Analytical Solution to the Constant Modulus Factorization Problem

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Iterative constant modulus algorithms have been used to blindly separate and retrieve interfering constant modulus signals impinging on an antenna array. These algorithms have several well-known but basically unsolved deficiencies. In this paper, we present an algorithm to analytically compute the solution to the underlying constant modulus (CM) factorization problem. With this new approach, it is possible to detect the number of CM signals present in the channel, and to retrieve all of them exactly, rejecting other, non-CM signals. Only a modest amount of samples are required. The algorithm is robust in the presence of noise, and is tested on real data, collected from an experimental set-up.

1. INTRODUCTION

A problem in sensor array signal processing with important applications to wireless communications is concerned with the case where there are several unknown constant modulus (CM) signals impinging on the array, and the objective is to copy each of them. Because of multipath effects, information on the array response vector cannot be used. Mathematically, we are given a data matrix $X : m{\times}n$, with x_{ij} the j-th sample of the i-th antenna, and we have to find a factorization of X, if it exists, as

$$X = A\,S = \mathbf{a}_1\mathbf{s}_1 + \cdots + \mathbf{a}_d\mathbf{s}_d\,, \tag{1}$$

$$(A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_d] \in \mathbb{C}^{m\times d},\ S = [\mathbf{s}_1^* \ \cdots \ \mathbf{s}_d^*]^* \in \mathbb{C}^{d\times n})\,,$$

such that A, S are full rank d, and all $\mathbf{s}_k \in \mathbb{C}^n$ are CM signals. The row vectors $\mathbf{s}_1, \cdots, \mathbf{s}_d$ contain samples of the d CM signals, the column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_d$ are the corresponding array response vectors. The CM factorization problem can be reformulated as finding all weight vectors $\mathbf{w}$ such that $\mathbf{w}X = \mathbf{s}$, for as many linearly independent CM signals $\mathbf{s}$ as possible. This formulation is more general: not all d signals present in X need to be CM signals, but only $\delta \le d$, say.

The CM factorization problem gained much interest in the area of communications, where many modulation or coding schemes produce signals that have the CM property, such as FM and phase-modulated signals. A reliable solution to this problem allows to receive and separate multiple co-channel CM signals impinging on an antenna array, without use of the structure of the array response matrix (*i.e.,* blindly). This leads to a direct increase in channel capacity. With some simplifying assumptions, the CM factorization problem is an appropriate mathematical formulation of this 'blind null-steering' scenario.

For a long time, the CM factorization problem was considered to be too non-linear to admit a closed-form analytical solution, and only iterative, gradient-descent schemes have been developed, mostly based on pioneering work by Godard, and Treichler, Agee and Larimore [1, 2]. Despite many efforts (we omit the references), most CMAs up to date have convergence problems, which seriously limit their practical and automated applicability. The speed of convergence is highly dependent on the initialization, but no suitable default initial points are known. The algorithms sometimes converge to local minima, which do not correspond to real signals. Only one signal at a time is retrieved; the other signals have to be found by starting from other initial points. Weak signals are hard to converge to in this way. The only way to detect the number of CM signals is a posteriori, by counting the number of independent CM signals that have been obtained.

Let $n > d^2$, and assume that there is a unique solution (modulo certain trivial transformations, viz. [4]). In this paper, we derive a new CM algorithm with the following properties.

- It is possible to determine the number of CM signals among all other signals present in X: δ equals the dimension of the kernel of a certain matrix constructed from X.
- The weight vectors and corresponding CM signals in X can be computed exactly, from a certain eigenvalue decomposition.
- With X distorted by additive noise, a generalization of the algorithm is robust in finding S, even when the number of samples is quite small. This is demonstrated with real data, measured from an experimental set-up.

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2. EXACT SOLUTION TO THE CM PROBLEM

2.1. Simultaneous quadratic equations

Denote by $\mathcal{R}'(X)$ the subspace spanned by the rows of X (the co-range of X) and by $\mathcal{CM}$ the set of CM matrices. Assuming that there is a unique solution, the CM factorization problem is precisely equivalent to the following problem:

Problem P1. *Find all linearly independent signals* $\mathbf{s}$ *that satisfy*

$$\begin{array}{ll} (A) & \mathbf{s} \in \mathcal{CM}\,, \\ (B) & \mathbf{s} \in \mathcal{R}'(X)\,. \end{array}$$

In a series of steps, this problem is translated into an equivalent but more tractable form. Let $X = U\Sigma V : U \in \mathbb{C}^{m\times m}, \Sigma \in \mathbb{R}^{m\times n}, V \in \mathbb{C}^{n\times n}$ be a singular value decomposition of X: U and V are unitary matrices, and Σ is a real diagonal matrix with non-negative entries. Suppose that $\text{rank}(X) = d$. We can write

$$X = \hat{U}\hat{\Sigma}\hat{V}; \qquad \hat{U} \in \mathbb{C}^{m\times d},\ \hat{\Sigma} \in \mathbb{R}^{d\times d},\ \hat{V} \in \mathbb{C}^{d\times n}$$

where $\hat{U}$, $\hat{\Sigma}$, $\hat{V}$ are submatrices of U, Σ, V, respectively, corresponding to the non-zero singular values of X. The rows of $\hat{V}$ form an orthonormal basis of the row span of X:

$$(B): \quad \mathbf{s} \in \mathcal{R}'(X) \quad \Leftrightarrow \quad \mathbf{s} = \mathbf{w}\hat{V}, \qquad \hat{V} : d \times n\,.$$

Here, the weight vector $\mathbf{w}$ is not precisely the same as before: it is now acting on the orthogonal basis vectors of $\mathcal{R}'(X)$, rather than directly on X. This reduces the number of parameters to estimate from m to d, and ensures that linearly independent $\mathbf{w}$ result in linearly independent $\mathbf{s}$.

To satisfy condition (A): $\mathbf{s} = \mathbf{w}\hat{V} \in \mathcal{CM}$, put $\hat{V} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$, where $\mathbf{v}_i \in \mathbb{C}^d$ is the i-th column in $\hat{V}$. Then

$$\begin{array}{rl} (A): & \mathbf{s} = [\,(\mathbf{s})_1 \ \cdots \ (\mathbf{s})_n\,] \in \mathcal{CM} \\ \Leftrightarrow & [\ |(\mathbf{s})_1|^2 \ \cdots \ |(\mathbf{s})_n|^2\ ] = [\ 1 \ \cdots \ 1\ ] \\ \Leftrightarrow & \left\{ \begin{array}{rcl} \mathbf{w}\mathbf{v}_1\mathbf{v}_1^*\mathbf{w}^* & = & 1 \\ & \vdots & \\ \mathbf{w}\mathbf{v}_n\mathbf{v}_n^*\mathbf{w}^* & = & 1 \end{array} \right. \\ \Leftrightarrow & \mathbf{w}P_k\mathbf{w}^* = 1\,, \qquad k = 1, \cdots, n\,, \end{array}$$

where $P_k = \mathbf{v}_k\mathbf{v}_k^* \in \mathbb{C}^{d\times d}$, for $k = 1, \cdots, n$. The CM problem is thus equivalent to the simultaneous solution of n quadratic equations in the entries of $\mathbf{w}$, corresponding to the intersection of n ellipsoids. To find all solutions, the approach is to expand these equations in the entries of $\mathbf{w}$, which gives rise to Kronecker products. At this point, we introduce the notation, for $Y \in \mathbb{C}^{d\times d}$, $\mathbf{y} \in \mathbb{C}^{d^2}$,

$$\text{vec}(Y) := \begin{bmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{21} \\ \vdots \\ Y_{dd} \end{bmatrix}, \quad \text{vec}^{-1}(\mathbf{y}) := \begin{bmatrix} (\mathbf{y})_1 & \cdots & (\mathbf{y})_d \\ (\mathbf{y})_{d+1} & \cdots & (\mathbf{y})_{2d} \\ \vdots & \ddots & \vdots \\ & \cdots & (\mathbf{y})_{d^2} \end{bmatrix}.$$

With these definitions, the quadratic expression $\mathbf{w}P_k\mathbf{w}^*$ is 'linearized' as

$$\mathbf{w}P_k\mathbf{w}^* = \mathbf{p}_k\mathbf{y}\,,$$

where $\mathbf{y} = \text{vec}(\mathbf{w}^*\mathbf{w}) \in \mathbb{C}^{d^2\times 1}$, $\mathbf{p}_k = \text{vec}(P_k)^T$.

The CM problem is thus: solve

$$\begin{array}{rcl} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_n \end{bmatrix} \mathbf{y} & = & \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \\ \mathbf{y} & = & \text{vec}(\mathbf{w}^*\mathbf{w})\,, \end{array}$$

i.e., we have to find vectors $\mathbf{y}$ that satisfy a linear system and can be factored as $\mathbf{y} = \text{vec}(\mathbf{w}^*\mathbf{w})$ as well. For each solution $\mathbf{w}$, the corresponding CM signal is given by $\mathbf{s} = \mathbf{w}\hat{V}$.

A description of all solutions to the linear system has in general the form

$$\mathbf{y} = \alpha_1\mathbf{y}_1 + \alpha_2\mathbf{y}_2 + \cdots + \alpha_\delta\mathbf{y}_\delta\,, \quad (\alpha_1 + \cdots + \alpha_\delta = 1)\,, \tag{2}$$

and can be constructed from the kernel of $\hat{P}$ in

$$\hat{P}\begin{bmatrix} \mathbf{y} \\ -1 \end{bmatrix} = 0\,, \quad \hat{P} = \begin{bmatrix} \mathbf{p}_1 & 1 \\ \vdots & \\ \mathbf{p}_n & 1 \end{bmatrix}. \tag{3}$$

We can argue that, *generically, there are precisely as many linearly independent solutions as there are CM signals.* Indeed, suppose that there are δ CM signals. Vectors $\{\mathbf{w}_1, \cdots \mathbf{w}_\delta\}$ are linearly independent if and only if vectors $\{\text{vec}(\mathbf{w}_1^*\mathbf{w}_1), \cdots, \text{vec}(\mathbf{w}_\delta^*\mathbf{w}_\delta)\}$ are. These vectors satisfy the linear system: it must have at least δ linearly independent solutions $\{\mathbf{y}_1, \cdots, \mathbf{y}_\delta\}$. Generically, the system is overdetermined and will not have other solutions, unless there are specific relations between the signals.

In case X is distorted by additive noise, δ and the basis $\{\mathbf{y}_i\}_1^\delta$ can be estimated from the approximate kernel of $\hat{P}$, using an SVD.

The remaining problem is to find a change of basis: transform $\{\mathbf{y}_1, \cdots, \mathbf{y}_\delta\}$ into a basis with "Kronecker structure". We have to find *all* values for $[\alpha_1, \cdots, \alpha_\delta]$ in equation (2) such that

$$\begin{array}{rl} \mathbf{y} = & \text{vec}(\mathbf{w}^*\mathbf{w}) \\ \Leftrightarrow & Y = \text{vec}^{-1}(\mathbf{y}) = \mathbf{w}^*\mathbf{w} \ \ (\text{rank } 1, \text{ symm.}) \\ \Leftrightarrow & \mathbf{w}^*\mathbf{w} = \alpha_1 Y_1 + \cdots + \alpha_\delta Y_\delta\,. \end{array} \tag{4}$$

Generically, there are precisely δ solutions $[\alpha_1, \cdots, \alpha_\delta]$ that generate rank-1 symmetric matrices. We have thus reduced the CM problem to a kind of generalized eigenvalue problem, which can be solved using standard linear algebra tools. Indeed, if $d = \delta = 2$, then there are two matrices Y_1 and Y_2, each of size 2×2, and we have to find $\lambda = \alpha_2/\alpha_1$ such that $Y_1 + \lambda Y_2$ has its rank reduced by one (to become one). For larger δ, there are more than two matrices, and the rank should be reduced to one by taking linear combinations of all of them. This can be viewed as an extension of the generalized eigenvalue problem. Its solution is better described from the opposite perspective, as follows.

Suppose that the solutions of the CM problem are $\mathbf{w}_1, \cdots, \mathbf{w}_\delta$. Then we must be able to write the matrix basis $Y_1, \cdots Y_\delta$ in terms of the rank-1 basis $\mathbf{w}_1^*\mathbf{w}_1, \cdots, \mathbf{w}_\delta^*\mathbf{w}_\delta$, *i.e.*,

$$\begin{aligned} Y_1 &= \lambda_{11}\mathbf{w}_1^*\mathbf{w}_1 + \cdots + \lambda_{1\delta}\mathbf{w}_\delta^*\mathbf{w}_\delta &&= W^*\Lambda_1 W \\ &\vdots && \vdots \\ Y_\delta &= \lambda_{1\delta}\mathbf{w}_1^*\mathbf{w}_1 + \cdots + \lambda_{\delta\delta}\mathbf{w}_\delta^*\mathbf{w}_\delta &&= W^*\Lambda_\delta W \end{aligned}$$

where

$$W = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_\delta \end{bmatrix}, \qquad \Lambda_k = \begin{bmatrix} \lambda_{k1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{k\delta} \end{bmatrix}$$

Hence, by the existence of a solution to the CM problem, there must be a matrix W whose inverse simultaneously diagonalizes $Y_1, \cdots, Y_\delta$. The rows of W, scaled to have norm $n^{1/2}$, are the weight vectors that solve the CM problem.

Generically, Y_1 and Y_2 are $d \times d$ matrices of rank δ, and not less than δ. In this case, a generalized eigenvalue decomposition of just Y_1 and Y_2 will already determine W. Numerically, and in the presence of noise, it is better to take all Y_k into account. Such an algorithm is described in the next section.

2.2. Simultaneous diagonalization

Assume, for the moment, that there is no noise added to X. As we have seen, there exists a full rank matrix $W \in \mathbb{C}^{\delta\times d}$ such that

$$\begin{aligned} Y_1 &= W^*\Lambda_1 W \qquad (\Lambda_1, \cdots, \Lambda_\delta \in \mathbb{C}^{\delta\times\delta}, \text{ diagonal}) \\ &\vdots \\ Y_\delta &= W^*\Lambda_\delta W. \end{aligned} \tag{5}$$

Bring in a QR factorization of W^* and an RQ decomposition of W: $W^* = Q^*R'$, $W = R''Z^*$, where Q, Z are unitary $d\times d$ matrices, and $R' \in \mathbb{C}^{d\times\delta}$, $R'' \in \mathbb{C}^{\delta\times d}$ are upper triangular. If $\delta < d$, then we can make sure that only the leading $\delta \times \delta$ blocks of R' and R'' are non-zero. Substitution into (5) leads to

$$\begin{aligned} QY_1Z &= R_1 \qquad (R_1, \cdots, R_\delta \in \mathbb{C}^{d\times d}, \text{ upper tr.}) \\ &\vdots \\ QY_\delta Z &= R_\delta \end{aligned} \tag{6}$$

with

$$R_1 = R'\Lambda_1 R'', \ \cdots, \ R_\delta = R'\Lambda_\delta R''. \tag{7}$$

Only the top-left $\delta\times\delta$ block of each R_k is non-zero. Hence, there exists Q, Z such that QY_kZ is upper triangular, for $k = 1, \cdots, \delta$, which is some kind of generalized Schur decomposition. With this decomposition, it is seen that a parameter vector $[\alpha_1 \ \cdots \ \alpha_\delta]$ satisfies (4) if and only if it satisfies

$$\alpha_1 R_1 + \cdots + \alpha_\delta R_\delta \quad \text{is rank 1.} \tag{8}$$

With the model of $R_1, \cdots, R_\delta$ in (7), we obtain

$$R'(\alpha_1\Lambda_1 + \cdots + \alpha_\delta\Lambda_\delta)R'' \quad \text{is rank 1.}$$

Since all the Λ_k are diagonal, the α_k are straightforward to compute: only one entry of the diagonal matrix $\alpha_1\Lambda_1 + \cdots + \alpha_\delta\Lambda_\delta$ can be non-zero. Setting this entry equal to one, all possible parameter vectors $[\alpha_1 \ \cdots \ \alpha_\delta]$ follow by constructing a matrix whose columns consist of the diagonal entries of the Λ_k,

$$\Lambda = \begin{bmatrix} (\Lambda_1)_{11} & \cdots & (\Lambda_1)_{\delta\delta} \\ \vdots & & \vdots \\ (\Lambda_\delta)_{11} & \cdots & (\Lambda_\delta)_{\delta\delta} \end{bmatrix}.$$

The rows of Λ^{-1} are the independent vectors $[\alpha_1 \ \cdots \ \alpha_\delta]$. It is not necessary to compute the factorization (7): the α_i can directly be obtained from (6), as follows.

Proposition 1. *For given $Y_1, \cdots, Y_\delta$, assume the decomposition (6). All independent parameter vectors $[\alpha_1 \ \cdots \ \alpha_\delta]$ such that $\alpha_1 Y_1 + \cdots + \alpha_\delta Y_\delta$ has rank 1 are given by the rows of A:*

$$A = R^{-1}, \qquad R = \begin{bmatrix} (R_1)_{11} & \cdots & (R_1)_{\delta\delta} \\ \vdots & & \vdots \\ (R_\delta)_{11} & \cdots & (R_\delta)_{\delta\delta} \end{bmatrix}.$$

Factoring each of the δ rank-1 matrices that is obtained in this way gives δ independent vectors $\mathbf{w}$, which form the rows of the matrix W that we were looking for in equation (5). Hence, in the noise-free case, the computation of a super-generalized Schur decomposition, *i.e.*, two unitary matrices Q, Z that satisfy (6), gives the solution to the simultaneous diagonalization problem.

When X is distorted by noise, there is no Q, Z which simultaneously makes all matrices Y_k upper triangular.

However, we can try to find Q, Z to make these matrices as much upper triangular as possible, by minimizing the Frobenius norm of the residual lower triangular entries. One approach for doing this goes via an extension to more than two matrices of the usual QZ iteration for computing the generalized Schur decomposition of two matrices, described in the next paragraph. With Q, Z and hence $R_1, \cdots, R_\delta$ obtained this way, we can compute all independent parameter vectors $[\alpha_1 \cdots \alpha_\delta]$ as in proposition 1. The resulting matrices Y have approximately the form $Y = \mathbf{w}^*\mathbf{w}$, and each $\mathbf{w}$ can be estimated as the singular vector corresponding to the largest singular value of each Y. It remains to scale $\mathbf{w}$ to ensure that $||\mathbf{w}|| = n^{1/2}$.

The QZ iteration for computing the Schur decomposition of *two* matrices [3] starts with setting $Q^{(0)} = I$, $Z^{(0)} = I$. At the k-th iteration step, a unitary matrix $Q^{(k)}$ is computed such that $Q^{(k)}(Y_1 Z^{(k-1)})$ is upper triangular, and a unitary matrix $Z^{(k)}$ is computed to make $(Q^{(k)}Y_2)Z^{(k)}$ upper triangular. As an extension to more than two matrices, we propose the following two step iteration. Denote by $|| \cdot ||_{LF}$ the Frobenius norm of the strictly lower triangular part of a matrix. Then $Q^{(k)}$ and $Z^{(k)}$ are chosen to be unitary matrices solving

$$\begin{aligned} Q^{(k)} &= \arg\min_Q || \, Q(Y_1 Z^{(k-1)}) \, ||_{LF}^2 + \cdots + || \, Q(Y_\delta Z^{(k-1)}) \, ||_{LF}^2 \,, \\ Z^{(k)} &= \arg\min_Z || \, (Q^{(k)} Y_1) Z \, ||_{LF}^2 + \cdots + || \, (Q^{(k)} Y_\delta) Z \, ||_{LF}^2 \,. \end{aligned}$$

Each of these steps is a least squares problem with an exact solution, which can be obtained using SVDs. We omit the details. The resulting QZ iteration is observed to converge fast, usually quadratically in 3–5 iterations. Because the inner loop consist of SVDs, the scheme is only practical if d is small, which is certainly the case for the currently envisioned applications ($d \leq 6$, say).

3. EXPERIMENTAL EVALUATION

To assess the performance of the algorithm, we have applied it to a number of test matrices, based both on computer generated data and on real data collected from an experimental set-up. In this paper, we will report on only one such example, using measurement data collected from a rooftop antenna set-up at ArgoSystems, Inc (Sunnyvale, CA). In this experiment, there are $m = 6$ receiver antenna's, arranged in a certain nondescript pattern, and $d = 4$ FM transmitters, marked *A*–*D*, broadcasting music, speech, and modulated tones. The angles of the transmitters with respect to the array broadside were $-1.5^\circ, 0^\circ, 7.1^\circ, 42.6^\circ$, for A, B, C, D, respectively, and their signal-to-background noise levels were 19.1 dB, 17.6 dB, 17.9 dB, 16.7 dB. In a second experiment, the power of B was lowered to 7.6 dB.

In figure 1 (a) and (b), the singular values of X and the condition matrix $\hat{P}$ in (3) are shown. For $n = 100$ and $n = 50$ samples, it is clear that there are 4 signals, and that in this example all of them have constant modulus. Figure 1(c) shows the modulus error of the solutions, $\mathrm{dist}(\mathbf{s}, \mathcal{CM}) = \sum_1^n (|(\mathbf{s})_k|^2 - 1)^2$. In order to assess the optimality of the solutions in the presence of noise, we apply the Gerchberg-Saxton algorithm [4] to the obtained signals, taking $d = 4$ and $\delta = 4$, for $n = 25$ samples. (The GS-algorithm is a block-iterative CM algorithm, which alternatingly projects an approximate solution $\mathbf{s}$ onto $\mathcal{R}'(X)$ and onto $\mathcal{CM}$.) The solid lines show the modulus error that result when the iterations are started from the $\delta = 4$ analytically computed weight vectors, the dashed lines is the error when we start with a number of randomly selected weight vectors. The Gerchberg iteration improve the analytically computed weight vectors only marginally: they are already close to optimal. For $n = 50$, straight lines occurred (not shown).

In a second experiment, the power of signal B was lowered to 7.6 dB. As the spacing of the B-antenna to the A-antenna is still only 1.5°, this is a challenging test of the algorithm. The results are depicted in 2. The detection of the other three signals from the singular values of $\hat{P}$ remained the same, but the fourth singular value (apparently corresponding to B) is raised and now somewhere in the middle of the gap between the large and small singular values. The detection that there are four independent signals from the singular values of X is also more difficult now, even for $n = 100$.

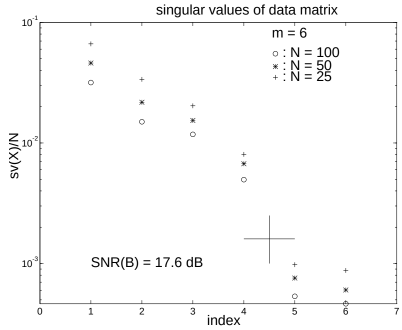
4. CONCLUDING REMARKS

In this paper, we have described an analytic method for solving the constant modulus factorization problem. The method condenses all conditions on the weight vectors $\mathbf{w}$ into a single matrix $\hat{P}$, and finds all independent vectors in the kernel of this matrix that have a certain (Kronecker) structure. This problem, in turn, is shown to be a simultaneous diagonalization problem, or super-generalized eigenvalue problem, which may be formulated in terms of a super-generalized Schur decomposition: for given matrices $Y_1, \cdots, Y_\delta$, find Q, Z (unitary) such that

$$QY_1Z = R_1\,, \quad \cdots \quad QY_\delta Z = R_\delta\,,$$

where $R_1, \cdots, R_\delta$ are as much upper triangular as possible. We have proposed a modified QZ iteration which treats all Y_k equally, converges to upper triangular matrices R_k in the absence of noise, and usually has quadratic convergence. Other interesting iterations might be devised as well. There are other approaches to the generalized eigenvalue problem (4): Cardoso [5], and Papadias and Slock [6] solve similar problems, using iterative techniques.

Important advantages of the analytic approach are



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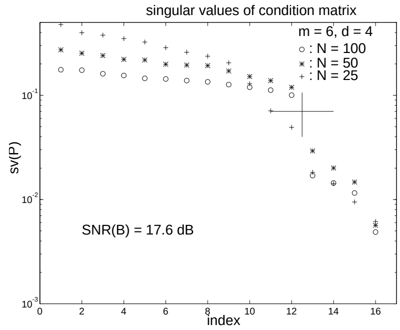

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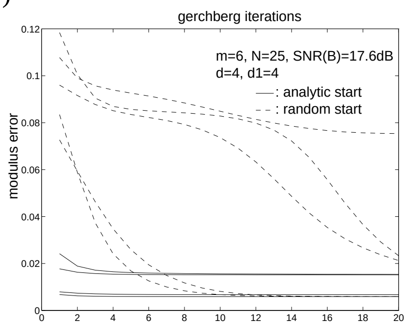

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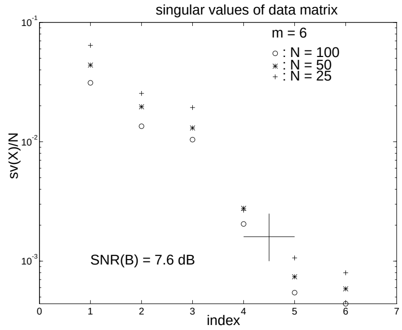
Figure 1. Experiment with d = 4 FM broadcasters and an array of m = 6 receiving antenna's. (*a*) svd(X), (*b*) svd($\hat{P}$), (*c*) dist($\mathbf{s}, \mathcal{CM}$)/n during Gerchberg iterations, with analytically computed and random initial starts.

1. It is less blind: the number of CM signals can be detected beforehand, from the close-to-zero singular values of $\hat{P}$. Not all signals have to be CM signals.
2. It is deterministic: it does not rely on lucky initial choices of $\mathbf{w}$. All CM signals are found. It does not lock on other signals (local minima). The only parameters that have to be set are the total number of signals, and the number of CM signals.

These two properties make the algorithm more reliable, so that it can operate with a lower number of samples and at a lower SNR.



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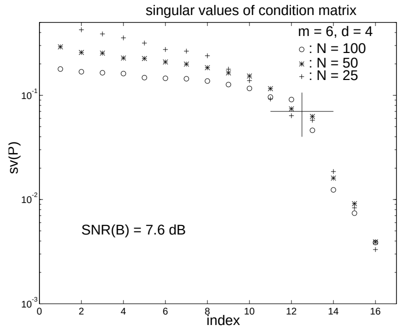

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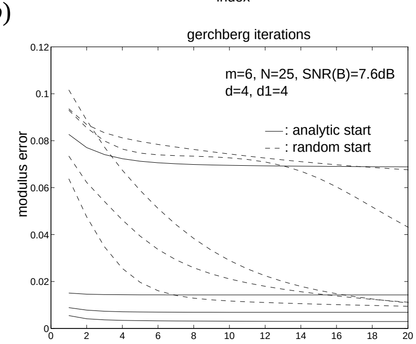

(*c*)

Figure 2. Same experiment as in figure 1, but now with SNR(B) lowered to 7.6 dB.

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