_c^{(-1)}B_1^*w^* + 2uB_2M_c^{(-1)}B_1^*w^* + 2xAM_c^{(-1)}B_2^*u^*
\end{aligned}
$$

Adding and subtracting the term $w\Gamma_cw^*$, $uD_{21}D_{21}^*u^*$, $2xC_1D_{21}^*u^*$ and using the expressions in Eq. (15), yields:

$$
\begin{aligned}
0 &= w\Gamma_cw^* - zz^* + x\left[U_3U_2^{-1}U_3^* - \tilde{B}_1U_1^{-1}\tilde{B}_1^*\right]x^* \\
&\quad \underline{- wU_1w^*} + uU_2u^* - u\tilde{B}_3U_1^{-1}\tilde{B}_3^*u^* \\
&\quad + 2x\tilde{B}_2u^* + \underline{2(x\tilde{B}_1 + u\tilde{B}_3)w^*}
\end{aligned}
$$

Completing the squares with the underlined terms, using the expression for U_3 in Eq. (15) and F now denoted as $-U_3U_2^{-1}$, leads to:

$$
\begin{aligned}
0 &= w\Gamma_cw^* - zz^* \\
&\quad - \left[(w - (x\tilde{B}_1 + u\tilde{B}_3)U_1^{-1})U_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}}\right]\Gamma_c\left[\Gamma_c^{-\frac{1}{2}}U_1^{\frac{1}{2}}(w^* - U_1^{-1}(\tilde{B}_1^*x^* + \tilde{B}_3^*u^*))\right] \\
&\quad + (u - xF)U_2(u^* - F^*x^*)
\end{aligned}
$$

Therefore, if we define the quantities v and r as:

$$v = (u - xF)U_2^{\frac{1}{2}} \tag{16}$$

$$r = (w - (x\tilde{B}_1 + u\tilde{B}_3)U_1^{-1})U_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \tag{17}$$

then the above identity can be written compactly as,

$$w\Gamma_cw^* - zz^* = r\Gamma_cr^* - vv^* \tag{18}$$

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Using Eq. (16), a first new LTV system P that can be derived from the LTV system T has the following input-output relationship,

$$\begin{bmatrix} z & r \end{bmatrix} = \begin{bmatrix} w & v \end{bmatrix} P \tag{19}$$

The latter system P has the state space representation:

$$xZ^{-1} = x(A + FB_2) + wB_1 + vU_2^{-\frac{1}{2}}B_2 \tag{20}$$

$$z = x(C_1 + FD_{21}) + vU_2^{-\frac{1}{2}}D_{21} \tag{21}$$

$$r = -xC_2' + wU_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} - vU_2^{-\frac{1}{2}}\tilde{B}_3 U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \tag{22}$$

with $C_2' = \tilde{B}_1 U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} + F\tilde{B}_3 U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}})$.

In the same way, we can define a second LTV system $\overline{T}$, such that,

$$\begin{bmatrix} v & y \end{bmatrix} = \begin{bmatrix} r & u \end{bmatrix} \overline{T} \tag{23}$$

$\overline{T}$ has the state space representation,

$$\begin{aligned} xZ^{-1} &= x(A + \tilde{B}_1 U_1^{-1}B_1) &+ r\Gamma_c^{\frac{1}{2}}U_1^{-\frac{1}{2}}B_1 &+ u(\tilde{B}_3 U_1^{-1}B_1 + B_2) \\ v &= -xFU_2^{\frac{1}{2}} &+ &+ uU_2^{\frac{1}{2}} \\ y &= x(C_2 + \tilde{B}_1 U_1^{-1}D_{12}) &+ r\Gamma_c^{\frac{1}{2}}U_1^{-\frac{1}{2}}D_{12} &+ u(\tilde{B}_3 U_1^{-1}D_{12} + D_{22}) \end{aligned}$$

denoted compactly as,

$$\begin{aligned} xZ^{-1} &= x\overline{A} &+ r\overline{B}_1 &+ u\overline{B}_2 \\ v &= x\overline{C}_1 + & &+ u\overline{D}_{21} \\ y &= x\overline{C}_2 &+ r\overline{D}_{12} &+ u\overline{D}_{22} \end{aligned} \tag{24}$$

5.2. An intermediate robust state reconstruction problem for the LTV system $\overline{T}$.

For the LTV system $\overline{T}$, given by the state space realization in Eq. (24), our aim of this section is to design an observer $\overline{K}$ to reconstruct the quantity xF. Following the outline of section 5, the observer $\overline{K}$ has the state space representation:

$$\begin{aligned} \hat{x}Z^{-1} &= \hat{x}\overline{A} &+ u\overline{B}_2 &+ (y - \hat{x}\overline{C}_2 - u\overline{D}_{22})\overline{L} \\ u &= \hat{x}F \end{aligned} \tag{25}$$

and the error $\zeta = x - \hat{x}$ on the reconstructed state quantities satisfies,

$$\zeta Z^{-1} = \zeta(\overline{A} - \overline{C}_2\overline{L}) + r(\overline{B}_1 - \overline{D}_{12}\overline{L}) \tag{26}$$

With the expression for F given as $-U_3 U_2^{-1}$, we can express the output v of the LTV system $\overline{T}$ as:

$$v = \zeta U_3 U_2^{-\frac{1}{2}} = \zeta\overline{C}_1 \tag{27}$$

Taking into account that the robust static state feedback problem of section 4 has been solved, that the assumptions 11(2-3) are satisfied and therefore the operator $\overline{D}_{12}^{*}\overline{D}_{12} \gg 0$, we address the following intermediate robust state reconstruction problem: For a given level of disturbance attenuation $\overline{\Gamma}_o \gg 0$, $\overline{\Gamma}_o \in \mathcal{D}(\mathcal{M}_2, \mathcal{M}_2)$, find (if it exists) a bounded operator $\overline{L}$, with $\overline{L} \in \mathcal{D}(\mathcal{N}_2, \mathcal{B}^{(-1)})$, such that:

1. The A-operator, $\overline{A} - \overline{C}_2\overline{L}$, of the filter $\overline{K}$ in Figure 3 is a.s., and

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Figure 3. Block-schematic representation of the intermediate robust state reconstruction problem.

2. The operator $\overline{T}_{rv}$ between r and v in Figure 3 with realization $\{\overline{A} - \overline{C}_2\overline{L}, \overline{B}_1 - \overline{D}_{12}\overline{L}, \overline{C}_1, 0\}$ satisfies:

$$\overline{\Gamma}_o I - \overline{T}_{rv}^* \overline{T}_{rv} \gg 0$$

A solution to this intermediate problem is provided by the observable dual of Theorem 9. This solution is summarized in the next Theorem, stated without proof.

Theorem 12. *Let $\overline{T}$ be a locally finite operator with state realization (24) and satisfying Assumptions 11(2-3). Furthermore, let $\overline{\Gamma}_o \in \mathcal{D}(\mathcal{M}_2, \mathcal{M}_2)$, be a prescribed disturbance attenuation level such that $\overline{\Gamma}_o \gg 0$. Then an operator $\overline{L} \in \mathcal{D}(\mathcal{N}_2, \mathcal{B}^{(-1)})$ solves the <u>intermediate</u> robust state reconstruction problem if and only if there exists a solution $\overline{M}_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of*

$$\overline{M}_o^{(-1)} = \overline{A}^* \overline{M}_o \overline{A} + [\overline{B}_1^* \overline{D}_E + \overline{A}^* \overline{M}_o \overline{C}_E] \left(\begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_E^* \overline{D}_E - \overline{C}_E^* \overline{M}_o \overline{C}_E \right)^{-1}$$

$$[\overline{D}_E^* \overline{B}_1 + \overline{C}_E^* \overline{M}_o \overline{A}] + \overline{B}_1^* \overline{B}_1 \tag{28}$$

with $\overline{D}_E = \begin{bmatrix} 0 & \overline{D}_{12} \end{bmatrix}$ and $\overline{C}_E = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix}$, such that

$$\overline{\Gamma}_o I - \overline{C}_1^* \overline{M}_o \overline{C}_1 \gg 0, \quad \overline{M}_o \text{is unique and } \geq 0$$

and the operator $\overline{A}_o^\times$, defined as:

$$\overline{A}_o^\times = \overline{A} + \overline{C}_E \left(\begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_E^* \overline{D}_E - \overline{C}_E^* \overline{M}_o \overline{C}_E \right)^{-1} [\overline{D}_E^* \overline{B}_1 + \overline{C}_E^* \overline{M}_o \overline{A}]$$

is a.s. With this solution $\overline{M}_o$ of Eq. (28), the observer gain operator $\overline{L}$ is given as,

$$\overline{L} = -\begin{bmatrix} 0 & I \end{bmatrix} \left(\begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_E^* \overline{D}_E - \overline{C}_E^* \overline{M}_o \overline{C}_E \right)^{-1} [\overline{D}_E^* \overline{B}_1 + \overline{C}_E^* \overline{M}_o \overline{A}] \tag{29}$$

5.3. A solution to the robust output feedback problem.

The LTV system P defined in the previous subsection has some interesting properties highlighted in the following Lemma.

Lemma 13. *Let the LTV system $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ be defined as in Eq. (19), with state space realization given in Eq. (20), then the following conditions hold:*

1. *$P \in \mathcal{U}^2$.*

2. $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = \begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}.$

3. $P_{12}^{-1} \in \mathcal{U}.$

In the following lemma, we consider LTV systems P satisfying the conditions 1 to 3 of Lemma 13 operating in closed-loop with a LTV system Q as depicted in Figure 4. In this lemma, we make use of

Figure 4. Closed-loop configuration of a LTV system P satisfying the conditions of Lemma 18 with a LTV system Q.

the following definition of internal stability.

Definition 14.   *The closed-loop configuration depicted in Figure 4 is internally stable if and only if,*

$$T_{wz} \in \mathcal{U}, \ T_{wr} \in \mathcal{U}, \ T_{wv} \in \mathcal{U} \text{ and } Q \in \mathcal{U}$$

Lemma 15.   *Let* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ *be a given LTV system satisfying the conditions 1 to 3 of Lemma 13. Let* $Q \in \mathcal{U}$. *Then, the system is internally stable, well-posed and* $\Gamma_c I - T_{wz} T_{wz}^* \gg 0$ *if and only if* $\Gamma_c I - QQ^* \gg 0$

Proof ($\Longleftarrow$) From the relationship $\begin{bmatrix} z & r \end{bmatrix} = \begin{bmatrix} w & v \end{bmatrix} P$ and $v = rQ$, we derive,

$$
\begin{array}{rcll}
r & = & wP_{12}(I - QP_{22})^{-1} & = & wT_{wr} \\
v & = & wP_{12}(I - QP_{22})^{-1}Q & = & wT_{wr}Q \\
z & = & w\left[P_{11} + P_{12}(I - QP_{22})^{-1}QP_{21}\right] &
\end{array}
\tag{30}
$$

Since $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = \begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}$, it follows that $P_{22}\Gamma_c P_{22}^* = I - P_{21}P_{21}^* \leq I$. Hence, $I \geq P_{22}\Gamma_c P_{22}^* \gg P_{22}QQ^* P_{22}$ and $(I - QP_{22})^{-1} \in \mathcal{U}$. Therefore, the closed-loop system is well posed. Since $P \in \mathcal{U}^2$ and $Q \in \mathcal{U}$, internal stability follows.

Again, from the property $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = \begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}$, we derive that,

$$zz^* + r\Gamma_c r^* = w\Gamma_c w^* + vv^*$$

Using the definition of r and v in terms of w as given in Eq. (30), this is equivalent to,

$$zz^* = w\left(\Gamma_c I + T_{wr}QQ^* T_{wr}^* - T_{wr}\Gamma_c T_{wr}^*\right)w^*$$

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$$= w \left(\Gamma_c I - T_{wr} (\Gamma_c I - QQ^*) T_{wr}^* \right) w^*$$

Since $\Gamma_c I - QQ^* \gg 0$ and T_{wr}, T_{wr}^{-1} are both in $\mathcal{U}$, it follows that,

$$zz^* < w\Gamma_c w^*$$

for $\forall w \neq 0$. This is equivalent to,

$$\Gamma_c I - T_{wz} T_{wz}^* \gg 0$$

($\Rightarrow$) Suppose there exists a non-zero ℓ_2 sequence r defining the ℓ_2 sequences $v = rQ$, $w = r(I - QP_{22})P_{12}^{-1}$ and $z = wT_{wz}$, such that,

$$vv^* - r\Gamma_c r^* \geq 0$$

Then, the relationship $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = \begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}$, shows that,

$$zz^* - w\Gamma_c w^* = vv^* - r\Gamma_c r^* \geq 0$$

And therefore, there exists an ℓ_2 sequence w, such that,

$$w(T_{wz} T_{wz}^* - \Gamma_c I)w^* \geq 0$$

However, this is a contradiction and the lemma is proved. $\qquad\square$

The above Lemma is the key towards the solution of the robust output feedback problem. In order to apply this Lemma, we consider the feedback configuration in Figure 4 with the LTV system Q replaced by the LTV system of Figure 3. This is depicted in our final Figure 5. The solution to robust output feedback problem is summarized in our final Theorem.

Figure 5. Block-schematic representation of the solution to robust output feedback problem.

Theorem 16.   *Let T be a locally finite operator with state space realization in Eqs. (12) and satisfying the Assumptions 11. Furthermore, let $\Gamma_c = \gamma I_{\mathcal{M}_1}$ be a prescribed disturbance attenuation level with $\gamma > 0$. For this Γ_c, let M_c be a solution to the Riccati equation (10) satisfying the conditions of Theorem 9. Let this M_c define the state space representation of the LTV system $\overline{T}$ as in Eq. (24). Let $\overline{\Gamma}_o = \gamma I_{\mathcal{M}_2}$ and let $\overline{M}_o$ be a solution to the Riccati equation (28) satisfying the conditions stated in Theorem 12, then the controller $\overline{K}$ defined in Eq. (25) by the observer gain operator $\overline{L}$ of Eq. (29) solves the robust output*

feedback problem.

Proof In addition to the system $\overline{T}$, the operator M_c defines the LTV system P in Figure 5 with state space representation as in Eq. (20). Since this system P satisfies conditions 1 to 3 of Lemma 13, we only have to show that the LTV system within the dashed box of Figure 5 satisfies the conditions stipulated on the LTV system Q in Lemma 15, in order to apply this Lemma. These are:

1. $\Gamma_c I - \overline{T}_{rv}\overline{T}_{rv}^* \gg 0$: Since the solution $\overline{M}_o$ of Theorem 12 guarantees that $\overline{\Gamma}_o I - \overline{T}_{rv}^*\overline{T}_{rv} \gg 0$ and since $\Gamma_c = \gamma I_{\mathcal{M}_1}$ and $\overline{\overline{\Gamma}}_o = \gamma I_{\mathcal{M}_2}$, the identity,

$$\left[\Gamma_c I - \overline{T}_{rv}\overline{T}_{rv}^*\right]^{-1} = \Gamma_c^{-1}\left[I + \overline{T}_{rv}(\overline{\Gamma}_o I - \overline{T}_{rv}^*\overline{T}_{rv})^{-1}\overline{T}_{rv}^*\right]$$

shows that this condition holds.

2. $\overline{T}_{rv} \in \mathcal{U}$: To show that the operator $\overline{T}_{rv}$ belongs to $\mathcal{U}$, we derive a state space representation of this operator. Recall the state space realization for $\overline{T}$ in Eq. (24) with $u = \hat{x}F$:

$$\begin{aligned} xZ^{-1} &= x\overline{A} &+& r\overline{B}_1 &+& \hat{x}F\overline{B}_2 \\ v &= x\overline{C}_1 &+& &+& \hat{x}F\overline{D}_{21} \\ y &= x\overline{C}_2 &+& r\overline{D}_{12} &+& \hat{x}F\overline{D}_{22} \end{aligned}$$

Substituting the last output equation in the state equation (25) for the observer $\overline{K}$ yields:

$$\hat{x}Z^{-1} = \hat{x}\overline{A} + \hat{x}F\overline{B}_2 + (x\overline{C}_2 - \hat{x}\overline{C}_2 + r\overline{D}_{12})\overline{L}$$

And therefore, the state space representation for $\overline{T}_{rv}$ becomes:

$$\begin{bmatrix} xZ^{-1} & \hat{x}Z^{-1} \end{bmatrix} = \begin{bmatrix} x & \hat{x} \end{bmatrix} \begin{bmatrix} \overline{A} & \overline{C}_2\overline{L} \\ F\overline{B}_2 & \overline{A} + F\overline{B}_2 - \overline{C}_2\overline{L} \end{bmatrix} + r\begin{bmatrix} \overline{B}_1 & \overline{D}_{12}\overline{L} \end{bmatrix}$$

$$v = \begin{bmatrix} x & \hat{x} \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ F\overline{D}_{21} \end{bmatrix}$$

We now perform the following constant similarity transformation to this state realization:

$$\begin{bmatrix} I & I & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} \overline{A} & \overline{C}_2\overline{L} & \overline{C}_1 \\ F\overline{B}_2 & \overline{A} + F\overline{B}_2 - \overline{C}_2\overline{L} & F\overline{D}_{21} \\ \overline{B}_1 & \overline{D}_{12}\overline{L} & \end{bmatrix} \begin{bmatrix} I & -I & \\ & I & \\ & & I \end{bmatrix} =$$

$$\begin{bmatrix} \overline{A} + F\overline{B}_2 & 0 & \overline{C}_1 + F\overline{D}_{21} \\ F\overline{B}_2 & \overline{A} - \overline{C}_2\overline{L} & F\overline{D}_{21} \\ \overline{B}_1 & -\overline{B}_1 + \overline{D}_{12}\overline{L} & \end{bmatrix}$$

From this state representation we conclude that $\overline{T}_{rv}$ is in $\mathcal{U}$ if $\overline{A} + F\overline{B}_2$ is a.s and $\overline{A} - \overline{C}_2\overline{L}$ is a.s. The latter condition is guaranteed by the solution $\overline{M}_o$ in Theorem 12. The first condition holds by Theorem 9, since by the definition of the quantities $\overline{A}$ and $\overline{B}_2$ in Eq. (24):

$$\overline{A} + F\overline{B}_2 = A + \tilde{B}_1 U_1^{-1} B_1 + F\tilde{B}_3 U_1^{-1} B_1 + F B_2$$

and the right hand side equals the operator $A_c^\times$ defined in Theorem 9.

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Hence, we conclude by Lemma 15, that with the controller $\overline{K}$, the closed-loop system in Figure 5 is well-posed, internally stable and satisfies,

$$\Gamma_c I - T_{wz} T_{wz}^* \gg 0$$

6. CONCLUDING REMARKS.

The ∞-horizon robust output feedback control problem for LTV systems under standard assumptions has been addressed in the present paper. The strategy of the solution follows that outlined in the keynote paper [1]. However, contrary to [1], which derives a solution for the continuous time-invariant counterpart based on operator theoretic results of mixed Hankel-Toeplitz operators, the bounded real lemma in the proper time-varying context plays the key role in solving the robust output feedback problem.

Taking into account that the latter lamm plays a fundamental role in the solution of a large number of engineering problems, such as demonstrated e.g. in [21] for the time-invariant case and later on in [22] for the time-variant case, it might be expected that the solution devised in this way becomes more easily accessible to the practitioner engineer interested in the theoretical background.

REFERENCES

[1] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis, "State-Space Solutions to Standard H_2 and H_∞ Control problems," *I.E.E.E. Trans. on Autom. Control*, vol. 34, no. 8, pp. 831–847, 1989.

[2] A.J. van der Veen and M. Verhaegen, "On Spectral Factorization and Riccati Equations for Time-Varying Systems in Discrete time," *(submitted to) SIAM J. Control and Optimization*, June 1993.

[3] P. Dewilde and H. Dym, "Interpolation for Upper Triangular Operators," in *Operator Theory: Advances and Applications* (I. Gohberg, ed.), vol. OT 56, pp. 153–260, Birkhaüser Basel, 1992.

[4] B.A. Francis, *A course in H_∞ control theory*, vol. 88 of *Lecture notes in control and information sciences*. Berlin: Springer Verlag, 1987.

[5] T. Basar, "Disturbance attenuation in LTI plants with finite horizon: Optimality of nonlinear controllers," *System & Control Letters*, vol. 13, pp. 183–191, 1989.

[6] D.W. Gu, M.C. Tsai, and I. Postelhwaite, "State space formulae for discrete time H_∞ optimization," *Int. J. Control*, vol. 49, pp. 1683–1723, 1989.

[7] A.A. Stoorvogel, "The Discrete Time H_∞ control problem with measurement Feedback," *SIAM J. Control and Optimization*, vol. 30, pp. 182–202, 1992.

[8] C.E. de Souza and L. Xie, "On the Discrete-time Bounded Real Lemma with application in the the characterization of static state feedback H_∞ controllers," *System & Control Letters*, vol. 18, pp. 61–71, 1992.

[9] A. Feintuch and B.A. Francis, "Uniformly Optimal Control of Linear Feedback Systems," *Automatica*, vol. 21, no. 5, pp. 563–574, 1985.

[10] M. Verhaegen and P. Dewilde, "Sensitivity Minimization for causal, linear, discrete time-varying systems through Nevanlinna-Pick interpolation and outer-inner factorization Time-Varying Systems in Discrete time," *(submitted to) SIAM J. Control and Optimization*, July 1993.

[11] A. Feintuch, P.P. Khargonekar, and A. Tannenbaum, "On the Sensitivity Minimization Problem for Linear Time-varying Systems," *SIAM J. Control and Optimization*, vol. 24, no. 5, pp. 1076–1085, 1986.

[12] T.T. Georgiou and P.P. Khargonekar, "A constructive algorithm for sensitivity optimization of periodic systems," *SIAM J. Control and Optimization*, vol. 25, no. 2, pp. 334–340, 1987.

[13] D. Limebeer, B.D.O. Anderson, P.P. Khargonekar, and M. Green, "A game theoretic approach to H_∞ control for time varying systems," *SIAM J. Control and Optimization*, vol. 30, pp. 262–283, 1992.

[14] T. Basar and P. Bernhard, "H_∞ optimal control and related min-max design problems," *Birkhauser Boston*, 1991.

[15] G. Tadmor, "Worst-case design in the time domain: The maximum principle and the standard H_∞ problem," *Math. Control Signals Systems*, vol. 3, pp. 301–324, 1989.

[16] R. Ravi, K.M. Nagpal, and P.P. Khargonekar, "H_∞ control of linear time-varying systems: a state space approach," *SIAM J. Control and Optimization*, vol. 29, pp. 1394–1413, 1991.

[17] K. Furuta and S. Phoojaruenchanachai, "An algebraic approach to discrete-time H_∞ Control problems," in *Proc. 1990 American Control Conference*, pp. 2067–2072, 1990.

[18] M. Verhaegen and A.J. van der Veen, "On spectral factorization for Time-Varying systems in Discrete time and its application to robust state reconstruction," in *Proc. 11th ECCTD*, August 1993.

[19] G. De Nicolao, "On the Time-Varying Riccati Difference Equation of Optimal Filtering," *SIAM J. Control and Optimization*, vol. 30, no. 6, pp. 1251–1269, 1992.

[20] A.A. Stoorvogel and A.J.T.M. Weeren, "The Discrete Time Riccati equation related to the H_∞ control problem," tech. rep., Eindhoven University of Technology, 1993.

[21] B.D.O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*. Prentice Hall, 1973.

[22] A.J. van der Veen, *Time-Varying System Theory and Computational Modeling: Realization, Approximation and Factorization*. PhD thesis, Delft University of Technology, 1993.