

Stellingen

behorende bij het proefschrift

Semi-separability and Operators of Mixed Causality

Edin Alijagić

1. Zonder genoeg ontwikkelde wiskundige inzichten kunnen algoritmische resultaten niet als betrouwbaar beschouwd worden.
[dit proefschrift]
2. 'Lelikhheid' van de wiskundige uitdrukkingen dat door een probleem is geïntroduceerd, is in grote mate te verwijten aan het menselijk tekort om het onderliggende probleem te ervaren als een restrictie van algemenere eigenschappen die, op hun beurt, aanleiding geven tot 'mooiere' uitdrukkingen.
[dit proefschrift]
3. De gegevens die *a priori* bepaald kunnen worden zouden nooit gegeneerd moeten worden door een algoritme - zelfs niet als zij gegeneerd kunnen worden in een stabiele en ideaal nauwkeurige manier. Ofschoon de moeite die noodzakelijk is om deze redundantie te verbannen soms ongelofelijk groot is, is het de moeite waard om dit te doen omdat dat evenredig is met de hoeveelheid van nieuwe ingezamelde inzichten.
[Gevolg van de vorige stelling]
4. De studie van lokaal-eindige, semi-separabele tijdsvariërende lineaire systemen is gebaseerd op een analyse van geschikte tweezijdig oneindige (begrensd) matrices. Alhoewel het ontmoedigend en hopeloos mag lijken om zulke matrices te analyseren, komt systeemtheorie te hulp: een zeventvoud van geschikte (blok) diagonalen blijkt genoeg te zijn om elke van de matrices te beschrijven.
[dit proefschrift]
5. Een gelukte, wetenschappelijke bijdrage van een promovendus die tegelijkertijd zelfvoldoend is, hangt in grote mate af of hij in staat is een duidelijke lijn te trekken tussen eigen ideeën en ideeën van zijn promotor.
6. Algebraïsche meetkunde blijkt een reputatie te hebben als ingewikkeld, uitsluitend en zeer abstract, met aanhangers die geheimzinnig samenzweren om de rest van de wiskunde over te nemen.
[- David B. Mumford]
7. Alhoewel de leer van lineaire systemen is ontstaan als een uitvloeisel van functionele analyse en operatortheorie, blijkt het dat de leer genoeg structuur toont om een winnende plek voor de algebraïsche aanpak te garanderen en niet voor de analytische.
[Gevolg van de vorige stelling uitleggend de algebraïsche 'smaak' van de inhoud van het proefschrift en eventuele toekomstige perspectieven]
8. De verslechtering van de economie is in direct verband met het niet in staat te zijn van een mens om tegelijkertijd hoger opgeleid te zijn en een daarvoor geschikte positie te bekleeden.
9. De noodzaak voor nieuwe uitvindingen laat vaak uitvinden van nieuwe noodzaken achter zich.
10. Een geschikte maat om de bevolkingstoename van een stad te meten is de toename in het getal van overleidingsacten in de lokale krant.
11. Het vrijheidsgevoel in stabiele, goed georganiseerde landen neemt af met de noodzaak nieuwe wetten tot stand te brengen voor een betere organisatie.
12. Te vergelijken met alle **spam**-filters, is een postbode de beste.
13. Als dit me niet kapot krijgt, zal het me sterker maken.
[- F. Nietzsche]

Deze stellingen worden verdedigbaar geacht en zijn als zodanig goedgekeurd door de promotor Prof. dr. ir. Patrick M. Dewilde

Propositions

accompanying the thesis

Semi-separability and Operators of Mixed Causality

Edin Alijagić

1. Without **sufficiently** developed mathematical insights, algorithmic results can not be interpreted in a reliable way.
[this thesis]
2. 'Ugliness' of the mathematical expressions are to a great extent due to one's inability to experience the underlying problem as a restriction of a general one that gives rise to 'nicer' expressions.
[this thesis]
3. The data that can be determined beforehand should never be generated by an algorithm - not even when they can be generated in a stable and ideally accurate way. Although the effort needed to banish this redundancy is often tremendously high, it is worth doing because it is in direct proportion with the quantity of newly gathered insights.
[Corollary of the previous proposition]
4. The study of locally finite time-varying linear systems relies upon analysis of appropriate doubly infinite (bounded) matrices. Although it may seem discouraging and hopeless to analyze the matrices as such, system theory comes to the rescue: a seven-tuple of appropriate (block) diagonals appears to be sufficient to describe any of them.
[this thesis]
5. A successful scientific contribution of a promovendus which is at the same time satisfactory, depends highly on his ability to make a clear distinction between his own ideas and his supervisor's ones.
6. Algebraic geometry seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics.
[- David B. Mumford]
7. Although the theory of linear systems has arisen as an outgrowth of functional analysis and in particular operator theory, it seems that the theory exhibits enough structure to ensure the winning position for the algebraic approach and not for the analytic one.
[Corollary of the previous proposition explaining the algebraic flavor of the thesis's content and possible future perspectives]
8. The worsening of the economy is in direct proportion with one's inability to be over-educated and employed accordingly at the same time.
9. The need for new inventions often succeeds invention of new needs.
10. A good measure of the population growth of a city is the growth of the number of obituaries in the local newspaper.
11. The feeling of freedom in stable socially well-organized countries often drops as the need for a better organization aims at introducing new laws.
12. In comparison with all **spam-filters**, a postman scores far the best.
13. If this does not kill me, it shall make me stronger!
[- F. Nietzsche]

These propositions are considered defensible and as such have been approved by the supervisor, Prof. dr. ir. Patrick M. Dewilde

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Semi-separability and Operators of Mixed Causality



Semi-separability and Operators of Mixed Causality

Proefschrift

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elektrotechnisch ingenieur
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*To Smail, Alma, Belma
and the rest of my family.*

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This not only helps in tracking expenses but also ensures compliance with tax regulations. The document further outlines the steps for recording these transactions, from identifying the nature of the expense to the final posting to the ledger.

Next, the document addresses the process of reconciling bank statements. It explains how to compare the bank's records with the company's books to identify any discrepancies. Common reasons for these differences include timing issues, such as deposits in transit or outstanding checks. The document provides a detailed guide on how to investigate these differences and adjust the books accordingly.

The final section of the document focuses on the preparation of financial statements. It describes how the accumulated data from the ledger is used to generate the income statement, balance sheet, and cash flow statement. Each statement is explained in terms of its purpose and the key components it includes. The document also offers tips on how to present these statements clearly and accurately to management and other stakeholders.

Acknowledgments

After all those years, I would like to express thanks to a list of people who contributed in some way to this thesis.

First of all, I am greatly indebted to my supervisor Prof. Dr. P. M. Dewilde for his guidance, weekly instructions, scientific support and countless revisions of the thesis's content I am very proud of. Without him this thesis would never have seen daylight.

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Special thank goes to Dr. Dan Volok for discussions on J-unitary operators and on math in general that happened during his six-months staying (Sept. 2004.-Feb. 2005.) at Department of Networks and Systems at TU Delft.

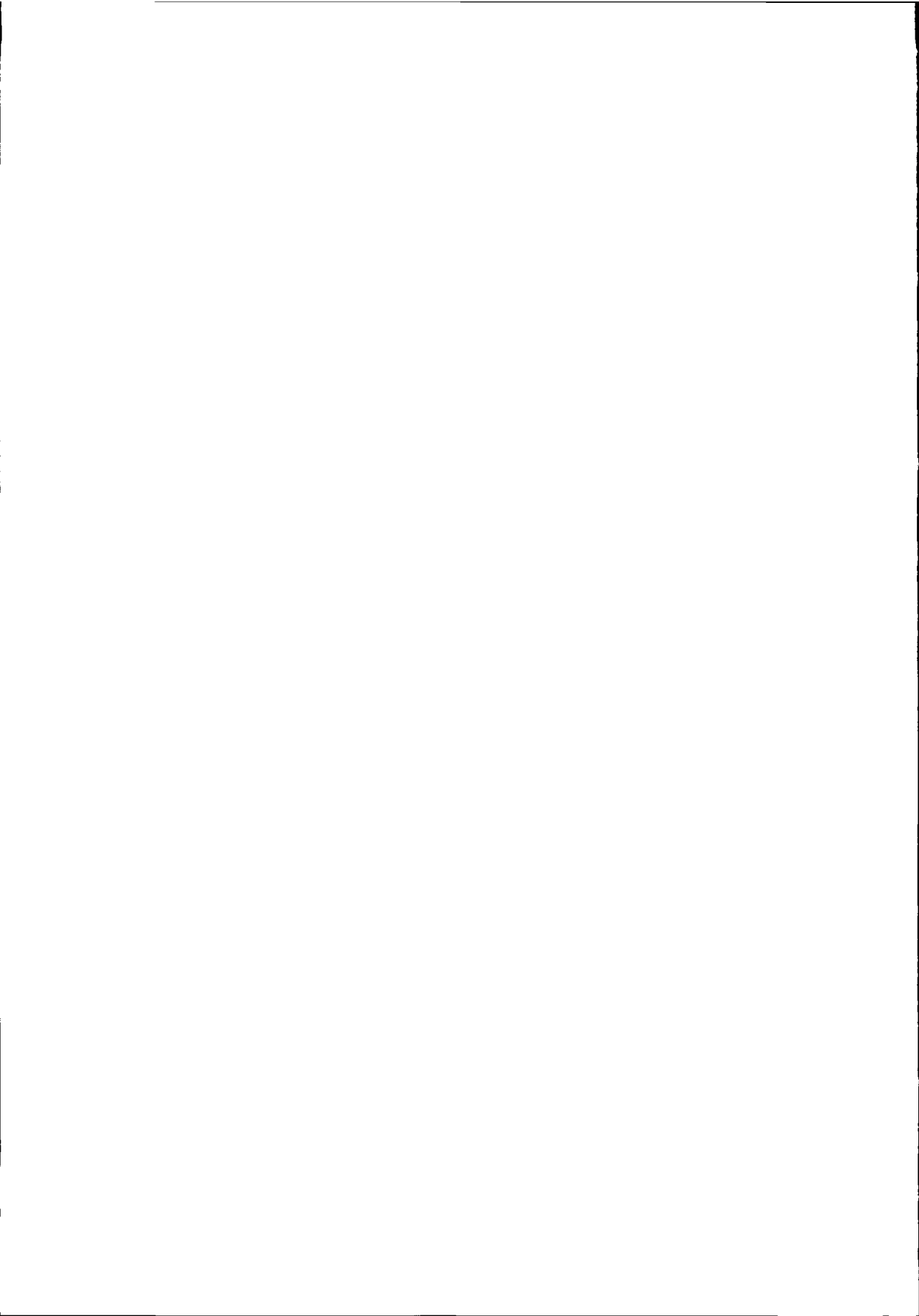
Finally, I would like to give credit to the committee members for their advice, which helped me through the last phase of this process.

My gratitude goes out to all colleagues and former colleagues at Department of Networks and Systems.

I can not end without thanking my family. Their love, understanding and a monumental unwavering support have helped me to complete this mission. I dedicate this work to them.

*Delft,
August 2005*

Edin Alijagić



Contents

Part I INTRODUCTION

1	INTRODUCTION	3
1.1	Background	3
1.2	Problem formulation	6
1.3	Outline	8
1.4	Contributions	9
2	PRELIMINARIES	11
2.1	Non-uniform vector spaces and s-dim sequence	11
2.2	The norm and the inner product on $\mathbb{C}^{\mathcal{M}}$	14
2.3	Linear mappings between non-uniform Hilbert spaces	15
2.4	Hilbert-Schmidt operators and orthogonal projections	17
2.5	Bilateral shift	19
2.6	State-space description	20
2.7	Reachability and observability	25
2.8	Hankel map and the spaces associated to it	27
2.9	Conclusions	30

Part II INVERSION

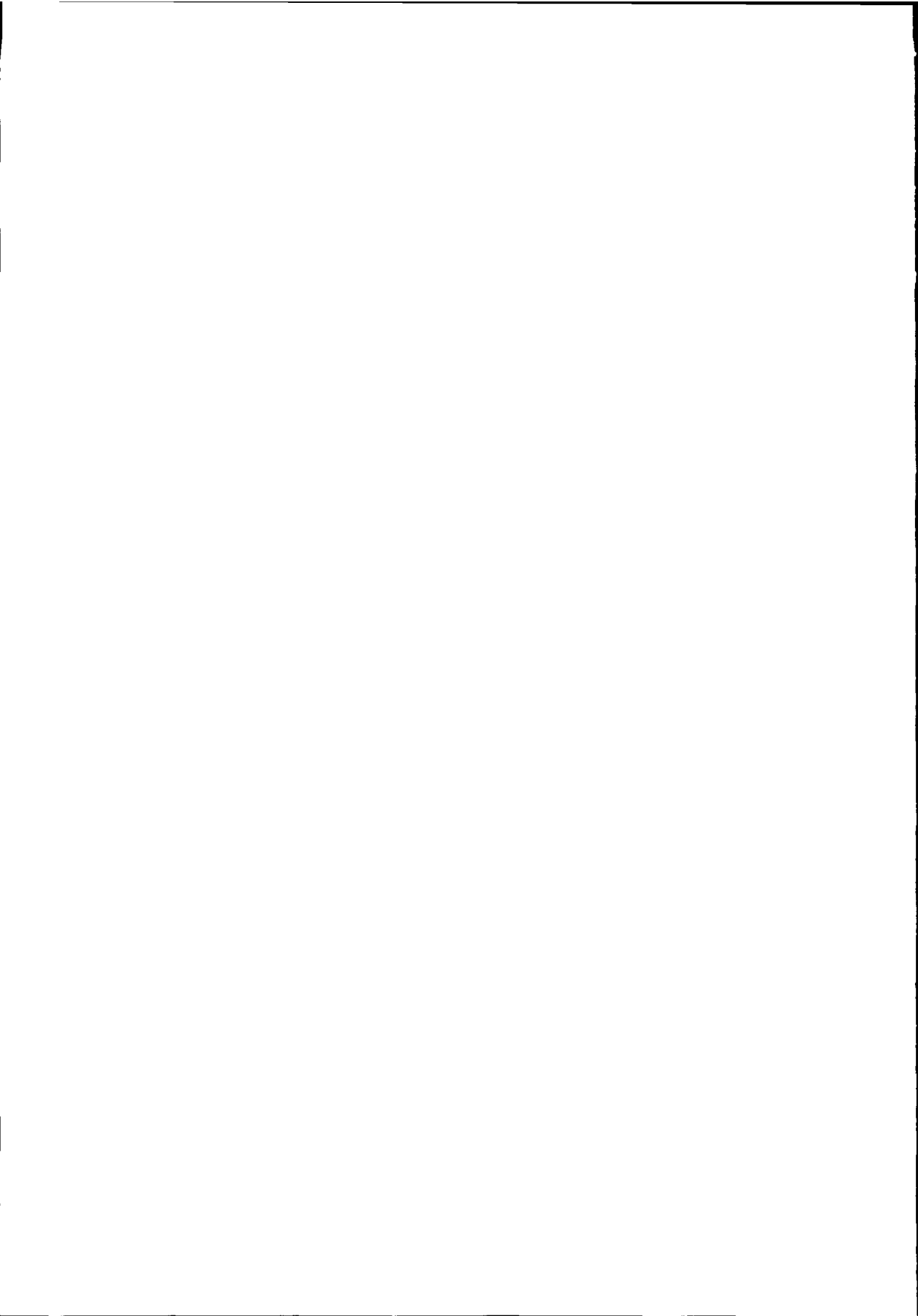
3	OPERATOR INVERSION	33
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3.1	Introduction	33
3.2	External and inner/outer representations	34
3.3	The inversion	45
3.4	The general case of a causal invertible operator	52
3.5	Inversion via URV-decomposition	53
3.6	System inversion and the inner/outer square-root algorithm	55
3.7	Minimal inverse of a mixed invertible operator	61
3.8	Conclusions	67
4	MAXIMAL-PHASE OPERATORS	69
4.1	Introduction	69
4.2	Construction of a minimal degree maximal-phase operator	70
4.3	Evaluation of the inverse	76
4.4	Outer/inner factorization of maximal-phase operators through	82
4.5	Interchanging inner and outer factors	86
4.6	Completion of a causal operator to achieve maximal-phase	99
4.7	Conclusions	119

Part III FACTORIZATION

5	INNER/ANTI-INNER FACTORIZATIONS	125
5.1	Introduction	125
5.2	Properties of isometric and coisometric operators	127
5.3	Isometric realizations for isometric operators from \mathcal{X}	143
5.4	One-pass unitary embedding algorithm for isometries from \mathcal{X}	146
5.5	State-space properties of partitioned unitary operators	149
5.6	Conjugation of unitary operators	159
5.7	Conclusions	168
6	MINIMAL FACTORIZATIONS OF OPERATORS FROM \mathcal{X} ...	171

6.1	Introduction	171
6.2	Product of operators from \mathcal{X}	172
6.3	Completion of a partially specified inverse	181
6.3.1	Construction of realizations for maximal-phase operators G and g	183
6.3.2	Evaluation of the unknown data from $[T_f]$ and $[t_f]$	195
6.4	A recipe on how to perform the search	199
6.5	Minimal causal/anti-causal factorization of J-unitary operators	201
6.6	Extraction of intrinsic factors from a J-unitary operator	208
6.7	Conclusions	210
7	POLAR DECOMPOSITION IN PONTRYAGIN SPACES	211
7.1	Introduction	212
7.2	Preliminaries	212
7.3	State-space properties of causal chain-scattering operators	218
7.4	Left-outer/isometric characterization	222
7.5	Outer/inner factorization for a causal J-unitary operator	230
7.6	Conclusions	235
8	CONCLUSIONS	239
8.1	Main results	239
8.2	Future research	240
	References	243
	Index	247
	Summary	251
	Samenvatting	253
	About the Author	255



INTRODUCTION



1

INTRODUCTION

The impact of linear system theory in now-days engineering practice is perhaps best explained by T. Kailath in his work [Kai80] where, in the preface, he states

Linear systems have been under study for a long time, and from several different points of view, in physics, mathematics, engineering, and many other fields. But the subject is such a fundamental and deep one that there is no doubt that linear systems will continue to be an object of study for as long as one can foresee.

The fundament of the subject is not only recognized by Kailath himself. Many other researchers such as I. Gohberg and W. Averson pointed at it during the past decades as well. The attention that linear systems receive may be seen as strange if one recalls that the majority of problems from engineering practice has a non-linear character. One of the explanations for the vast attractivity is due to our ability to apply extensive analysis when coping with such systems.

1.1 Background

This thesis deals with time-varying linear systems. Loosely speaking, existing control theory relies mostly on the study of time-invariant systems. This study, in turn, is based on linear operator theory where concepts of zeros and poles play the crucial

role, and may be regarded to as well-understood (Zames). The great popularity in engineering circles for dealing with such systems is due to the fact that time-invariant systems are thought simpler. In reality however, almost nothing is time-invariant. Due to change in circumstances such as aging, climate change etc. the systems inevitably change their properties. At some scale time-varying models can still exhibit time-invariant properties making them attractive and useful to be dealt with. If the time scale of the model is small compared to the life span of the modeled process, time-invariance is a good assumption.

But there are systems when the time-invariance is not a good assumption at all. In the design of *gain scheduling* controllers the presence of non-linear dynamics is *a priori* presumed. In communications, processing of time-discrete signals is frequently involved. Depending on the way a signal must be processed, digital signal processors (DSP) use different routines to accomplish their task. Depending on circumstances, each routine then represents a different process and hence, give rise to time-varying character of the process.

Another reason for considering time-varying systems is the connection between linear and non-linear system theory. Non-linear system theory has a reputation of still being highly undeveloped. The reputation is owed to the fact that the local properties of such systems hardly ever provide information about the dynamics at some other point. To say something about their dynamics, one is many times forced to rely on computer simulations.

Design of oscillators is an obvious example where non-linear dynamics is exploited in generation of sinusoidal or rectangular waves. It is known, that varying the trajectory of a non-linear system around some trajectory, gives rise to a linear time-varying system. Because linear time-varying systems offer a nice trade-off between model simplicity and the ability to describe the behavior of certain processes, a natural question arises: up to which extent can a non-linear system be described in this way?

Most of the work in the research literature for linear time-varying systems is carried out on a purely theoretical level. Almost all theoretical breakthroughs for linear time-invariant systems have been followed by generalizations into the time-varying framework [Rugh96]. The power of time-varying system theory relies on the fact that it moves beyond the widely known concept of poles and zeros used in frequency analysis and, by passing to the geometry of Hilbert spaces, provides us with an insightful and unifying character. This has led to increased insight in, among others, connections between H_∞ control and game theory.

In this thesis we adopt time-varying state equation descriptions to model and describe operators under consideration. This notation appears to be efficient in analyzing operators. Formerly, it is introduced in the pioneering work in this field [ADe90],[ADeD90]. It experiences a full development in [VdV93] in the context of problems from system theory.

Other ways to describe an operator such as input/output description or by systems of difference equations do not provide a compact and insightful mean to study the operators. Contrary to the latter descriptions, the state-space approach is based on the geometric background of problems. This property is due to the geometry of so-called Hankel operators.

By an input/output representation for an operator¹ T on a separable Hilbert space of complex valued sequences ℓ_2 we mean that, for two sequences (row vectors) $u = [\dots, u_{-1}, \boxed{u_0}, u_1, \dots]$ and $y = [\dots, y_{-1}, \boxed{y_0}, y_1, \dots]$ from ℓ_2 , it holds that

$$y = uT.$$

The vectors u and y are special in the sense that each element from $\{u_i\}$ and $\{y_i\}$ consists of row vectors of finite dimension. This notation provides the Hilbert space setting for time-varying systems we use throughout. Sequences such as u or y are referred to as *non-uniform* sequences.

Since ℓ_2 is separable, T in the standard basis has a matrix representation in the form

$$T = \begin{bmatrix} \ddots & & & & \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \\ \dots & T_{0,-1} & \boxed{T_{0,0}} & T_{0,1} & \\ & T_{1,-1} & T_{1,0} & T_{1,1} & \\ & & \vdots & & \ddots \end{bmatrix}$$

for some T_{ij} where i and j are integers. The quantities $\{T_{ij}\}$ are complex valued matrices (blocks) of finite or even vanishing rank. The boxed term is for the house-keeping purpose only - in a doubly infinite matrix such as T , it is not clear what is the position of the (0,0)-term. Hence we put that term in the box to distinguish it from the other terms. Some frequently used classes of operators are *causal* ($T_{ij} = 0, i \geq j$) and *anti-causal* ($T_{ij} = 0, i \leq j$). We also have *strictly causal* ($T_{ij} = 0, i > j$), *strictly anti-causal* ($T_{ij} = 0, i < j$) and *diagonal* ($T_{ij} = 0, i > j$ and $i < j$) operators. An operator consisting of both causal and anti-causal parts is called a *mixed causality* operator. An important operator to mention is a *block Toeplitz* operator $T_{ij} = T_{i-1, j-1}$ and is a slight generalization of an ordinary Toeplitz operator mapping between ℓ_2 spaces.

A state-space description of T introduces a properly partitioned intermediate state-space variable $x_k = [x_{c,k} \ x_{a,k}]$ from some non-uniform space such that the input/output description $y = uT$, can be replaced by the set equations of the type

¹In the following chapter, the formal definitions for the concepts such as a bounded operator and a non-uniform ℓ_2 space are given.

$$[T]_k := \left[\begin{array}{c|c} A_{c,k} & C_{c,k} \\ \hline & A_{a,k} \\ \hline B_{c,k} & B_{a,k} \\ & D_k \end{array} \right],$$

$$[x_{c,k+1} \ x_{a,k} | y_k] = [x_{c,k} \ x_{a,k+1} | u_k] [T]_k$$

for some set of finite dimensional matrices $\{A_{c,k}, B_{c,k}, C_{c,k}, A_{a,k}, B_{a,k}, C_{a,k}, D_k\}$. The sequence of matrices $[T]_k$ is called a sequence of *additive realizations* for T . The name comes from the fact that the part of the realization indexed with 'c' realizes the strictly causal part T_c of T , the part indexed with 'a' strictly anti-causal part T_a and $\{D_k\}$ realizes the constant part from T .

The problem of determining $A_{c,k}, B_{c,k}, C_{c,k}, A_{a,k}, B_{a,k}, C_{a,k}$ and D_k from the measured pairs (u, y) related through $y = uT$ is called the *system identification* problem.

The problem of determination of $[T]_k$ through $\{T_{ij}\}$ is studied by many authors. In [KFA70] a modern insight into the problem is developed where the famous Kronecker theorem from linear algebra is used in describing the Hankel operator associated to an operator that represents a time-invariant system. In [DvdV98] the extension to time-varying systems has been made. The Hankel operator associated to a causal T is the portion of T that maps between the past signal inputs to the future outputs. Analysis of the Hankel operator forms the crux in analyzing the operator itself. The power of this approach relies on the fact that a geometric interpretation can be associated to the action of the operator and, as the consequence, linear algebra techniques [GC96], [HJo91] such as *orthogonalization*, *Cholesky*-RQ-, SVD- and URV-*decompositions* can be used in characterization of the operator. In [VdV93] and [DvdV98], process of obtaining the data from $[T]_k$ is explained in the case when T is either causal or anti-causal. The procedure extends to a mixed causality bounded T as well and $\{T_{ij}\}$ when expressed in terms from $[T]_k$ have the form

$$T_{ij} = \begin{cases} B_{c,i} A_{c,i+1} \dots A_{c,j-1} C_j, & i < j, \\ D_i, & i = j, \\ B_{a,i} A_{a,i-1} \dots A_{a,j+1} C_{a,j}, & i > j. \end{cases}$$

1.2 Problem formulation

This thesis focuses on linear operators of mixed causality having state-space descriptions as introduced in the previous paragraph in realm of solving two problems:

- (1) operator inversion and
- (2) minimal factorization problem.

The motivation for approaching the problems is explained by the following.

Roughly said, almost all linear algebra problems can be expressed through matrix inversion problems. Although techniques from linear algebra such as algorithms for doing QR-decomposition can be used to determine the inverse of an invertible matrix, they book only a limited success in inverting large matrices arising from e.g.: applications of *finite elements* methods, simulations of *lumped* networks, solving partial differential equations numerically that arise in forecasting, geoscience and oil-winning technology. The involved matrices often exhibit an additional structure whereby its entries exhibits some repeating pattern within the matrix and, in turn, introduce remarkable simplification within algorithms.

The causality as a physical property of both a modeled process from practice and its designed controller forms then an important simplification in formulation of algorithms on one side. On the other hand however, it may pose serious restrictions. In the *chain scattering approach* to H_∞ -control, operators of mixed causality are derived from the causal ones by, among other, inverting some portions of the operators. To provide answers regarding robust stability is then important to know when the inverse exists and if it can be evaluated in a stable way. If the answer is affirmative, another question arises: is it possible to find the inverse in an one-pass way?

The first part of the thesis addresses the inversion of large (doubly infinite) semi-separable matrices of mixed causality.

Important issue in analyzing operators is their ability to be represented as a product of two factors that have prescribed properties. One of such factorizations is the famous *inner/outer* factorization [LoPi98],[Er80],[Ma02]. In system theoretic setting, it plays a similar role as a QR-decomposition in linear algebra. In the robust control theory of [Kim97], operator inversion problems [DvdV00] and interpolation, factorizations play a fundamental role. Some of them can be derived from the terms involved in an inner/outer factorization. Other, such as *external* and *spectral* factorization are substantially different [DvdV98].

Throughout the thesis, heavy use of factorizations such as inner/outer and external is made. For purpose of analyzing mixed causality operators, external factorization will be applied first. Then, when considering operator inversion problems, we introduce so-called *maximal-phase/inner* factorization as an alternative representation.

The second part of the thesis addresses so-called *minimal* factorizations of mixed causality operators.

In the definition of a minimal factorization the degree of an operator plays a role [Son98]. For LTI-systems the Smith-McMillan degree is defined as a characteristic of the Smith-McMillan canonical form. It turns out to be equivalent to the minimal dimension of the state-space required in a state-space representation. In the LTV-theory the Smith-McMillan form does not exist anymore and the degree theory has to be adapted accordingly based on minimal state-space representations. The minimal factorization can then naturally be defined as a factorization in which the degrees of the factors add up to the degree of the operator in question. The minimal factorization

theory in the realm of the geometry of Hankel operators is grounded in the work [DeJV75] where causal operators having normal rank at infinity are considered in the LTI setting. The same problem was considered in [BartGohKaa79] from a different point of view and the Bart-Gohberg-Kaashoek theorem has been established that provide us with necessary and sufficient conditions for the existence of a minimal factorization.

1.3 Outline

The outline of the thesis is as follows.

In Chapter 2 we introduce the relevant preliminary notations and nomenclature of linear time-varying discrete systems.

In Chapter 3 we deal with the inverse of a mixed causality operator. We first start with a causal invertible systems, find its inner and the outer factor from the inner/outer representations and then try to assemble the inverse by inverting the factors. The important tool here is the algorithm for doing inner/outer factorization. We investigate various algebraic properties of the terms involved by the algorithm in order to obtain a closed form for the inverse. Later on, we extend the theory to the mixed causality operators. It appears that both the algorithm for inner/outer factorization and an external factorization algorithm are needed in obtaining the inverse. The method that is used here is an operator-setting equivalent to URV-decomposition and consists of finding an external factor of the operator by postmultiplying the operator with appropriate inner factor. Then the inner/outer factorization is applied on the external factor.

In Chapter 4 we introduce a new representation for causal operators: *maximal-phase/inner* factorization. An invertible maximal-phase operator is those that has the anti-causal inverse. This factorization provides us with a mean to characterize a minimal additive realization for the inverse of a invertible operator in a way that provides a close relationship with the Hankel operator associated to the operator at hand. We also provide an algorithm for an one-pass construction of a maximal-phase operator.

Chapter 5 considers mixed causality isometries and coisometries. We show that these operators always admit a minimal causal/anti-causal factorization in which one of the factors is either isometry or coisometry whileas the other is inner or anti-inner. Characterization of these operators is then done by considering properties of minimal additive realizations of them. We provide also an one-pass algorithm that make possible to extend a mixed causality isometry into a unitary operator. Finally, we introduced an additional partitioning in the unitary operators and realizations. This made possible to evaluate an important state-space transformation that reveals connections between solutions of Sylvester equations and Lyapunov-Stein equations.

On the other side, certain appropriately partitioned unitary operators give rise to, so called, J-unitary operators. The importance of the latter is recognized in H_∞ -control and interpolation.

Chapter 6 deals with minimal factorizations of locally finite operators. We concentrate on invertible operators and provide necessary and sufficient conditions for a minimal factorization to exist. The factorization do not necessarily have a unique solution and a search is needed first to generate a possible candidate for a factor. Then, by checking this candidate on an algebraic condition, we produce the answer if the candidate is indeed a minimal factor. The method used here aims at completing of the inverse of the factor when the factor itself is partially specified. The approach via maximal-phase factors as introduced in Chapter 4 appears to be useful in characterizing the inverse. Finally, a minimal causal/anti-causal factorization of a J-unitary operator of mixed causality is considered and it is shown that this factorization exists if and only the bounded solution of a certain indefinite Lyapunov-Stein equation is invertible. Moreover, all possible causal/anti-causal factorizations for this operators involve causal and anti-causal maximal-phase operators of minimal degree.

In Chapter 7 we consider polar decompositions of locally finite causal operators defined on Pontryagin spaces. Starting from the state-space description of causal invertible operators, we determine necessary and sufficient conditions for such operators to admit an outer/causal-J-factorization . Standard outer/inner factorization setting forms the crux in determination of the conditions. Finally, a stable one-pass algorithm for the indefinite polar decomposition for left invertible causal operators is stated.

The thesis is concluded by Chapter 8. Therein we enlist the main results obtained during this research. Finally we enlist some open problems that were not considered in the thesis.

1.4 Contributions

On the basis of the thesis three publications have been produced:

- E.ALIJAGIC, P.DEWILDE, *Minimal semi-separable representation of the inverse of a semi-separable matrix*, MTNS2004-Proceedings available on CD-ROM,
- P.DEWILDE, E.ALIJAGIC, *One-pass square-root algorithm for the inverse of a semi-separable matrix*, submitted for Linear Algebra and its Applications.
- E.ALIJAGIC, P.DEWILDE, *Minimal factorizations of quasi-separable isometric operators*, submitted for Operator theory: Advances and Applications 2005.

In the first paper we consider the problem of computing the inverse of infinite systems of linear equations specified by a finite set of data when no intrinsic factors are involved.

The second paper is an extension of the former one. We consider the problem of computing the closed formula for the inverse of a mixed causality operator in presence of intrinsic inner factors. The linear operator representing the system matrix is represented by a discrete time-varying system Smith-McMillan degree of which is time-varying but finite at each time instant. This setting leads to the generalization of the classical linear system theory in which the QR-decomposition such as is replaced by its operator-setting counterpart; inner/outer factorization. Other factorization such as URV-decomposition have also analogons in the operator setting and we make a heavy use of them in the paper to derive a closed formula for minimal state-space representations for the inverses of matrices. We first consider inverting a causal operator with possibly mixed inverse. It turns out that that a simple one-pass square root algorithm is capable of producing an efficient closed form for the general non-causal system at the cost of a reduced Lyapunov-Stein recursion that runs in the opposite direction. It is, however, possible to provide efficient, non-minimal representations of the inverse system with a simple unilateral recursion. Finally, we present minimal realizations for semi-separable inverses of semi-separable originals. All algorithms presented are backward numerically stable.

In the last one, we show that every isometric, semi-separable operator Σ of mixed causality, given by a minimal, uniformly exponentially stable realization $[\Sigma]$, admits a minimal isometric/anti-inner factorization. To show this we first discover the state-space properties of unitary operators of mixed causality. Then, we work out these properties further to produce isometric, u.e.s realizations for Σ . We also state dual results concerning coisometric operators. Finally, an one-pass algorithm for the unitary extension for Σ is stated.

The other still unpublished contributions refer to the results from Chapter 6. They state necessary and sufficient conditions for an invertible operator to admit a minimal factorization.

2

PRELIMINARIES

This chapter presents some general results from time-varying systems theory and the related theory of quasi-separable operators that play a role throughout the thesis. It reviews the setting and nomenclature of linear time-varying systems. Most of the results presented in this chapter are already known and treat the system theory using a Hilbert space theory setting. We refer the reader to [LoPi98],[Er80] and [Ma02] for details in Hilbert space theory.

2.1 Non-uniform vector spaces and s-dim sequence

In this section we introduce the concept of non-uniform vector space. This kind of spaces provides foundations for the time-varying system theory that shall be gradually introduced throughout this chapter. We follow the framework set up by Dewilde and Van de Veen in their pioneering work on time-varying system theory [DvdV98].

Because the spaces we will work on are vector spaces over the field of complex numbers \mathbb{C} , let us first give the definition of a vector space over \mathbb{C} .

Definition 2.1. *A vector space over \mathbb{C} is a nonempty set \mathcal{S} with two operations*

- *a mapping $(x, y) \rightarrow x + y$ from $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ called addition,*
- *a mapping $(\alpha, x) \rightarrow \alpha x$ from $\mathbb{C} \times \mathcal{S} \rightarrow \mathcal{S}$ called scalar multiplication*

such that for every $u, v, w \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{C}$ it holds that

$$(1) u + v = v + u,$$

$$(2) (u + v) + w = u + (v + w),$$

$$(3) \alpha(\beta u) = (\alpha\beta)u,$$

$$(4) (\alpha + \beta)u = \alpha u + \beta u,$$

$$(5) \alpha(u + v) = \alpha u + \alpha v,$$

$$(6) 1u = u \text{ and}$$

$$(7) \text{there exists } z \in \mathcal{S} \text{ such that } u + z = v,$$

Elements of a vector space are naturally called *vectors*. The following definition introduces the notion of a *subspace* of a vector space.

Definition 2.2. A subset \mathcal{E} of a vector space \mathcal{S} from Definition 2.1 is called a *subspace* if for every $\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathcal{E}$ it holds that $\alpha u + \beta v \in \mathcal{E}$.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ denote respectively the set of natural numbers and the set of integers. Let u be a (row) vector

$$u = [\dots u_{-1} u_0 u_1 \dots]$$

of infinite length each component of which

$$u_i = [u_{i,1} u_{i,2} \dots u_{i,M_i}], \quad \begin{cases} M_i \in \mathbb{N} \\ u_{i,k} \in \mathbb{C} \end{cases}, \quad \begin{cases} i = \dots, -1, 0, 1, \dots \\ k = 1, \dots, M_i \end{cases}$$

is an M_i -dimensional vector with entries from \mathbb{C} . The dimension of u at the index i is a natural number M_i . It is called the *local dimension* of the i -th component of u . The index i will be referred to as a *time index*. If $M_i = 0$ at some time i , then we say that there is no component of u at this time. In this case we put a dot¹ in u at the time i . The integers from the sequence $\{M_i\}$ need not be equal. Allowing this kind of freedom we are able to introduce the notion of 'time-variability' for u . Because u is an infinitely long vector, for the sake of either vector and matrix multiplications, or shifting it in the forward or backward direction, it is convenient to choose an element from u to play the role of a 'reference point'. Let us adopt u_0 to be this reference point. We put it into the box and write $\boxed{u_0}$ to differ it from the remaining components. With this convention, u looks like

$$u = [\dots u_{-1} \boxed{u_0} u_1 \dots].$$

¹This convention is common in MATLAB where vectors and matrices of vanishing dimensions frequently occur.

Similarly, the integers $\{M_i\}$ when recast into a vector form introduce

$$M = \left[\dots M_{-1} \boxed{M_0} M_1 \dots \right].$$

By construction, M belongs to

$$\mathcal{M} := \dots \times \mathcal{M}_{-1} \times \boxed{\mathcal{M}_0} \times \mathcal{M}_1 \times \dots$$

where² $\mathcal{M}_i \in \mathbb{N}$.

Throughout we write

$$u \in \mathbb{C}^{\mathcal{M}}$$

to denote that u belongs to the *non-uniform space* to which the s-dim vector M belonging to the space \mathcal{M} is associated.

The sequence $\{M_k\}$ is called *s-dim sequence* associated to u . It reflects the non-uniform nature of the sequence $\{u_i\}$ and hence will be our model in introducing non-uniform vector spaces. A special case occurs when M specifies a vector space which is one-dimensional at each i , i.e if

$$M = \left[\dots 1 \boxed{1} 1 \dots \right] \quad (2.1)$$

It is a well-known starting point for the study of the space ℓ_2 over \mathbb{C} .

Since the vectors from our considerations are not necessarily of uniform s-dim sequence, it is clear that if we add them then the information over their s-dim sequences is most likely lost in the addition. Equivalently said if u and v are two vectors having s-dim sequences M_u and M_v respectively, then we say that the addition

$$w := u + v$$

is well defined if $M_u = M_v$. In that case we naturally take $M_w := M_u$.

So far we have mentioned the algebraic properties of the non-uniform linear vector spaces. However, nothing about the 'closeness' of two points from such a space has been said neither their 'lengths' were the subject of the consideration. The standard infinite dimensional linear algebra comes then into rescue after noting that the non-uniform spaces can be made uniform if the corresponding s-dim sequence is set to be equal (2.1). In other words, the non-uniform vector space is isometrically isomorph to the ordinary infinite dimensional Euclidean space.

²The case $\mathcal{M}_i = \mathbb{N}$ includes the infinite vectors which are 'locally' infinite too. This case shall not be studied throughout.

2.2 The norm and the inner product on $\mathbb{C}^{\mathcal{M}}$

Let $u \in \mathbb{C}^{\mathcal{M}}$ where the space $\mathbb{C}^{\mathcal{M}}$ be as defined in the previous section. Let \mathbb{R}^+ denote the set of non-negative real numbers.

Definition 2.3. *The bounded function*

$$\|\cdot\|_{\mathcal{M}}: \mathbb{C}^{\mathcal{M}} \rightarrow \mathbb{R}^+$$

is called a norm on $\mathbb{C}^{\mathcal{M}}$ if it satisfies the following conditions

- (1) $\|u\|_{\mathcal{M}} = 0 \Leftrightarrow u = 0_{\mathcal{M}}$,
- (2) $\|\alpha u\|_{\mathcal{M}} = |\alpha| \|u\|_{\mathcal{M}} \quad \forall u \in \mathbb{C}^{\mathcal{M}}, \forall \alpha \in \mathbb{C}$,
- (3) $\|u + v\|_{\mathcal{M}} \leq \|u\|_{\mathcal{M}} + \|v\|_{\mathcal{M}} \quad \forall u, v \in \mathbb{C}^{\mathcal{M}}$.

The frequently used norm on a finite dimensional uniform Euclidean space \mathbb{C}^n , $n \in \mathbb{N}$ is the standard 2-norm (vector norm). For the (finite dimensional) component $u_i = [u_{i,1} \ u_{i,2} \ \dots \ u_{i,M_i}]$ of $u \in \mathbb{C}^{\mathcal{M}}$ at the time instant i , the 2-norm is defined by

$$\|u_i\|_2^2 = \sum_{j=1}^{M_i} |u_{i,j}|^2, \quad i = \dots, -1, 0, 1, \dots$$

We then define $\|u\|_2$ for u through

$$\|u\|_2^2 := \sum_{-\infty}^{+\infty} \|u_i\|_2^2.$$

The space $\mathbb{C}^{\mathcal{M}}$ endowed with the function³ $\|\cdot\|_2$ is a normed space. It is easy to prove that $\|\cdot\|_2$ fulfills the conditions from Definition 2.3. By standard arguments from functional analysis, it easily proves that each Cauchy sequence from $\mathbb{C}^{\mathcal{M}}$ converges to a single⁴ point in this space. Because it is then complete with respect to this norm, it is a Banach space by definition.

Perhaps the greatest disadvantage in using Banach space theory here is lack of the notion of orthogonality. In a finite dimensional vector space such as \mathbb{C}^n , $n \in \mathbb{N}$, the standard inner product between two vectors is defined to be the map

$$\langle \cdot, \cdot \rangle: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}, \quad n \in \mathbb{N}.$$

When the norm of the vectors is appropriate, it is a scalar 'measure' of how the vectors are aligned with respect to each other.

³In fact, it is a functional.

⁴Because this is a Hausdorff space, a converging sequence can have at most one limit.

In order to be an inner product space, a vector space \mathcal{S} have to satisfy the properties listed in the following definition.

Definition 2.4. Let \mathcal{S} be a vector space over \mathbb{C} . If there is a function

$$\langle \cdot, \cdot \rangle_{\mathcal{S}}: \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$$

such that

$$(1) \langle x, x \rangle_{\mathcal{S}} \geq 0 \quad \forall x \in \mathcal{S} \quad \text{and} \quad \langle x, x \rangle_{\mathcal{S}} = 0 \Leftrightarrow x = 0,$$

$$(2) \langle x, y \rangle_{\mathcal{S}} = \overline{\langle y, x \rangle_{\mathcal{S}}}, \quad \forall x, y \in \mathcal{S},$$

$$(3) \langle \alpha x + \beta y, z \rangle_{\mathcal{S}} = \alpha \langle x, z \rangle_{\mathcal{S}} + \beta \langle y, z \rangle_{\mathcal{S}}, \quad \forall x, y, z \in \mathcal{S}, \forall \alpha, \beta \in \mathbb{C}.$$

then \mathcal{S} is called an inner product space.

It is not hard to show that $\mathbb{C}^{\mathcal{M}}$ is an inner product space. An inner product provides us with the information about the size of the projection of one vector when projected onto the other. The projection is again a vector and, if its length vanishes, we say that the vectors are *orthogonal*. It turns out that every inner product space is also a normed space. In the case $\mathcal{S} = \mathbb{C}^{\mathcal{M}}$ the 2-norm of a vector x from this space is related to the inner product via the well-known formula

$$\langle x, x \rangle_{\mathbb{C}^{\mathcal{M}}} = \|x\|_2^2, \quad x \in \mathbb{C}^{\mathcal{M}}.$$

The space $\mathcal{S} = \mathbb{C}^{\mathcal{M}}$ belongs to the class of finite dimensional Hilbert spaces the formal definition of which follows.

Definition 2.5. A complete⁵ inner product space is called a Hilbert space.

Finally, the space of nonuniform sequences in $\mathbb{C}^{\mathcal{M}}$ with finite 2-norm is defined as

$$\ell_2^{\mathcal{M}} = \{u \in \mathbb{C}^{\mathcal{M}}: \|u\|_2 < \infty\}.$$

2.3 Linear mappings between non-uniform Hilbert spaces

In this section we introduce the notion of a linear operator mapping between two non-uniform Hilbert spaces. The notion of boundedness is used in the following definition, but its definition shall be given thereafter.

⁵Here we mean the completeness of the space as a normed space.

Definition 2.6. Let \mathcal{M} and \mathcal{N} be two non-uniform Hilbert spaces. The map

$$\cdot T: \mathcal{M} \rightarrow \mathcal{N}$$

is called a linear operator if it satisfies the following properties

- (1) $(u + v)T = uT + vT, \quad \forall u, v \in \mathcal{M}$ and
 (2) $(\alpha u)T = \alpha(uT), \quad \forall u \in \mathcal{M}, \forall \alpha \in \mathbb{C}.$

In Definition 2.6 we used the notion of the *boundedness* for the operator T . Since \mathcal{M} and \mathcal{N} as mentioned in the definition are normed spaces with their respective norms $\|\cdot\|_{\mathcal{M}}$ and $\|\cdot\|_{\mathcal{N}}$, we say that the linear map T is bounded if and only if for each $u \in \ell_2^{\mathcal{M}}$ the result $y = uT$ is in $\ell_2^{\mathcal{N}}$ and the *induced operator norm* defined as

$$\|T\| = \sup_{u, u \neq 0} \frac{\|uT\|_2}{\|u\|_2}$$

is bounded.

Definition 2.7. Let T be an operator whose map is defined as $\cdot T: \mathcal{M} \rightarrow \mathcal{N}$. The operator $\cdot T^*: \mathcal{N} \rightarrow \mathcal{M}$ and defined by

$$\langle uT, y \rangle_{\mathcal{N}} = \langle u, yT^* \rangle_{\mathcal{M}} \quad \forall u \in \mathcal{M}, y \in \mathcal{N}$$

is called the *adjoint operator* of T .

The adjoint of a bounded operator is again a bounded operator. This fact easily follows after considering the Cauchy-Schwartz inequality⁶ which holds in any inner product space.

The space of bounded linear operators mapping between signal spaces will be denoted by $\mathcal{X}(\mathcal{M}, \mathcal{N})$ or, for simplicity, by \mathcal{X} whenever it is clear what is meant. The spaces $\|\cdot\|_{\mathcal{M}}$ and $\|\cdot\|_{\mathcal{N}}$ have complete orthonormal bases. By the definition they are both *separable* Hilbert spaces. The operator itself, since bounded and defined everywhere on the separable Hilbert space, has a matrix representation which, with respect to the standard bases of the involved spaces, takes the form of a doubly infinite matrix

$$T = \begin{bmatrix} \ddots & & \vdots & & \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \\ \dots & T_{0,-1} & \boxed{T_{0,0}} & T_{0,1} & \dots \\ & T_{1,-1} & T_{1,0} & T_{1,1} & \\ & & \vdots & & \ddots \end{bmatrix}. \quad (2.2)$$

⁶ $|\langle u, v \rangle| \leq \|u\| \|v\|$

The block entry on the i, j position of T is the matrix $T_{ij}: \mathbb{C}^{M_i} \rightarrow \mathbb{C}^{N_j}$ of the size $(M_i \text{ rows}) \times (N_j \text{ columns})$. The position of the 'boxed' reference term in the above representation for T is induced by similar terms from u and y .

We define the following subspaces of bounded operators:

$$\begin{aligned} \mathcal{U} &= \{T \in \mathcal{X} : T_{ij} = 0 \text{ for } i \leq j\}, \\ \mathcal{L} &= \{T \in \mathcal{X} : T_{ij} = 0 \text{ for } i \geq j\} \text{ and} \\ \mathcal{D} &= \mathcal{U} \cap \mathcal{L} \end{aligned}$$

which will be referred to as subspaces of *causal*-, *anti-causal* and *operators reducing to a constant* respectively.

An important operator to remember is a *isometric* operator. It is a mapping $U \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ that maps \mathcal{M} isomorphic onto \mathcal{N} . Equivalently said for any $u, v \in \mathcal{M}$ it holds that

$$\langle u, v \rangle_{\mathcal{M}} = \langle uU, vU \rangle_{\mathcal{N}}.$$

It is an injective mapping. If, in addition, its adjoint U^* is also an isometric operator, then U is called *unitary*. A unitary operator from \mathcal{U} is called *inner*.

2.4 Hilbert-Schmidt operators and orthogonal projections

Let us recall the definition of the *Frobenius norm* of a finite dimensional matrix A . It is defined as $\|A\|_F = \sum_{i,j} |A_{ij}|^2$ where A_{ij} is the scalar entry of A at the ij -th position. For linear operators we extend this definition as follows.

Definition 2.8. Let $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ with the matrix representation given by (2.2). T is said to be a *Hilbert-Schmidt operator* if the norm defined as

$$\|T\|_{HS}^2 = \sum_{i,j} \|T_{ij}\|_F^2$$

is finite.

It is clear that if T is a Hilbert-Schmidt operator then T is bounded. The space of operators bounded in the Hilbert-Schmidt norm is defined as

$$\mathcal{X}_2 = \{T \in \mathcal{X}(\mathcal{M}, \mathcal{N}) : \|T\|_{HS}^2 < \infty\}.$$

The Hilbert-Schmidt operators are said to belong to the *trace-class* operators because the mapping $\langle \cdot, \cdot \rangle_{HS}: \mathcal{X}_2 \times \mathcal{X}_2 \rightarrow \mathbb{C}$ defined as

$$\langle T, G \rangle_{HS} = \text{trace}(TG^*) \quad \forall T, G \in \mathcal{X}$$

is indeed an inner product for which $\langle T, T \rangle_{HS} = \|T\|_{HS}^2$. We introduce the following subspaces of \mathcal{X}_2 :

$$\begin{aligned}\mathcal{U}_2 &= \mathcal{X}_2 \cap \mathcal{U}, \\ \mathcal{L}_2 &= \mathcal{X}_2 \cap \mathcal{L} \text{ and} \\ \mathcal{D}_2 &= \mathcal{U}_2 \cap \mathcal{D}.\end{aligned}$$

Conformably to spaces \mathcal{U}_2 , \mathcal{L}_2 and \mathcal{D}_2 , we define *orthogonal projections*

$$\begin{aligned}\mathbf{P}(\cdot) &: \mathcal{X}_2 \rightarrow \mathcal{U}_2, \\ \mathbf{P}'(\cdot) &: \mathcal{X}_2 \rightarrow \mathcal{X}_2 \ominus \mathcal{U}_2 \text{ and} \\ \mathbf{P}_0(\cdot) &: \mathcal{X}_2 \rightarrow \mathcal{D}_2\end{aligned}$$

In the previous definition for the projection $\mathbf{P}'(\cdot)$ we used the symbol ' \ominus '. Its definition is given in the sequel. Let \mathcal{S}'_2 and \mathcal{S}''_2 be two spaces of Hilbert-Schmidt operators. The *direct sum space* of \mathcal{S}'_2 and \mathcal{S}''_2 is defined as

$$\mathcal{S}'_2 \dot{+} \mathcal{S}''_2 = \{s' + s'' : s' \in \mathcal{S}'_2, s'' \in \mathcal{S}''_2\}.$$

The *orthogonal sum space* of orthonormal spaces \mathcal{S}'_2 and \mathcal{S}''_2 is defined as

$$\mathcal{S}'_2 \oplus \mathcal{S}''_2 = \{s' + s'' : s' \in \mathcal{S}'_2, s'' \in \mathcal{S}''_2\}.$$

Finally, we use the symbol \perp in $\mathcal{S}'_2 \perp \mathcal{S}''_2$ to denote that \mathcal{S}'_2 is 'orthogonal' to \mathcal{S}''_2 i.e

$$\mathcal{S}'_2 \perp \mathcal{S}''_2 \Leftrightarrow \langle s', s'' \rangle_{HS} = 0 \quad \forall s' \in \mathcal{S}'_2, \forall s'' \in \mathcal{S}''_2.$$

If \mathcal{S}_2 is a space of Hilbert-Schmidt operators and \mathcal{S}'_2 its subspace, then

$$\mathcal{S}_2 \ominus \mathcal{S}'_2 = \{s \in \mathcal{S}_2 : \forall s' \in \mathcal{S}'_2 \text{ it holds that } \langle s, s' \rangle_{HS} = 0\}.$$

The projections \mathbf{P} , \mathbf{P}' and \mathbf{P}_0 are obviously bounded in the Hilbert-Schmidt sense. It is worth noting that the projections are not defined on all \mathcal{X} . It is quite possible that, for instance, the causal part of some T defined on \mathcal{X} is unbounded. Just taking this causal part from T could be considered as an orthogonal projection but there is no guarantee that this projection is bounded.

In \mathcal{X}_2 we introduce the *diagonal inner product* (see [ADD90] and [DvdV98]). It is a \mathcal{D}_2 -valued mapping $\{\cdot, \cdot\} : \mathcal{X}_2 \times \mathcal{X}_2 \rightarrow \mathcal{D}_2$ defined as

$$\{X_1, X_2\} := \mathbf{P}_0(X_1 X_2^*) \quad X_1, X_2 \in \mathcal{X}_2.$$

It is an inner product in a Hilbert space modulo theory.

2.5 Bilateral shift

One extremely important and widely known operator is a shift operator. On $\ell_2(\mathbb{C})$ it is commonly denoted with z and, when applied on some u from the latter space it enforces u to shift all its entries for one position to the right as

$$[\dots u_{-1} \boxed{u_0} u_1 \dots] z = [\dots \boxed{u_{-1}} u_0 u_1 \dots].$$

The definition of the shift on ℓ_2^M is similar and a bit more involved. The extra job encounters handling the s-dim sequence M associated to \mathcal{M} properly. Let u be now an element of ℓ_2^M and let

$$M = [\dots M_{-1} \boxed{M_0} M_1 \dots]$$

be the s-dim sequence associated to \mathcal{M} . We define

$$M^{(k)} = [\dots M_{-k-1} \boxed{M_{-k}} M_{-k+1} \dots], \quad k \in \mathbb{Z}$$

and $\ell_2^{M^{(k)}}$ to be another space of, in the 2-norm bounded, non-uniform sequences having $M^{(k)}$ as its s-dim sequence. The *right shift* $Z_{\mathcal{M}}$ on ℓ_2^M is then the operator $Z_{\mathcal{M}}: \ell_2^M \rightarrow \ell_2^{M^{(1)}}$ whose action is defined by

$$[\dots u_{-1} \boxed{u_0} u_1 \dots] Z_{\mathcal{M}} = [\dots u_{-2} \boxed{u_{-1}} u_0 \dots].$$

The matrix representation for $Z_{\mathcal{M}}$ is

$$Z_{\mathcal{M}} = \left[\begin{array}{cccccccc} \dots & \dots & & & & & & \\ & 0 & I_{M_{-1} \times M_{-1}} & & & & & \\ & & \boxed{0} & & & & & \\ & & & I_{M_0 \times M_0} & & & & \\ & & & 0 & I_{M_1 \times M_1} & \dots & & \\ & 0 & & & 0 & & \dots & \\ & & & & & & & \dots \end{array} \right].$$

The block on the $(l, l+1)$ -position of $Z_{\mathcal{M}}$ is the identity matrix of the size $(M_l \text{ rows}) \times (M_{l+1} \text{ columns})$. All other terms are zero. For simplicity reasons we will omit the index \mathcal{M} and write simply Z whenever there is no danger of misinterpretation. We also adopt the notation Z^k to denote k -times repeated application of Z :

$$Z^k = Z_{\mathcal{M}} Z_{\mathcal{M}^{(1)}} \dots Z_{\mathcal{M}^{(k-1)}}, \quad k \in \mathbb{Z}.$$

Viz. the matrix representation given above we simply check that $ZZ^* = I_{\mathcal{M}}$ and $Z^*Z = I_{\mathcal{N}}$. For the spaces of bounded operators \mathcal{U} and \mathcal{L} the following properties are easy to check

$$\begin{aligned} ZU &\subset \mathcal{U}, & UZ &\subset \mathcal{U}, \\ Z^{-1}\mathcal{L} &\subset \mathcal{L}, & \mathcal{L}Z^{-1} &\subset \mathcal{L}, \\ ZU \cap \mathcal{L} &= \{0\}, & \mathcal{L}Z^{-1} \cap \mathcal{U} &= \{0\}. \end{aligned}$$

For the proof see [ADD90]. The same literature provides the proof that \mathcal{X}_2 can be represented as

$$\mathcal{X}_2 = \mathcal{L}_2 Z^{-1} \oplus \mathcal{D}_2 \oplus \mathcal{U}_2 Z.$$

The matrix representation (2.2) of T from \mathcal{X} and the shift Z allow to view T as a sequence of diagonals. Note first that $T_{[k]}$ for $k \in \mathbb{Z}$ defined as

$$T_{[k]} = \mathbf{P}_0(Z^{-k}T)$$

select the k -th off-diagonal of T . The entire T is not necessarily representable as the series

$$T = \sum_{k=-\infty}^{\infty} Z^k T_{[k]}$$

unless the uniform convergence of the partial series is ensured. In the case T is a Hilbert-Schmidt operator, the convergence however is guaranteed.

A widely used operator in characterization of operators from \mathcal{X} is called *outer* operator. It is an operator from \mathcal{U} . Because the algebra of operators we deal with is non-commutative, we differ between a *left outer* T_{ol} and a *right outer* T_{or} operator. They are implicitly defined through the following conditions

$$\begin{aligned} T_{ol} \in \mathcal{U}(\mathcal{M}, \mathcal{N}) & & \overline{\mathcal{U}_2 T_{ol}} &= \mathcal{U}_2 \\ T_{or} \in \mathcal{U}(\mathcal{M}, \mathcal{N}) & & \overline{T_{or} \mathcal{U}_2} &= \mathcal{U}_2. \end{aligned}$$

The bar in '̄' means the *closure* of the space denoted with '·'. If an operator is both left- and right outer, it is called outer and, for simplicity reasons, we omit the indices 'l' and 'r'.

2.6 State-space description

In this section we deal with the state-space description of an operator $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$. In [DvdV98] it is shown that a mapping $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$, $y = uT$ has a state-space representation if all snapshots of the Hankel operator corresponding to T are of finite rank. In that case, it is shown that there exist sequences of matrices of finite dimensions A_k, B_k, C_k, D_k and a sequence of vectors $x_k \in \ell_2^{\mathcal{B}}$ on some non-uniform s -dim space \mathcal{B} such that

$$\begin{aligned} x_{k+1} &= x_k A_k + u_k B_k, \\ y_k &= x_k C_k + u_k D_k, \quad k = \dots, -1, 0, 1, \dots \end{aligned} \tag{2.3}$$

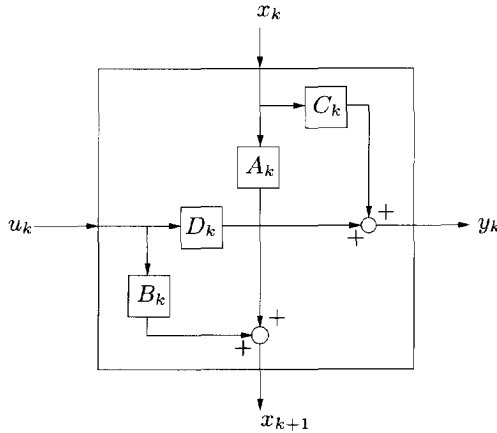


Fig. 2.1. A block-description of the state-space equations (2.3) at the time k .

Such a T is called a *locally finite* bounded causal operator. A block-description of the latter state-space equations at the time k is given in Figure 2.1.

In the same literature it is specified how the sequence can be obtained from the matrix representation (2.2) of the operator. Similar claim can be derived for $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ we started from. Since T , $y = uT$ is defined on the semi-separable space, it is possible to find a set of intermediate state subspaces: $\ell_2^{\mathcal{B}_c}$ and $\ell_2^{\mathcal{B}_a}$ having both as small as possible dimension at each time instant, and sequences of finite dimensional matrices

$$\{(A_c)_k, (B_c)_k, (C_c)_k, (A_a)_k, (B_a)_k, (C_a)_k, (D)_k\}, \quad k \in \mathbb{Z} \quad (2.4)$$

defined as

$$\begin{cases} (A_c)_k \in \mathcal{D}(\mathcal{B}_{c,k}, \mathcal{B}_{c,k+1}), & (A_a)_k \in \mathcal{D}(\mathcal{B}_{a,k+1}, \mathcal{B}_{a,k}), \\ (B_c)_k \in \mathcal{D}(\mathcal{M}_k, \mathcal{B}_{c,k+1}), & (B_a)_k \in \mathcal{D}(\mathcal{M}_k, \mathcal{B}_{a,k}), \\ (C_c)_k \in \mathcal{D}(\mathcal{B}_{c,k}, \mathcal{N}_k), & (C_a)_k \in \mathcal{D}(\mathcal{B}_{a,k}, \mathcal{N}_k), \\ (D)_k \in \mathcal{D}(\mathcal{M}_k, \mathcal{N}_k) & \forall k \in \mathbb{Z} \end{cases} \quad (2.5)$$

such that for the intermediate variable $x_k := [x_{c,k} \ x_{a,k}]$, $k \in \mathbb{Z}$ whereby $x := [x_c \ x_a] \in \ell_2^{\mathcal{B}_c} \oplus \ell_2^{\mathcal{B}_a}$, it holds that

$$[T]_k = \left[\begin{array}{c|c} (A_c)_k & (C_c)_k \\ \hline (A_a)_k & (C_a)_k \\ \hline (B_c)_k & (B_a)_k \\ \hline & (D)_k \end{array} \right], \quad (2.6)$$

$$[x_{c,k+1} \ x_{a,k} | y_k] = [x_{c,k} \ x_{a,k+1} | u_k] [T]_k, \quad k \in \mathbb{Z} \quad (2.7)$$

where u_k s.t. $u \in \ell_2^{\mathcal{M}}$ and y_k s.t. $y \in \ell_2^{\mathcal{N}}$ are the k -th components of u and y . Although it is defined implicitly, the sequence of maps $\{[T]_k\}$ defined by (2.6)-(2.7) will be referred throughout as a sequence of *locally minimal additive realizations* for the *mixed causality* operator T .

Definition 2.9. *The sequence of realizations defined by (2.6)-(2.7) is called uniformly exponentially stable (u.e.s) if the limits defined as*

$$\ell_{A_c} := \lim_{l \rightarrow \infty} \left(\sup_k \|(A_c)_{k+1}(A_c)_{k+2} \dots (A_c)_{k+l}\|^{1/l} \right) \text{ and}$$

$$\ell_{A_a} := \lim_{l \rightarrow \infty} \left(\sup_k \|(A_a)_{k-1}(A_a)_{k-2} \dots (A_a)_{k-l}\|^{1/l} \right)$$

are such that $\ell_{A_c} < 1$ and $\ell_{A_a} < 1$. With $\ell_{A_c} < 1$ (resp. $\ell_{A_a} < 1$) we denote that the causal (resp. anti-causal) part of T has a minimal u.e.s realization.

In the case of uniform exponential stability, $\ell_{A_c} < 1$, it can be shown (see [DvdV98]) that the series $\sum_{i=0}^{\infty} (A_c Z)^i$ converges to the bounded causal operator $(I - A_c Z)^{-1}$. The key feature in the proof is the spectral radius for $A_c Z$. A spectral radius for $X \in \mathcal{X}$ is defined as

$$r(X) := \lim_{n \rightarrow \infty} \|X^n\|^{1/n}.$$

As a well-known fact [Tak93],[Yos71] and [Er80], if $r(X) < 1$ then $(I - X)^{-1}$ exists as a bounded operator, and $(I - X)^{-1} = \sum_{i=0}^{\infty} (X)^i$ is known as *Neumann series expansion* partial series of which converge uniformly to $(I - X)^{-1}$. When $X := A_c Z$ is substituted we obtain that $(I - A_c Z)^{-1}$ exists as a bounded causal operator. Similar claim holds if $\ell_{A_a} < 1$. In this case $(I - A_a Z^*)^{-1}$ exists as a bounded anti-causal operator. It is worth noting that, in the case of u.e.s $I - A_c Z$ and $(I - A_a Z^*)^*$ are both outer operators.

The following definition introduces the notion of the *degree* of a mixed causality operator.

Definition 2.10. *The degree $\#(T)$ of $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ given by a sequence of minimal, additive realization $[T]_k$ from (2.6)-(2.7), is the s -dim sequence corresponding to the state-space vector $x \in \ell_2^{\mathcal{B}_c \oplus \mathcal{B}_a}$.*

To avoid working with the time index k , it is notationally convenient to put the set of matrices given by (2.4) into a diagonal form. For the set $\{(D)_k\}$, $k \in \mathbb{Z}$ we define

$$D := \begin{bmatrix} \ddots & & & & & \\ & \ddots & & & & \\ & & (D)_{-1} & & & \\ & & & \boxed{(D)_0} & & \\ & & & & (D)_1 & \\ & & & & & \ddots \end{bmatrix}$$

and introduce the shift along the main diagonal in the South-East direction for one position as

$$D^{(-1)} := ZDZ^* = \begin{bmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ & & (D)_0 & & & & \\ & & & \boxed{(D)_1} & & & \\ & & & & (D)_2 & & \\ & & & & & \ddots & \\ & & & & & & \ddots \end{bmatrix}.$$

The shift for k positions is defined recursively as

$$D^{(k)} = Z^{*k} D^{(k+1)} Z^k \quad k \in \mathbb{Z}.$$

The block diagonals A_c, B_c, C_c, A_a, B_a and C_a are defined similarly. Together with these representations, and by means of the shift operator Z defined on $\mathcal{B}_{c,k}$ as

$$[\dots, \boxed{x_{c,k}}, \dots]Z = [\dots, \boxed{x_{c,k-1}}, \dots]$$

and on $\mathcal{B}_{a,k}$ as

$$[\dots, \boxed{x_{a,k}}, \dots]Z = [\dots, \boxed{x_{a,k-1}}, \dots],$$

it is possible to define realization for T in the form

$$[T] = \left[\begin{array}{c|c} A_c & C_c \\ \hline A_a & C_a \\ \hline B_c & B_a \\ \hline & D \end{array} \right], \quad (2.8)$$

$$\begin{bmatrix} x_c^{(-1)} & x_a | y \end{bmatrix} = \begin{bmatrix} x_c & x_a^{(-1)} | u \end{bmatrix} [T].$$

The map given by (2.8) defines an additive realization for T and is given in a block diagonal form. In Figure 2.2 a block-description of the realization (2.8) is depicted. Therein and throughout this book we adopt the convention that the energy of the causal states propagates from top to bottom and the opposite for the energy of the anti-causal states. Further on, $[T_c]$ and $[T_{sa}]$ constitute block diagonal representations of realizations for respectively the causal part of T including the diagonal term D and the strictly anti-causal part for T . For instance, the k -th snapshot of $[T_c]$ has the structure as given by the block-description from Figure 2.1. The state-space structure for $[T_{sa}]$ can be analogously deduced.

The shorter notation $[T] = \{A_c, B_c, C_c, A_a, B_a, C_a, D\}$, will be used throughout to denote the mapping given by (2.8). The u.e.s property for $[T]$ implies the existence of a *transfer function representation* for T in the form

$$T = D + B_c Z (I - A_c Z)^{-1} C_c + B_a Z^* (I - A_a Z^*)^{-1} C_a \quad (2.9)$$

where

$$T_{ij} = \begin{cases} (B_c)_i (A_c)_{i+1} \dots (A_c)_{j-1} (C)_j, & i < j, \\ (D)_i, & i = j, \\ (B_a)_i (A_a)_{i-1} \dots (A_a)_{j+1} (C_a)_j, & i > j. \end{cases}$$

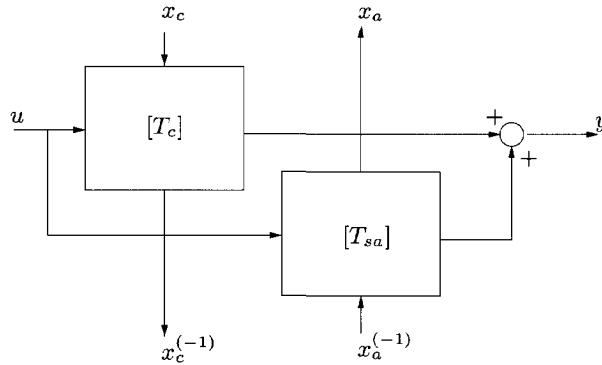


Fig. 2.2. A block-description of the state-space equations (2.8).

constitute the matrix representation (2.2) of T .

The minimal realization $[T]$ obtained above is not unique. Given an additive minimal realization $[T]$, all other additive minimal realizations can be derived from $[T]$ by applying appropriate boundedly invertible *state-space transformations*. To define such a transformation, it suffices to start from (2.9) and to note that for any pair of boundedly invertible block diagonals R_c and R_a of appropriate s-dim sequences, it holds that

$$T = D + (B_c R_c^{(-1)}) Z \left(I - (R_c^{-1} A_c R_c^{(-1)}) Z \right)^{-1} (R_c^{-1} C_c) \\ + (B_a R_a^{-1}) Z^* \left(I - (R_a^{(-1)} A_a R_a^{-1}) Z^* \right)^{-1} (R_a^{(-1)} C_a)$$

whereby the s-dim sequences of A_c and A_a are the same as those of $R_c^{-1} A_c R_c^{(-1)}$ and $R_a^{(-1)} A_a R_a^{-1}$ respectively. Then,

$$[T]_{alt.} := \left[\begin{array}{cc|c} R_c^{-1} A_c R_c^{(-1)} & & R_c^{-1} C_c \\ & R_a^{(-1)} A_a R_a^{-1} & R_a^{(-1)} C_a \\ \hline B_c R_c^{(-1)} & B_a R_a^{-1} & D \end{array} \right] \quad (2.10)$$

is an alternative minimal realization for T . It is important to stress that R_c and R_a do not generate all possible realizations for T of the degree $\#(A_c) + \#(A_a)$. A more general realization for $[T]$ will include the terms of mixed causality. In this way, it is quite possible that, for appropriate choice of non-trivial block diagonals A_{ca} and A_{ac} , the realization

$$[T]_{general} := \left[\begin{array}{cc|c} A_c & A_{ca} & C_c \\ A_{ac} & A_a & C_a \\ \hline B_c & B_a & D \end{array} \right]$$

is a minimal realization for T too. But it is evident that $[T]_{general}$ can not be related to $[T]$ by an appropriate choice of R_c and R_a . Clearly, $[T]_{general}$ gives rise to another

representation for T namely

$$\begin{aligned} T &= D + B_c Z (I - A_c Z)^{-1} C_c + B_a Z^* (I - A_a Z^*)^{-1} C_a \\ &\quad + B_c Z (I - A_c Z)^{-1} A_{ca} Z^* (I - A_a Z^*)^{-1} C_a \\ &\quad + B_a Z^* (I - A_a Z^*)^{-1} A_{ac} Z (I - A_c Z)^{-1} C_c. \end{aligned}$$

2.7 Reachability and observability

Similarly as in the linear time-invariant system theory [Kai80],[Rugh96], a realization for a time-varying system is minimal if and only if it is both reachable and observable. Algebraic tests on the reachability and observability consider positive definiteness of the solutions of certain matrix equations. In the context of the diagonal algebra, these equations are given in terms of elements from \mathcal{D} .

Definition 2.11. A self-adjoint (or Hermitian) operator T , $T = T^*$ is called semi-positive $T \geq 0$ if

$$\langle uT, u \rangle \geq 0 \quad \forall u \in \ell_2^M.$$

T is called positive $T > 0$ if

$$\langle uT, u \rangle > 0 \quad \forall u \in \ell_2^M$$

and T is called uniformly strictly positive $T \gg 0$ if there is a real $\epsilon > 0$ such that

$$\langle uT, u \rangle \geq \epsilon \langle u, u \rangle \quad \forall u \in \ell_2^M.$$

Definition for reachability and observability are given in the following two definitions (see also [Xi96]).

Definition 2.12. Let (A_c, B_c) be the AB-pair from a u.e.s realization for a causal operator. Define

$$C := \begin{bmatrix} B_c^{(1)} \\ B_c^{(2)} A_c^{(1)} \\ B_c^{(3)} A_c^{(2)} A_c^{(1)} \\ \vdots \end{bmatrix}.$$

and define $\Lambda_c := C^* C \in \mathcal{D}$. Then

- (A_c, B_c) is said to be partially unreachable if Λ_c is singular.
- (A_c, B_c) is said to be reachable if $\Lambda_c > 0$.
- (A_c, B_c) is said to be uniformly reachable if $\Lambda_c \gg 0$.

The observability is defined dually.

Definition 2.13. Let (A_c, C_c) be the AC-pair from a u.e.s realization for a causal operator. Define

$$\mathcal{O} := \left[C_c \ A_c C_c^{(-1)} \ A_c A_c^{(-1)} C_c^{(-2)} \ \dots \right].$$

and define $\Lambda_{o,c} := \mathcal{O}\mathcal{O}^* \in \mathcal{D}$. Then

- (A_c, C_c) is said to be partially unobservable if $\Lambda_{o,c}$ is singular.
- (A_c, C_c) is called observable if $\Lambda_{o,c} > 0$.
- (A_c, C_c) is called uniformly observable if $\Lambda_{o,c} \gg 0$.

The following two lemmas are from [DvdV98].

Lemma 2.14. Let (A_c, B_c) be the AB-pair from a u.e.s realization for a causal operator. The block diagonal Λ_c as introduced in Definition 2.12 and associated to the pair (A_c, B_c) is the bounded Hermitian solution for the Lyapunov-Stein equation

$$A_c^* \Lambda_c A_c + B_c^* B_c = \Lambda_c^{(-1)}. \quad (2.11)$$

Lemma 2.15. Let (A_c, C_c) be the AC-pair from a u.e.s realization for a causal operator. The block diagonal $\Lambda_{o,c}$ as introduced in Definition 2.13 and associated to the pair (A_c, C_c) is the bounded Hermitian solution for the Lyapunov-Stein equation

$$A_c \Lambda_{o,c}^{(-1)} A_c^* + C_c C_c^* = \Lambda_{o,c}.$$

The diagonal operators Λ_c and $\Lambda_{o,c}$ are called the *reachability* and the *observability* Gramians. The u.e.s argument used in the lemmas 2.14 and 2.15 is used for convergence reasons. Although an explicit converging expression can be given [DvdV00] for Λ_c (resp. $\Lambda_{o,c}$), these expressions are not very useful because they aim at knowing all data from $\{A_k\}$, $\{B_k\}$ (resp. $\{A_k\}$, $\{C_k\}$). The expression then may be regarded as a restriction since, in most cases, the operators from consideration are gradually built up by choosing either time-forward or time-backward sequences of their state-space data. Numerically speaking, if $\ell_{A_c} < 1$ then, by the Banach fixed point theorem, it is guaranteed that Λ_c and $\Lambda_{o,c}$ from these definitions converge to their unique bounded solutions when started from some initial $\Lambda_{c,init} \in \mathcal{D}$ and $\Lambda_{o,c,init} \in \mathcal{D}$ respectively.

For reasons of numerical stability it is preferable not to work with unreachable or unobservable pairs. The following lemma shows that it is possible to 'extract' unreachable parts from a given u.e.s AB-pair. Its proof is given in [DvdV98]. It is a time-varying extension of the time-invariant case see [Rugh96] and [Kai80]. In these references, the standard notions from linear algebra such as *singular value decomposition* (SVD), *eigenvalue decomposition* and *QR-decomposition* are used. We refer the reader to [GC96] and [HJo91] for extensive and algorithmic approach to these decompositions.

Lemma 2.16. *Let (A_c, B_c) with $\ell_{A_c} < 1$ be an AB-pair coming from a realization of a causal locally finite operator. Let $\Lambda_c \in \mathcal{D}$ be the unique bounded solution of (2.11) from Lemma 2.14.*

If Λ_c is singular and has closed range, then:

- (1) *There exist properly partitioned unitary $Q = [Q_1 \ Q_2] \in \mathcal{D}^{1 \times 2}$ and an invertible Hermitian $\lambda_c \in \mathcal{D}$ such that*

$$\Lambda_c = [Q_1 \ Q_2] \begin{bmatrix} \lambda_c & \\ & \underline{0} \end{bmatrix} \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix}$$

is an eigenvalue decomposition of Λ_c where the underlined zero block has maximal rank.

- (2) *There are $a_{11}, a_{21}, a_{22} \in \mathcal{D}$ and $b \in \mathcal{D}$ such that*

$$Q^* A Q^{(-1)} = \begin{bmatrix} a_{11} & \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B Q^{(-1)} = [b \ 0] \quad (2.12)$$

where (a_{11}, b) is the reachable pair of maximal s -dim sequence having λ_c as its reachability Gramian.

Extraction of maximal unobservable part from a given unobservable AC-pair can be done after taking the dual statement to that from Lemma 2.16. For simplicity reasons this will be omitted.

The AB-pair given by (2.12) will be referred to as a *Kalman controller form*. A *Kalman observer form* is defined dually.

For a mixed causality operator T , testing on minimality of an additive realization $[T]$ for T must be done for both the causal and the anti-causal AB- and AC-pairs. The definitions 2.12 and 2.13 and the lemmas 2.14 and 2.15 consider a causal operator part from T only. Similar definitions and statements can be made for the anti-causal operator part if we start from T^* a realization of which is $[T]^*$ and consider its causal part.

2.8 Hankel map and the spaces associated to it

Recall \mathcal{C} and \mathcal{O} from the definitions 2.12 and 2.13. Let the triplet (A_c, B_c, C_c) constitute the strictly causal part T_c of $T \in \mathcal{X}$. Define k -th *snapshots* of \mathcal{C} and \mathcal{O} as

$$\mathcal{C}_k := \begin{bmatrix} B_{c,k-1} \\ B_{c,k-2} A_{c,k-1} \\ B_{c,k-3} A_{c,k-2} A_{c,k-1} \\ \vdots \end{bmatrix} \quad \text{and} \quad \mathcal{O}_k := [C_{c,k} \ A_{c,k} C_{c,k+1} \ A_{c,k} A_{c,k+1} C_{c,k+2} \ \dots].$$

The operator

$$(H_{T_c})_k := C_k \mathcal{O}_k$$

is called the k -th snapshot of the *Hankel operator* corresponding to the triple (A_c, B_c, C_c) . Avoiding the time index is possible if we pass to the diagonal algebra representation. The *overall* Hankel operator is then a map $\cdot H_{T_c} : \mathcal{L}_2 Z^{-1} \rightarrow \mathcal{U}_2$ defined as

$$Y H_{T_c} = \mathbf{P}(Y T_c), \quad Y \in \mathcal{L}_2 Z^{-1}.$$

Since T_c is bounded, H_{T_c} , as a contraction of T_c , is bounded too (see [DvdV98]).

In most of the thesis we shall deal with mixed causality operators. They can always be written additively as

$$T = T_a + D + T_c$$

where T_c is a strictly causal part of T , D is the constant term and T_a is the strictly anti-causal part. Since T is assumed bounded, these projections are all bounded. For T_c and T_a we introduce the following spaces (see [Xi96] and [DvdV98])

$$\begin{aligned} \mathcal{K}_c(T_c) &:= \{U \in \mathcal{L}_2 Z^{-1} : \mathbf{P}(U T_c) = 0\}, \\ \mathcal{H}_c(T_c) &:= \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 T_c^*), \\ \mathcal{H}_{o,c}(T_c) &:= \mathbf{P}(\mathcal{L}_2 Z^{-1} T_c), \\ \mathcal{K}_{o,c}(T_c) &:= \{Y \in \mathcal{U}_2 : \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(Y T_c^*) = 0\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_a(T_a) &:= \{U \in \mathcal{U}_2 Z : \mathbf{P}_{\mathcal{L}_2}(U T_a) = 0\}, \\ \mathcal{H}_a(T_a) &:= \mathbf{P}_{\mathcal{U}_2 Z}(\mathcal{L}_2 T_a^*), \\ \mathcal{H}_{o,a}(T_a) &:= \mathbf{P}_{\mathcal{L}_2}(\mathcal{U}_2 Z T_a), \\ \mathcal{K}_{o,a}(T_a) &:= \{Y \in \mathcal{L}_2 : \mathbf{P}_{\mathcal{U}_2 Z}(Y T_a^*) = 0\}. \end{aligned}$$

The spaces \mathcal{K}_c and \mathcal{H}_c are called the *input null space* and the *input state-space* of H_{T_c} . In fact, it holds [DvdV98] that $\mathcal{K}_c(T_c) = \ker(\cdot H_{T_c})$ and $\mathcal{H}_c = \text{ran}(\cdot H_{T_c}^*)$. Since, in general [Er80] and [Rud66], the kernel of a linear operator is orthogonal to the closure of the range of its adjoint and they both span the whole space on which the operator is defined, we have that

$$\overline{\mathcal{H}_c} \oplus \mathcal{K}_c = \mathcal{L}_2 Z^{-1}.$$

For the *output state-space* $\mathcal{H}_{o,c}$ and the *output null space* $\mathcal{K}_{o,c}$, the situation is similar and we have that

$$\overline{\mathcal{H}_{o,c}} \oplus \mathcal{K}_{o,c} = \mathcal{U}_2.$$

Define

$$\mathbf{F}_c^* := B_c Z (I - A_c Z)^{-1}, \quad (2.13)$$

$$\mathbf{F}_{o,c} := (I - A_c Z)^{-1} C_c, \quad (2.14)$$

$$\mathbf{F}_a^* := B_c Z^* (I - A_a Z^*)^{-1}, \quad (2.15)$$

$$\mathbf{F}_{o,a} := (I - A_a Z^*)^{-1} C_a. \quad (2.16)$$

The following lemma provides a useful tool in handling the products of operators of mixed causalities. Its proof is standard and can be found in [DvdV00].

Lemma 2.17. (The mixed partial fraction lemma) *Let X_i , $i = 1, 2$ be contractive operators from \mathcal{X} such that the spectral radius of either $X_1 U$ or $X_2 U$ (or both) is less than 1, where $U \in \mathcal{X}$ is a unitary operator of appropriate s -dim sequence.*

Then, $(I - X_i U)$, $i = 1, 2$ are boundedly invertible.

In addition, for an arbitrary operator $\Gamma \in \mathcal{X}$ of appropriate s -dim sequence, the equation

$$X_1 U M U^* X_2^* + \Gamma = M$$

has the unique bounded solution M and it holds that

$$(I - X_1 U)^{-1} \Gamma (I - U^* X_2^*)^{-1} = (I - X_1 U)^{-1} A_1 U M + M + M U^* X_2^* (I - U^* X_2^*)^{-1}.$$

By straightforward evaluation and applying the mixed partial fraction lemma 2.17 it easily shows that for the Gramians A_c and $A_{o,c}$ defined in the definitions 2.12 and 2.12 respectively, it holds that

$$A_c = \mathbf{P}_0(\mathbf{F}_c \mathbf{F}_c^*) \text{ and } A_{o,c} = \mathbf{P}_0(\mathbf{F}_{o,c} \mathbf{F}_{o,c}^*).$$

The operators indexed with 'c' from (2.13)-(2.16) generate bases for the column span and the row span of respectively reachability and the observability operators from Definitions 2.12 and 2.13. Similarly the operators indexed with 'a' generate bases for the column span and the row span of respectively the reachability and the observability operators corresponding to the triple (A_a, B_a, C_a) .

The following proposition is from [Xi96] (see also [DvdV98]).

Proposition 2.18. *Let $T = T_c + D + T_a$ be the decomposition of $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ an given by (2.9) into the strictly causal, block diagonal and strictly anti-causal operator part such that T_c is associated to the causal triplet (A_c, B_c, C_c) and T_a to the anti-causal triplet (A_a, B_a, C_a) . Let (2.13)-(2.16) be bases generated by the triplets. Then*

$$\mathcal{H}_{o,c} \subseteq \mathcal{D}_2 \mathbf{F}_{o,c}, \quad \mathcal{H}_c \subseteq \mathcal{D}_2 \mathbf{F}_c, \quad \mathcal{H}_{o,a} \subseteq \mathcal{D}_2 \mathbf{F}_{o,a} \text{ and } \mathcal{H}_a \subseteq \mathcal{D}_2 \mathbf{F}_a.$$

In particular,

(1) If (A_c, C_c) is uniformly observable, then $\mathcal{H}_{o,c} = \mathcal{D}_2 \mathbf{F}_{o,c}$.

(2) If (A_c, B_c) is uniformly reachable, then $\mathcal{H}_c = \mathcal{D}_2 \mathbf{F}_c$.

(3) If (A_a, C_a) is uniformly observable, then $\mathcal{H}_{o,a} = \mathcal{D}_2 \mathbf{F}_{o,a}$.

(4) If (A_a, B_a) is uniformly reachable, then $\mathcal{H}_a = \mathcal{D}_2 \mathbf{F}_a$.

The statements from Proposition 2.18 relate state-spaces of the Hankel operators to the notion of (uniform) reachability and observability. The geometry of these spaces is crucial in obtaining the operator features such as *external* or *inner/outer* factorizability [Xi96],[DvdV98],[DvdV00]. Not only for the theoretical reasons, but also for the practical ones such as the synthesis of *lossless impedances circuits* [ADD90], *interpolation* [DvdV98] and *robust control* [Kim97], the geometry of the Hankel maps provides us with additional insights.

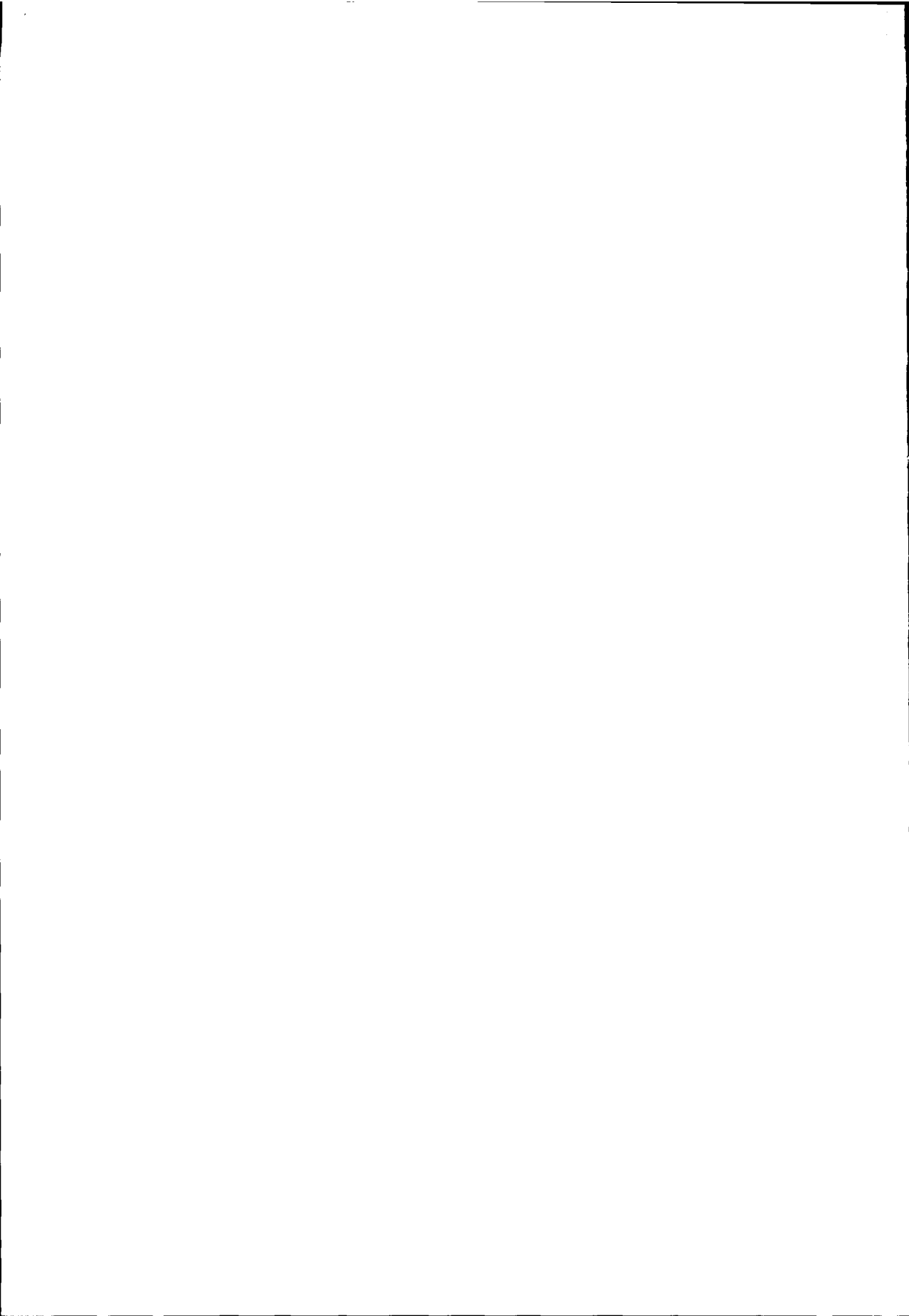
These concepts will be introduced in the subsequent chapters where, among others, the features of the Hankel map associated to the operator *inversion* and *minimal factorization* problem are studied.

2.9 Conclusions

This chapter provides the reader with the basic concepts and preliminaries of time-varying system theory. We started from the notion of a set on which a non-uniform vector space structure is posed. Then, we extended this algebraic concept to introduce the norm and to define the inner product underlying the given norm. In this way we obtained a Hilbert space of non-uniform ℓ_2 sequences. Finally, the passage to the diagonal algebra made possible to 'suppress' the time-varying nature of the underlying space and, by stacking non-uniform sequences on each other, to introduce spaces of operators. On these spaces we introduced the generic bilateral shift. This, in turn, made possible to define some concepts from the realization theory and to define notion of a uniform exponentially stable realization for a mixed causality operator. The notion of reachability and observability are closely related to realizations of operators. We have defined these notions and recalled how to obtain a minimal realization from a given nonminimal realization. The concept of the Hankel operator is also introduced and some basic relations between this map and the reachability and the observability operators associated to the given realization are stated. As we have seen the Hankel operator gives rise to the (state) spaces relevant in checking on (uniform) reachability and observability of a realization. The geometry of these spaces stays central in the subsequent chapters.

Part II

INVERSION



3

OPERATOR INVERSION

In the eighties, low rank approximations of submatrices of semi-separable type appeared in the study of integral equations [KKG85], where it was noted that such equations could be inverted efficiently if that structure was used systematically. In the early nineties, the connection with time-varying dynamical system theory was established and new ways of handling the numerics were proposed [DvdV91]. In particular, it was shown in [vdV93] and [vdV95] that backward stable numerical algorithms for the inversion of such structured systems could be obtained using orthogonal transformations, yielding transformation matrices or operators with the same efficient structural properties in the process. These basic methods were later refined by a number of authors [EidelmanGohberg1997, DvdV98, ChandrasekaranGu00, EidelmanGohberg2002], yielding a variety of precise characterizations for efficient inversion of the system of equations.

3.1 Introduction

This chapter is concerned with the evaluation of a minimal realization for the inverse of a boundedly invertible locally operator given by a sequence of minimal local realizations. To be able to evaluate the inverse, we first factorize the operator into invertible factors by working on the data from a state-space representation of the operator in question. Based on the state-space data, we then investigate how these factors and their inverses cancel and what is an algebraic characterization or a test

that the cancellation takes place. Finally, we assemble a minimal realization for the inverse starting from minimal realizations for the inverses of the factors. To obtain a closed formula for the inverse, we consider two methods. The first one arises from the outer/unitary factorization of the operator. The second one relies on the operator theoretical concept of URV-decomposition the definition of which will be given later in the text. Both methods are very much alike in the sense that they both start from the minimal external factorization step to produce a causal external factor. This factor is then subjected to either inner/outer or outer/inner decomposition. All such obtained factors are then inverted and the inverse is assembled by considering their product. Hence, the methods may be regarded as equivalent up to the order of the factors in inner/outer (resp. outer/inner) factorization. During the decomposition and the assembling, cancellations between the terms are possible. From the point of view of numerical stability, one is highly recommended to eliminate this kind of redundancies from (nonminimal) realizations of the factors before proceeding to the next step. For practical purposes however, the URV-decomposition method deserves more attention. As the final goal is to state an one-pass algorithm, it appears that this method is able to cope with cancellations properly. Extraction of the redundant subspaces online aims for algorithms as well but they fit perfectly into the one-pass sequence of events giving rise to an one-pass square-root algorithm.

3.2 External and inner/outer representations

It is well known [Du70], [Ko98] from the Beurling-Lax theory on Hardy spaces that complex valued functions of single variable allow for the inner/outer decomposition. Later on, the theory has been extended to the Hardy space of vector valued functions. That the theory holds true as well in the Hardy space of non-uniform sequences is shown in [DvdV98], [DvdV00]. Let $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be such an operator. Then, for some intermediate non-uniform space \mathcal{M}' , there is a coisometric $U \in \mathcal{U}(\mathcal{M}, \mathcal{M}')$ and a right outer $G_{or} \in \mathcal{U}(\mathcal{M}', \mathcal{N})$ such that $G = UG_{or}$ is the inner/outer representation for G . The factors have the property

$$\begin{aligned}\overline{GU_2} &= UU_2, \\ \overline{G_{or}U_2} &= U_2\end{aligned}$$

where because $\mathcal{K}(T) = \overline{GU_2}$ and because \mathcal{K} is *DZ-shift invariant* [DvdV98], the first equation is actually a non-stationary equivalent to the Beurling-Lax characterization of shift invariant subspaces. The second equation, however, is a range condition for the non-stationary equivalent to the notion of *outerness* defined on polydisc [Hel64].

When bases for the reachability and the observability spaces for G are chosen, realizations for the factors can easily be evaluated. In [DvdV00], the dual to the following theorem is proved for the case of a coisometric/right-outer factorization.

Theorem 3.1. *Let a causal operator $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s realization*

$$[G] = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \quad (3.1)$$

Let $[U]_k$ be unitary matrices, and Y_k be an uniformly bounded sequence of matrices such that

$$\begin{bmatrix} Y_k A_k & Y_k C_k \\ B_k & D_k \end{bmatrix} = [U]_k \begin{bmatrix} 0 & 0 \\ Y_{k+1} & 0 \\ B_{or,k} & D_{or,k} \end{bmatrix} \quad (3.2)$$

and such that

- (1) Y_k is such that $\ker(\cdot Y_k) = 0$ and has maximal row rank, and
- (2) $\ker(\cdot D_{or,k}) = 0$.

Assume that the partitioning in $[U]_k$ is $[U]_k = \begin{bmatrix} C'_{U,k} & A_{U,k} & C_{U,k} \\ D'_{U,k} & B_{U,k} & D_{U,k} \end{bmatrix}$.

Then $\{A_{U,k}, B_{U,k}, C_{U,k}, D_{U,k}\}$ is a coisometric realization for U and $\{A_k, B_{or,k}, C_k, D_{or,k}\}$ is a realization for the outer $[G_{or}]$ in the coisometric/right-outer factorization

$$G = U G_{or}.$$

In addition, $\{A_{U,k}, B_{U,k}, C'_{U,k}, D'_{U,k}\}$ is a realization for a causal coisometric U' of maximal degree which is such that $U'^* G = 0$.

The quantity $Y \in \mathcal{D}$ from Theorem 3.1 deserves some attention. By passing the unitary realization $[U]$ to the left side of (3.2) it shows that Y is the unique bounded solution for

$$A_U^* Y A + B_U^* B = Y^{(-1)}$$

or equivalently,

$$Y = P_o(QF^*)$$

where $Q^* := B_U Z(I - A_U Z)^{-1}$ and $F^* := B Z(I - A Z)^{-1}$ are uniform sliced bases representations for the reachability spaces corresponding to $[U]$ and $[G]$ respectively.

Since G we are dealing with is assumed boundedly invertible, the sequence of square-root equations from Theorem 3.1 produces a sequence of realizations for the outer factor which is both left and right invertible although not necessarily in a bounded fashion. Hence, the dimension of a coisometric U' which must be such that $U'^* G = 0$, vanishes enforcing the coisometric U to be unitary with $[U]$ that is not necessarily a u.e.s realization. This fact simplifies the QR-decomposition (3.2) and turns it into

$$\begin{bmatrix} YA & YC \\ B & D \end{bmatrix} = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix} \begin{bmatrix} Y^{(-1)} & 0 \\ B_o & D_o \end{bmatrix} \quad (3.3)$$

where $[U] = \{A_U, B_U, C_U, D_U\}$ is unitary, u.e.s¹ realization and $[G_o] = \{A, B_o, C, D_o\}$ with locally square invertible D_o is minimal and u.e.s if and only if it is reachable. As before Y is boundedly right invertible and of maximal row rank at each of its snapshots.

Starting from the realization $[G_o]$ for the outer G_o , a realization for G_o^{-1} evaluates easily as

$$[G_o^{-1}] := \begin{bmatrix} \Delta_o & -CD_o^{-1} \\ D_o^{-1}B_o & D_o^{-1} \end{bmatrix} \quad (\Delta_o := A - CD_o^{-1}B_o). \quad (3.4)$$

As $[G_o]$ shares its AC-pair with the realization $[G]$, a nonminimality, if any, is induced by an eventual nonreachable part from the reachability pair (A, B) from $[G_o]$. Also, it is obvious that if $[G_o]$ is a minimal realization then

$$\text{s-dim}(G) = \text{s-dim}(G_o)$$

and U cancels completely in G_o , i.e $\text{s-dim}(UG_o) = \text{s-dim}(G_o)$.

Later on, we shall encounter that the minimality of the realization $[G_o]$ depends only on the intrinsicness of the inner U .

The following proposition is proved in [DvdV98] and its statement will be frequently used in the sequel.

Proposition 3.2. *Let $[T_o] = \{A_o, B_o, C_o, D_o\}$ be a minimal, u.e.s realization for a boundedly invertible outer operator T_o . Then, the realization*

$$[T_o^{-1}] = \{A_o - C_oD_o^{-1}B_o, -C_oD_o^{-1}, D_o^{-1}B_o, D_o^{-1}\}$$

for T_o^{-1} is a minimal, u.e.s realization as well.

Notice the importance of the assumption 'boundedly invertible' in Proposition 3.2. If this is not the case, then the Schur complement of A with respect to $[T_o]$ defined as $\Delta_o := A_o - C_oD_o^{-1}B_o$ is not necessarily u.e.s. In theory of complex valued functions on the unit disc, this case corresponds to the analytic functions inverses of which are analytic everywhere except on the unit circle where there is only finite number of poles [Ga81].

Another way of looking at G is considering its inverse $G_o^{-1}U^*$. This inverse is, in general, of a mixed causality and because $G^{-1}U (= G_o^{-1})$ is causal, we note that G_o^{-1} is an external factor of G^{-1} for a fixed inner U .

Therefore, we may assume that a minimal, u.e.s realization for $G^{-1} \in \mathcal{X}$ exists and find its external factorization. Because we know that G is causal, the obtained

¹Hence minimal

terms from an external factorization algorithm shall yield u.e.s realizations for outer G_o^{-1} and anti-inner U^* . Our first goal here is to evaluate a u.e.s realization for G^{-1} starting from minimal, u.e.s realizations for the inner U and the outer G_o obtained from QR-decomposition (3.2).

Proposition 3.3. *Let $[G_o] = \{A, B_o, C, D_o\}$ and $[U] = \{A_U, B_U, C_U, D_U\}$ be minimal, u.e.s realizations for the outer G_o and the inner U from the representation $G := UG_o \in \mathcal{U}(\mathcal{M}, \mathcal{N})$. Let $L \in \mathcal{D}$ be the unique bounded solution for*

$$\Delta_o L^{(-1)} A_U^* - CD_o^{-1} C_U^* = L \quad (\Delta_o := A - CD_o^{-1} B_o). \quad (3.5)$$

Then,

$$[G^{-1}] = \left[\begin{array}{c|c} \Delta_o & -CD_o^{-1} D_U^* + \Delta_o L^{(-1)} B_U^* \\ \hline A_U^* & B_U^* \\ \hline D_o^{-1} B_o \ D_o^{-1} (C_U^* + B_o L^{(-1)} A_U^*) & D_o^{-1} (D_U^* + B_o L^{(-1)} B_U^*) \end{array} \right] \quad (3.6)$$

is a u.e.s, additive realization for $G^{-1} \in \mathcal{X}(\mathcal{N}, \mathcal{M})$ possibly having partially unobservable causal AC-pair.

PROOF: The proof follows by straightforward evaluation of G^{-1} as

$$\begin{aligned} G^{-1} &= G_o^{-1} U^* \\ &= (D_o^{-1} + D_o^{-1} B_o Z (I - \Delta_o Z)^{-1} (-CD_o^{-1})) (D_U^* + C_U^* Z^* (I - A_U^* Z^*)^{-1} B_U^*) \\ &= D_o^{-1} D_U^* + D_o^{-1} C_U^* Z^* (I - A_U^* Z^*)^{-1} B_U^* \\ &\quad + D_o^{-1} B_o Z (I - \Delta_o Z)^{-1} (-CD_o^{-1}) D_U^* \\ &\quad + D_o^{-1} B_o Z (I - \Delta_o Z)^{-1} (-CD_o^{-1}) C_U^* Z^* (I - A_U^* Z^*)^{-1} B_U^* \\ &= D_o^{-1} (C_U^* + B_o L^{(-1)} A_U^*) Z^* (I - A_U^* Z^*)^{-1} B_U^* \\ &\quad + D_o^{-1} (D_U^* + B_o L^{(-1)} B_U^*) \\ &\quad + D_o^{-1} B_o Z (I - \Delta_o Z)^{-1} (-CD_o^{-1} D_U^* + \Delta_o L^{(-1)} B_U^*). \end{aligned}$$

By using L from the statement and the mixed partial fraction lemma 2.17 we decompose the underlined term of mixed causality and end up with the realization $[G^{-1}]$ from the statement. Because the causal AB-pair $(\Delta_o, D_o^{-1} B_o)$ belongs to a minimal u.e.s realization for G_o^{-1} we conclude that an eventual nonminimality of $[G^{-1}]$ lies in its observability pair. This completes the proof. \blacksquare

At this point, we are capable of stating the following relationship between L and Y .

Lemma 3.4. *Let a boundedly invertible $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s realization $[G] = \{A, B, C, D\}$ and let for some intermediate space \mathcal{M}' , inner $U \in \mathcal{U}(\mathcal{M}, \mathcal{M}')$ and an outer G_o , the decomposition $G = UG_o$ be an inner/outer*

factorization for G . Let $[U]$ and $[G_o]$ be the corresponding u.e.s realizations U and G_o obtained from the QR-decomposition (3.3). Let $L \in \mathcal{D}$ be the unique bounded solution for (3.5) and let $\Omega \in \mathcal{D}$ be the unique boundedly invertible solution for

$$\Delta_o \Omega^{(-1)} \Delta_o^* + (-CD_o^{-1})(-CD_o^{-1})^* = \Omega \quad (\ell_{\Delta_o} < 1). \quad (3.7)$$

Then,

$$\left(\Delta_o (LY^{\dagger*} + \Omega)^{(-1)} \Delta_o^* - (LY^{\dagger*} + \Omega) \right) Y^* = 0$$

for any right inverse Y^{\dagger} of $Y \in \mathcal{D}$ from (3.3).

PROOF: Let us substitute $A_U = Y \Delta_o Y^{\dagger(-1)}$ and $C_U = Y C D_o^{-1}$ obtained from (3.3) into (3.5). This produces

$$\Delta_o (LY^{\dagger*})^{(-1)} \Delta_o^* Y^* + (-CD_o^{-1})(Y C D_o^{-1})^* = L$$

or equivalently

$$\left(\Delta_o (-LY^{\dagger*})^{(-1)} \Delta_o^* + (-CD_o^{-1})(-CD_o^{-1})^* \right) (-Y^*) = L. \quad (3.8)$$

From (3.7) one finds that $(-CD_o^{-1})(-CD_o^{-1})^* = \Omega - \Delta_o \Omega^{(-1)} \Delta_o^*$. When the latter equation is substituted into (3.8) and a little algebra is applied we end up with

$$\left(\Delta_o (LY^{\dagger*} + \Omega)^{(-1)} \Delta_o^* - (LY^{\dagger*} + \Omega) \right) Y^* = 0.$$

This completes the proof. ■

Using equations (3.5) and (3.7) it is possible to evaluate the singular observability Gramian for the AC-pair from the nonminimal realization (3.6) associated to the causal operator part of G^{-1} . We state this in the following proposition.

Proposition 3.5. *The observability Gramian for the part of the realization (3.6) for G^{-1} referring to the causal operator part of G^{-1} is the bounded solution for*

$$\Delta_o (\Omega - LL^*)^{(-1)} \Delta_o^* + \{-CD_o^{-1} D_U^* + \Delta_o L^{(-1)} B_U^*\} \{.\}^* = \Omega - LL^* \quad (3.9)$$

where $L, \Omega \in \mathcal{D}$ are unique bounded solutions for (3.5) and (3.7) respectively.

PROOF: We start from the equation (3.5) and postmultiply it with its transpose. This produces

$$\left[\Delta_o (-L)^{(-1)} C D_o^{-1} \right] \begin{bmatrix} A_U^* \\ C_U^* \end{bmatrix} \left[A_U \ C_U \right] \begin{bmatrix} (-L)^{*(-1)} \Delta_o^* \\ (C D_o^{-1})^* \end{bmatrix} = (-L)(-L)^*.$$

Since the realization $[U]$ is coisometric, the identity

$$\begin{bmatrix} A_U^* \\ C_U^* \end{bmatrix} [A_U \ C_U] = \begin{bmatrix} I \\ I \end{bmatrix} - \begin{bmatrix} B_U^* \\ D_U^* \end{bmatrix} [B_U \ D_U]$$

holds true. Let us substitute it in the latter equation. This produces

$$\Delta_o(LL^*)^{(-1)}\Delta_o^* + (CD_o^{-1})(CD_o^{-1})^* - \{-CD_o^{-1}D_U^* + \Delta_oL^{(-1)}B_U^*\}\{.\}^* = LL^*. \quad (3.10)$$

Finally, we solve the equation (3.7) in $(CD_o^{-1})(CD_o^{-1})^*$ and substitute it in (3.10). After a little algebra we obtain

$$\Delta_o(\Omega - LL^*)^{(-1)}\Delta_o^* + \{-CD_o^{-1}D_U^* + \Delta_oL^{(-1)}B_U^*\}\{.\}^* = \Omega - LL^*.$$

The unique bounded solution of this Lyapunov-Stein equation defines the singular observability Gramian for the causal AC-pair from G^{-1} as we needed to show. This completes the proof. ■

On the other side, the inner/outer factorization framework is not necessary to prove the following result on boundedly invertible *maximal-phase* operators. Those are the causal operators having bounded anti-causal inverses.

Lemma 3.6. (*External factorization of maximal-phase operators*) *Let a boundedly invertible and causal $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ whose inverse $G^{-1} \in \mathcal{L}(\mathcal{M}, \mathcal{N})$ is anti-causal be given by a minimal, u.e.s realization $[G] = \{A, B, C, D\}$. Let $[U] = \{A_U, B_U, C_U, D_U\}$ and $[G_o] = \{A, B_o, C, D_o\}$ be minimal u.e.s realizations for respectively the inner U and the outer G_o satisfying*

$$G = UG_o.$$

Then, for the bounded solution $L \in \mathcal{D}$ for

$$\Delta_oL^{(-1)}A_U^* - CD_o^{-1}C_U^* = L$$

a minimal, u.e.s realization for $G^{-1} \in \mathcal{L}(\mathcal{N}, \mathcal{M})$ can be obtained in the form

$$[G^{-1}] = \left[\begin{array}{c|c} A_U^* & B_U^* \\ \hline D_o^{-1}(C_U^* + B_oL^{(-1)}A_U^*) & D_o^{-1}(D_U^* + B_oL^{(-1)}B_U^*) \end{array} \right]. \quad (3.11)$$

PROOF: In Proposition 3.3 we have evaluated a u.e.s, additive realization for G^{-1} (see (3.6)). By assumption G^{-1} is anti-causal. Hence, the realization (3.11) for G^{-1} is obtained after omitting the causal part from (3.6). We show that $[G^{-1}]$ is a minimal realization by applying the contradiction argument. Assume that the realization $[G^{-1}]$ is not minimal. Then, $s\text{-dim}([G^{-1}]) < s\text{-dim}(A_U)$ and there exists a properly partitioned unitary state-space transformation $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \in \mathcal{D}^{2 \times 1}$ such that $A \in \mathcal{D}$, being the bounded singular solution for

$$A_U A^{(-1)}A_U^* + (C_U^* + B_oL^{(-1)}A_U^*)^* D_o^{-*} D_o^{-1} (C_U^* + B_oL^{(-1)}A_U^*) = A$$

admits an eigenvalue decomposition in the form

$$A = [R_1^* \ R_2^*] \begin{bmatrix} X & \\ & 0 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

for some invertible Hermitian $X \in \mathcal{D}$ of maximal rank. By applying the transformation R on the realization $[G^{-1}]$, we obtain

$$\left[\begin{array}{c|c} a_{11}^* & b_1^* \\ a_{12}^* \ a_{22}^* & b_2^* \\ \hline c_1^* \ 0 & D_o^{-1}(D_U^* + B_o L^{(-1)} B_U^*) \end{array} \right]$$

for some properly partitioned block diagonals a , b and c . After extraction of the maximal unreachable subspace, the previous realization becomes minimal. Define

$$[G^{-1}]_{alt.} := \left[\begin{array}{c|c} a_{11}^* & b_1^* \\ \hline c_1^* & D_o^{-1}(D_U^* + B_o L^{(-1)} B_U^*) \end{array} \right].$$

and note that $[G^{-1}]_{alt.}$ is a u.e.s realization. The proof completes after showing that G^{-1} given by $[G^{-1}]_{alt.}$ is such that $G_o G^{-1} \neq U^*$. Indeed, since $G_o G^{-1}$ is anti-causal, the s-dim sequence of its anti-causal part is fixed by the s-dim sequence of G^{-1} . The latter s-dim sequence equals $s\text{-dim}(a_{11}^*)$ and is smaller than $s\text{-dim}(U^*) (= s\text{-dim}(A_U))$. This produces the contradiction to the assumption that $[G^{-1}]$ is a nonminimal realization and completes the proof. ■

Proposition 3.7. *Let a boundedly invertible $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s realization $[G] = \{A, B, C, D\}$. Let an inner U and the outer G_o be such that $G = UG_o$ is the inner/outer representation for G . Assume that $s\text{-dim}(G) = s\text{-dim}(G_o)$. Let minimal u.e.s realizations; $[G_o] = \{A, B_o, C, D_o\}$ and unitary $[U] = \{A_U, B_U, C_U, D_U\}$ for G_o and U respectively be obtained from the QR-decomposition (3.3).*

Then the unique bounded solution $L \in \mathcal{D}$ for (3.5), is locally left invertible. In addition, there is a right inverse Y_c^\dagger for Y such that

$$L = -Y_c^\dagger.$$

PROOF: Since the outer factor G_o is such that $s\text{-dim}(G) = s\text{-dim}(G_o)$, there exists a properly partitioned boundedly invertible state-space transformation $R \in \mathcal{D}^{1 \times 2}$ and some block diagonals a_{21} , a_{22} and c such that

$$\left(R^{-1} \Delta_o R^{(-1)}, R^{-1}(C D_o^{-1}) \right) = \left(\begin{bmatrix} A_U & \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} C_U \\ c \end{bmatrix} \right).$$

Let us apply R on the equation (3.5) and employ the previous identity. This yields

$$\begin{bmatrix} A_U \\ a_{21} & a_{22} \end{bmatrix} (-R^{-1}L)^{(-1)} A_U^* + \begin{bmatrix} C_U \\ c \end{bmatrix} C_U^* = (-R^{-1}L). \tag{3.12}$$

The previous equation introduces the block row partitioning in $R^{-1}L$ as

$$-R^{-1}L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \tag{3.13}$$

for some $l_i \in \mathcal{D}, i = 1, 2$. It can easily be seen that l_1 admits the top block row of (3.12);

$$A_U l_1^{(-1)} A_U^* + C_U C_U^* = l_1.$$

Because $[A_U \ C_U]$ is isometric, we conclude that $l_1 = I$. Hence, $\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ is a locally left invertible block diagonal. Due to the equation (3.13) in which R is locally square invertible, we conclude that L is left invertible.

To prove the additional statement, we start from the square-root equation (3.3) and pass all matrices from the right to the left. After a little algebra we find that, for an arbitrary right inverse Y^\ddagger for Y , it holds that

$$Y(\Delta_o Y^{\ddagger(-1)} A_U^* + C D_o^{-1} C_U^*) = I.$$

Hence, $\Delta_o Y^{\ddagger(-1)} A_U^* + C D_o^{-1} C_U^*$ is a right inverse for Y too but is not necessarily equal to Y^\ddagger . But the previous Lyapunov-Stein equation has a unique bounded fixed point. This means that there is another right inverse Y_c^\ddagger for Y such that

$$\Delta_o Y_c^{\ddagger(-1)} A_U^* + C D_o^{-1} C_U^* = Y_c^\ddagger.$$

After subtracting the previous equation from the equation (3.5) we conclude that $L = -Y_c^\ddagger$. This completes the proof. ■

The reason why we have considered Y and L is twofold. On one side, algebraic relationships between these quantities enables us to see how U and G_o introduce cancellations. On the other however, because we prefer to work with minimal realizations, these quantities make possible to find minimal realization for the factors. To see how this works let us evaluate an alternative realization for the inner U by working on the product $U = G G_o^{-1}$. Because both G and G_o^{-1} are causal, we may consider $G G_o^{-1}$ by working on the product of their corresponding realizations evaluated previously. This produces the following identity

$$\left[\begin{array}{c|c} A & C \\ \hline I & D \end{array} \right] \left[\begin{array}{c|c} I & \\ \hline \Delta_o & -C D_o^{-1} \\ \hline D_o^{-1} B_o & D_o^{-1} \end{array} \right] = \left[\begin{array}{c|c} A & C D_o^{-1} B_o \\ \hline \Delta_o & -C D_o^{-1} \\ \hline B & D D_o^{-1} B_o \\ \hline & D D_o^{-1} \end{array} \right].$$

The realization on the right side of the previous identity is not minimal and, in turn, its state-space complexity can further be reduced by applying the state-space transformation

$$r := \begin{bmatrix} I & -I \\ & I \end{bmatrix}$$

on it. So doing, we introduce the following, still not necessarily minimal, realization

$$\left[\begin{array}{c|c} r^{-1} & \\ \hline & I \end{array} \right] \left[\begin{array}{c|c} A & CD_o^{-1}B_o \\ \hline \Delta_o & -CD_o^{-1} \end{array} \right] \left[\begin{array}{c|c} r^{(-1)} & \\ \hline & I \end{array} \right] = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \Delta_o \end{array} \middle| \begin{array}{c} 0 \\ -CD_o^{-1} \end{array} \right] \left[\begin{array}{c|c} B & -B + D_o^{-1}B_o \\ \hline & DD_o^{-1} \end{array} \right]$$

giving rise to

$$[U]_{n,n,r} := \left[\begin{array}{c|c} \Delta_o & -CD_o^{-1} \\ \hline -B + D_o^{-1}B_o & DD_o^{-1} \end{array} \right] \quad (3.14)$$

which, again, is not necessarily minimal. Since $[U]_{n,n,m}$ shares the same uniformly observable AC-pair with the realization $[G_o^{-1}]$, we conclude that the AB-pair from $[U]_{n,n,r}$ is not necessarily reachable. To extract the eventual unreachable part, we introduce a boundedly invertible state-space transformation $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \mathcal{D}^{2 \times 1}$ which brings the realization $[U]_{n,n,r}$ into a Kalman canonical controller form

$$[U]_{n,n,r,T} := \left[\begin{array}{c|c} a_U & c_U \\ \hline \Delta_{21} & \Delta_{22} \quad \Gamma \\ b_U & D_U \end{array} \right] \quad (3.15)$$

for some, block diagonals b_U, c_U and $[\Delta_{21} \quad \Delta_{22} \quad \Gamma]$ of possibly vanishing rank. After this triple has been omitted, we obtain an alternative, minimal, u.e.s and unitary realization

$$[U]_{alt.} := \left[\begin{array}{c|c} a_U & c_U \\ \hline b_U & D_U \end{array} \right] \quad (3.16)$$

for U .

Our goal now is to minimize the realization $[U]_{n,n,r}$ by applying T onto it.

We consider now the connection of T_1 with the (boundedly right invertible) block diagonal operator Y' from the square-root algorithm for G .

Lemma 3.8. *Let a boundedly invertible and causal operator G be given by a minimal u.e.s realization $[G] = \{A, B, C, D\}$ and let $G = UG_o$ for some inner U and outer G_o minimal, u.e.s realizations of which, $[U] = \{A_U, B_U, C_U, D_U\}$ and $[G_o] = \{A, B_o, C, D_o\}$ are obtained from the square-root equation (3.3) Then, for any unitary $W \in \mathcal{D}$ and a right invertible of possibly vanishing rank $T_2 \in \mathcal{D}$ for which $T := \begin{bmatrix} -Y \\ T_2 \end{bmatrix}$ is invertible, T is the state-space transformation which brings the not necessarily reachable, u.e.s realization*

$$[U]_{n,n,r} := \left[\begin{array}{c|c} A - CD_o^{-1}B_o & -CD_o^{-1} \\ \hline -B + DD_o^{-1}B_o & DD_o^{-1} \end{array} \right]$$

for the inner U into the Kalman controller form

$$[U]_{n,n,r,T} := \left[\begin{array}{c|c} a_U & c_U \\ \hline \Delta_{21} & \Delta_{22} \\ \hline b_U & D_U \end{array} \right].$$

for some $\Delta_{21}, \Delta_{22}, \Gamma \in \mathcal{D}$ and $\{a_U, b_U, c_U, D_U\}$ defining a minimal, u.e.s realization for the inner U .

PROOF: We start from the square-root equation and premultiply it with its transpose. Since $[U]$ is unitary, we obtain that

$$A^*(Y^*Y)A + B^*B = (Y^*Y)^{(-1)} + B_o^*B_o \quad (3.17)$$

$$A^*(Y^*Y)C + B^*D = B_o^*D_o \quad (3.18)$$

$$C^*(Y^*Y)C + D^*D = D_o^*D_o \quad (3.19)$$

To complete the proof we need to show that for T from the statement, the Hermitian form

$$A = [T_1^* \ T_2^*] \begin{bmatrix} I \\ \underline{0} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad (3.20)$$

where the underlined zero block may have vanishing rank, turns the Lyapunov-Stein equation

$$\Delta_o^* A \Delta_o + (-B + DD_o^{-1}B_o)^*(-B + DD_o^{-1}B_o) = A^{(-1)}. \quad (3.21)$$

into an identity. Indeed, let us substitute $A = T_1^* T_1 = Y^* Y$ into (3.21). We obtain

$$\Delta_o^*(Y^*Y)\Delta_o + (-B + DD_o^{-1}B_o)^*(-B + DD_o^{-1}B_o) = (Y^*Y)^{(-1)} \quad (3.22)$$

whose terms can be rearranged to produce an equivalent equation in the form

$$\left(A^*(Y^*Y)A + B^*B - (Y^*Y)^{(-1)} \right) - \left(A^*(Y^*Y)C + B^*D \right) D_o^{-1} B_o \quad (3.23)$$

$$- B_o^* D_o^{-*} \left(C^*(Y^*Y)A + D^*B \right) + B_o^* D_o^{-*} \left(C^*(Y^*Y)C + D^*D \right) D_o^{-1} B_o = 0 \quad (3.24)$$

Because of (3.17)-(3.19) the expression in the brackets vanish yielding the truth that (3.21) becomes an identity when the presumed T is substituted into. Finally, when T is applied on $[U]_{n,n,r}$ because of the choice of the eigenvalue decomposition (3.20) we obtain Kalman controller form $[U]_{n,n,r,T}$ with its state transition operator block lower triangularized. Extraction of the maximal unreachable space from the latter realization results in minimal, u.e.s realization $\{a_U, b_U, c_U, D_U\}$ for the inner U . This completes the proof. ■

The following observation is straightforward and reveals that the AC-pair of the realization $[U]_{alt.}$ is in the output normal form if and only if the AC-pair of the realization $[G]$ is in output normal form.

Lemma 3.9. *Assume that the assumptions from the Lemma 3.8 hold for a minimal, u.e.s realization $[G] = \{A, B, C, D\}$ for a causal invertible operator $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$. Let $[G]$ be in output normal form (i.e. $AA^* + CC^* = I$). Then, the boundedly invertible state-space transformation $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$, which brings $[U]_{n.n.r}$ into a non-minimal realization*

$$[U]_{n.n.r,T} = \left[\begin{array}{cc|c} a_U & & c_U \\ \Delta_{21} & \Delta_{22} & \Gamma \\ \hline b_U & & D_U \end{array} \right]$$

whose AC-pair is in the output normal form if and only if the rows of T_2 form an orthonormal basis.

PROOF: It proves easily by straightforward substitution and by noting that the observability Gramian corresponding to the (minimal) AC-pair from $[U]_{att}$ admits a Cholesky factorization in the form

$$\begin{bmatrix} W \\ Y' \lambda \end{bmatrix} \begin{bmatrix} W^* & Y'^* \\ & \lambda^* \end{bmatrix}$$

for some unitary $W \in \mathcal{D}$, bounded $Y' \in \mathcal{D}$ and boundedly invertible $\lambda \in \mathcal{D}$. Also, the observability Gramian of the realization $[G]$ is equal to

$$[T_1^\dagger \ T_2^\dagger] \begin{bmatrix} W \\ Y' \lambda \end{bmatrix} \begin{bmatrix} W^* & Y'^* \\ & \lambda^* \end{bmatrix} \begin{bmatrix} T_1^{\dagger*} \\ T_2^{\dagger*} \end{bmatrix}.$$

Hence, $[G]$ is in output normal form if and only if

$$[T_1^\dagger \ T_2^\dagger] \begin{bmatrix} W \\ Y' \lambda \end{bmatrix} (= [T_1^\dagger W + T_2^\dagger Y' \ T_2^\dagger \lambda])$$

is unitary. This in turn, is equivalent to the following set equations

$$\begin{aligned} (T_1^{\dagger*} W + T_2^{\dagger*} Y') (T_1^{\dagger*} W + Y'^* T_2^{\dagger*}) + T_2^{\dagger*} \lambda \lambda^* T_2^{\dagger*} &= I \\ (T_1^{\dagger*} W + Y'^* T_2^{\dagger*}) (T_1^{\dagger*} W + T_2^{\dagger*} Y') &= I \\ \lambda^* T_2^{\dagger*} (T_1^\dagger + T_2^\dagger Y') &= 0 \\ \lambda^* T_2^{\dagger*} T_2^\dagger \lambda &= I \end{aligned}$$

from which we find that $Y' = -(T_2^{\dagger*} T_2^\dagger)^{-1} T_2^{\dagger*} T_1^\dagger$. We see that the necessary and sufficient condition for the observability Gramian for the realization $[U]_{n.n.r,T}$ to be the identity operator is that $Y' = 0$ and that λ is unitary. This is only possible if and only if the rows of T_2 form an orthonormal basis. This completes the proof. \blacksquare

3.3 The inversion

In this section we state a closed formula for the minimal realization of the inverse G^{-1} . The case of an invertible causal operator containing an intrinsic inner factor will also be considered and a closed formula for its inverse will be stated.

In the previous section we have shown how to find an alternative minimal realization to the realization $[U]$ by completion of the intermediate variable Y by enough rows to form the boundedly invertible state-space transformation T .

Since $[U]_{n.n.m}$ and $[G_o^{-1}]$ share the same AC-pair, let us apply T on $[G_o^{-1}]$. The resulting realization is of course minimal because $[G_o^{-1}]$ is so. We obtain

$$[G_o^{-1}]_{alt.} = \left[\begin{array}{cc|c} a_U & & c_U \\ \Delta_{21} & \Delta_{22} & \Gamma \\ \hline D_o^{-1}B_oT_1^{\dagger(-1)} & D_o^{-1}B_oT_2^{\dagger(-1)} & D_o^{-1} \end{array} \right]. \quad (3.25)$$

The following proposition states necessary and sufficient conditions needed for, as we will see later on, turning the recursion in a certain algorithm.

Proposition 3.10. *Let a causal boundedly invertible G be given by a minimal, u.e.s realization $[G] = \{A, B, C, D\}$ and let minimal u.e.s realizations*

$$[U] = \{A_U, B_U, C_U, D_U\} \quad \text{and} \quad [G_o] = \{A, B_o, C, D_o\}$$

for respectively the inner U and the outer G_o satisfying $G = UG_o$ be obtained from the square-root equation (3.3). Let a locally right invertible $T_2 \in \mathcal{D}$ be any extension of the row span of $Y \in \mathcal{D}$ for which $T := \begin{bmatrix} Y \\ T_2 \end{bmatrix}$ is invertible. Let

$$T^{-1} = [Y^{\dagger} \ T_2^{\dagger}]$$

be the inverse of T with block partitioning which is conform to that of T^* . Let $N, K \in \mathcal{D}$ be unique bounded solutions for respectively

$$(T_2\Delta_oT_2^{\dagger(-1)})N^{(-1)}A_U^* + (-T_2CD_o^{-1})C_U^* = N \quad (3.26)$$

and

$$(T_2\Delta_oT_2^{\dagger(-1)})K^{(-1)}A_U^{*(-1)} - (T_2\Delta_oY^{\dagger(-1)}) = K \quad (3.27)$$

Then,

$$KA_U^* + N = 0$$

if and only if T_2 is chosen such that Y^{\dagger} is the unique bounded solution for

$$\Delta_oY^{\dagger(-1)}A_U^* + CD_o^{-1}C_U^* = Y^{\dagger} \quad (3.28)$$

In addition, T_2 is unique up to a locally square invertible factor on its left.

PROOF: Let us evaluate expression $KA_U^* + N$ first. We have that

$$\begin{aligned} KA_U^* + N &= KA_U^* + T_2 \Delta_o T_2^{\dagger(-1)} N^{(-1)} A_U^* - T_2 CD_o^{-1} C_U^* \\ &= T_2 \Delta_o \left(T_2^{\dagger} KA_U^* - Y^{\dagger} + T_2^{\dagger} N \right)^{(-1)} A_U^* - T_2 CD_o^{-1} C_U^* \\ &= T_2 \Delta_o \left(-Y^{\dagger} + T_2^{\dagger} (KA_U^* + N) \right)^{(-1)} A_U^* - T_2 CD_o^{-1} C_U^* \\ &= T_2 \Delta_o T_2^{\dagger(-1)} (KA_U^* + N)^{(-1)} A_U^* - T_2 (\Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^*). \end{aligned}$$

It is obvious that $KA_U^* + N = 0$ if and only if T_2 can be chosen such that

$$T_2 (\Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^*) = 0. \quad (3.29)$$

(\Rightarrow) Assume T_2 is chosen such that it gives rise to the bounded solution Y^{\dagger} for (3.28). Let us rewrite the latter equation as

$$-Y^{\dagger} + \Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^* = 0$$

and premultiply it with T_2 . When the identity $T_2 Y^{\dagger} = 0$ which, in turn, follows from $TT^{-1} = I$ is applied in the result, we obtain the equation (3.29). Since we have shown that, viz. $\ell_{\Delta_o} < 1$, the latter equation is satisfied if and only if $KA_U^* + N = 0$, we complete this part of the proof.

(\Leftarrow) Assume $KA_U^* + N = 0$ for some T_2 . Then, the equation (3.29) is fulfilled. From the identity $T^{-1}T = I$ we find that

$$T_2^{\dagger} T_2 = I - Y^{\dagger} Y. \quad (3.30)$$

Let us premultiply equation (3.29) with T_2^{\dagger} and employ identity (3.30) in the obtained result. We have that

$$(I - Y^{\dagger} Y) (\Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^*) = 0.$$

Because of $Y (\Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^*) = I$ which, in turn, is easily obtained from (3.3), we conclude that, for such a chosen T_2 , the block diagonal Y^{\dagger} must satisfy

$$\Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^* = Y^{\dagger}.$$

Hence, Y^{\dagger} is the unique bounded solution for (3.28). This completes this part of the proof.

Let us prove the additional statement.

Let T_2' be a representant of the variety of all elements which give rise to the unique

bounded solution Y^{\dagger} for (3.28) and where the block diagonal quantity $T' := \begin{bmatrix} Y \\ T_2' \end{bmatrix}$

is locally square invertible. This variety is not empty since it contains at least one element: T_2 , admitting the conditions from the statement. Then, for some T'^{\dagger} we have that $T^{-1} = [Y^{\dagger} \ T'^{\dagger}]$ is the inverse for T' . Since $T' T'^{-1} = I$, we have that

$$\begin{bmatrix} Y \\ T'_2 \end{bmatrix} [Y^\dagger \ T'^{\dagger}_2] = \begin{bmatrix} I \\ I \end{bmatrix}.$$

T'_2 is a locally right invertible block diagonal. Hence, for some locally square invertible $r \in \mathcal{D}$ and a locally isometric $q \in \mathcal{D}$ it holds that $T'_2 = rq$ is an RQ-decomposition for T'_2 . Since r is invertible, identity $T'T'^{-1} = I$ implies that

$$\begin{bmatrix} Y \\ q \end{bmatrix} [Y^\dagger \ q^\dagger] = \begin{bmatrix} I \\ I \end{bmatrix}.$$

Hence, the isometry q itself gives rise to Y^\dagger too. By pre- and postmultiplication of the latter equation with $\text{diag}(I, R)$ and $\text{diag}(I, R^{-1})$ respectively, where $R \in \mathcal{D}$ is locally square invertible and further arbitrary, we cover the variety entirely. Hence, we conclude that T_2 chosen as the statement announces is unique up to an arbitrary locally square invertible factor on its left side. This makes the proof complete. ■

The Lyapunov-Stein equation (3.28) from Proposition 3.10 is a matrix equation running time-backwardly. As we will see in the sequel the time backward nature of this equation can not be excluded in obtaining a time forward sequence of local realizations for, in general of a mixed causality, the inverse of a boundedly invertible causal operator. However, as stated in the following proposition, it is possible to reduce the complexity of this equation.

Proposition 3.11. *Assume the conditions from Proposition 3.10 hold true. Let T_2 be an extension for Y into a locally square invertible $T = \begin{bmatrix} Y \\ T_2 \end{bmatrix}$ the inverse of which $T^{-1} = [Y^\dagger \ T_2^\dagger]$ is such that Y^\dagger is the unique bounded solution for (3.28). Let for some locally isometric $Q_1 \in \mathcal{D}$ and locally square invertible termwise lower $R_{11} \in \mathcal{D}$ representation*

$$Y = R_{11}Q_1$$

be an RQ-decomposition for Y . Let $Q_2 \in \mathcal{D}$ be any isometry for which $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ is a unitary block diagonal matrix. Let $X \in \mathcal{D}$ be the unique bounded solution for

$$(Q_2 \Delta_o Q_2^{*(-1)}) X^{(-1)} A_U^* + Q_2 \{ \Delta_o (Q_1^* R_{11}^{-1}) A_U^* + C D_o^{-1} C_U^* \} = X. \tag{3.31}$$

Then, a possible candidate for T_2 is

$$T_2 = Q_2 - XY.$$

In addition, for the unique bounded solutions $N \in \mathcal{D}$ and $K \in \mathcal{D}$ of the Lyapunov-Stein equations (3.26) and (3.27), it holds that

$$K A_U^* + N = 0.$$

PROOF: Let, for some properly partitioned locally square termwise lower invertible R and block diagonal unitary Q , the representation

$$T = \begin{bmatrix} Y \\ T_2 \end{bmatrix} = \begin{bmatrix} R_{11} & \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

be an RQ-decomposition for the invertible state-space transformation T introduced in the statement of Proposition 3.10 whereby R_{ii} , $i = 1, 2$ are both locally invertible. The block diagonal quantity T_2 constituting T is assumed chosen such that

$$T^{-1} = [Y^\dagger \ T^\dagger]$$

is the inverse of T for some block diagonal T^\dagger whereas Y^\dagger is the unique bounded solution for (3.28).

From the previous RQ-decomposition we find that

$$T_2 = R_{21}Q_1 + R_{22}Q_2.$$

By postmultiplying the previous equation with Q_1^* and applying the invertibility argument for R_{ii} , $i = 1, 2$ we obtain

$$R_{22}^{-1}T_2Q_1^* = (R_{22}^{-1}R_{21}R_{11}^{-1})R_{11}.$$

After postmultiplication of the latter equation with Q_1 and employing the identity $Q_1^*Q_1 = I - Q_2^*Q_2$, we obtain

$$R_{22}^{-1}T_2 - R_{22}^{-1}T_2Q_2^*Q_2 = (R_{22}^{-1}R_{21}R_{11}^{-1})R_{11}Q_1.$$

Since $T_2Q_2^* = R_{22}$ and $Y = R_{11}Q_1$, we finally obtain

$$R_{22}^{-1}T_2 = Q_2 + (R_{22}^{-1}R_{21}R_{11}^{-1})Y. \quad (3.32)$$

On the other side, by using the above RQ-factorization for T , we evaluate Y^\dagger as

$$Y^\dagger = [Q_1^* \ Q_2^*] \begin{bmatrix} R_{11}^{-1} \\ -R_{22}^{-1}R_{21}R_{11}^{-1} \end{bmatrix}.$$

Let us substitute this representation for Y^\dagger into the equation (3.28) and premultiply the resulting equation with the unitary Q . We obtain that

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Delta_o \{ Q_1^* R_{11}^* + Q_2^* (-R_{22}^{-1} R_{21} R_{11}^*) \}^{(-1)} A_U^* \\ + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} C D_o^{-1} C_U^* = \begin{bmatrix} R_{11}^{-1} \\ -R_{22}^{-1} R_{21} R_{11}^{-1} \end{bmatrix}$$

which is an equation to (3.28). As one can see, the previous equation splits into two equations of a lower complexity

$$\begin{aligned} (Q_1 \Delta_o Q_1^{*(-1)}) R_{11}^{-1} A_U^* + Q_1 C D_o^{-1} C_U^* &= R_{11}^{-1} \\ (Q_2 \Delta_o Q_2^{*(-1)}) (-R_{22}^{-1} R_{21} R_{11}^{-1})^{(-1)} A_U^* \\ &+ Q_2 \{ \Delta_o (Q_1^* \quad R_{11}^{-1}) A_U^* + C D_o^{-1} C_U^* \} = -R_{22}^{-1} R_{21} R_{11}^{-1}. \end{aligned}$$

The first equation from the previous set is a Lyapunov-Stein equation in the Moore-Penrose right pseudo-inverse $Y^\dagger = Q_1^* R_{11}^{-1}$ and is known from the time-forward recursion (3.3). Hence, its solution may be regarded as known. The second one is a Lyapunov-Stein equation in $-R_{22}^{-1} R_{21} R_{11}^{-1}$.

Define

$$X = -R_{22}^{-1} R_{21} R_{11}^{-1}.$$

As Proposition 3.10 states, the block diagonal quantity T_2 is unique up to a locally square invertible block diagonal multiplicative factor. Hence, the block diagonal quantity $R_{22} T_2$ from the equation (3.32) is a possible candidate for the bottom block row for the state-space transformation

$$T = \begin{bmatrix} Y \\ Q_2 - XY \end{bmatrix}$$

which gives rise to the inverse $T^{-1} = [Y^\dagger \quad T_2^\dagger]$ the left column block of which is the unique bounded solution for (3.28) as we needed to prove.

The additional statement is a straightforward consequence of Proposition 3.10. This completes the proof. \blacksquare

Theorem 3.12. *Let a boundedly invertible $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s realization $[G] = \{A, B, C, D\}$ and let minimal, u.e.s realizations: $[U] = \{A_U, B_U, C_U, D_U\}$ and $[G_o] = \{A, B_o, C, D_o\}$ for respectively the inner U and the outer G_o satisfying $G = U G_o$ be obtained from (3.3). Let for some locally square termwise upper triangular R and an isometric block diagonal Q_1 the factorization*

$$Y = R Q_1$$

be an RQ -decomposition for $Y \in \mathcal{D}$ from (3.3). Let $Q_2 \in \mathcal{D}$ be any isometric factor for which $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \in \mathcal{D}^{2 \times 1}$ is unitary.

Let for the unique bounded solution X for the equation (3.31) from Proposition 3.11 the block diagonal

$$T = \begin{bmatrix} Y \\ T_2 \end{bmatrix} = \begin{bmatrix} Y \\ Q_2 - XY \end{bmatrix}$$

be a boundedly invertible state-space transformation and let $T^{-1} = [Y^\dagger \quad T_2^\dagger]$ be its inverse partitioned conformably.

Then, a u.e.s and minimal realization for the mixed causality G^{-1} can be found in the additive form

$$[G^{-1}] = \left[\begin{array}{c|c} T_2 \Delta_o T_2^{\dagger(-1)} & \begin{array}{c} c_c \\ B_U^* \end{array} \\ \hline \begin{array}{c} A_U^* \\ D_o^{-1} B_o T_2^{\dagger(-1)} \end{array} & \begin{array}{c} b_a \\ d \end{array} \end{array} \right]. \quad (3.33)$$

where

$$\begin{aligned} c_c &:= T_2 \left\{ (-CD_o^{-1})D_U^* - \Delta_o Y^{\dagger(-1)} B_U^* \right\}, \\ b_a &:= D_o^{-1} \left\{ C_U^* - B_o Y^{\dagger(-1)} A_U^* \right\}, \\ d &:= D_o^{-1} \left\{ D_U^* - B_o Y^{\dagger(-1)} B_U^* \right\}. \end{aligned}$$

PROOF: We complete the proof by showing that $G_o = G^{-1}U$ where G_o .

$$\begin{aligned} G^{-1}U &= D_o^{-1} \left\{ D_U^* - B_o Y^{\dagger(-1)} B_U^* \right\} \\ &+ D_o^{-1} \left\{ C_U^* - B_o Y^{\dagger(-1)} A_U^* \right\} Z^* (I - A_U^* Z^*)^{-1} B_U^* \\ &+ D_o^{-1} B_o T_2^{\dagger(-1)} Z (I - T_2 \Delta_o T_2^{\dagger(-1)} Z^*)^{-1} T_2 \{ (-CD_o^{-1})D_U^* - \Delta_o Y^{\dagger(-1)} B_U^* \} \\ &\quad \times (D_U + B_U Z (I - A_U Z)^{-1} C_U) \\ &= \dots \\ &= D_o^{-1} - D_o^{-1} B_o Y^{\dagger(-1)} Z (I - A_U Z) C_U \\ &\quad + D_o^{-1} B_o T_2^{\dagger(-1)} Z (I - (T_2 \Delta_o T_2^{\dagger(-1)}) Z)^{-1} \\ &\quad \quad \times T_2 \{ (-CD_o^{-1}) + (\Delta_o Y^{\dagger(-1)} A_U^* + (CD_o^{-1}) C_U) \} \\ &\quad + D_o^{-1} B_o T_2^{\dagger(-1)} Z (I - (T_2 \Delta_o T_2^{\dagger(-1)}) Z)^{-1} \\ &\quad \quad \times T_2 \{ (\Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^*) A_U - \Delta_o Y^{\dagger(-1)} \} Z (I - A_U Z)^{-1} C_U. \end{aligned}$$

By virtue of Proposition 3.11, we have that T from the statement is such that

$$T_2 (\Delta_o Y^{\dagger(-1)} A_U^* + CD_o^{-1} C_U^*) = 0.$$

When the previous equation together with $Y \Delta_o Y^{\dagger(-1)} = A_U$ and $YCD_o^{-1} = C_U$ is included into the evaluation of $G^{-1}U$ we obtain the realization for the causal operator

$$[G_o^{-1}]_{alt.} := \left[\begin{array}{cc|c} Y \Delta_o Y^{\dagger} & & Y(-CD_o^{-1}) \\ T_2 \Delta_o Y^{\dagger(-1)} & T_2 \Delta_o T_2^{\dagger(-1)} & T_2 (-CD_o^{-1}) \\ \hline D_o^{-1} B_o Y^{\dagger(-1)} & (D_o^{-1} B_o) T_2^{\dagger(-1)} & D_o^{-1} \end{array} \right]$$

which obviously can be related to the realization $[G_o^{-1}]$ given by (3.25) by means of a boundedly invertible state-space transformation. Since we know that $[G_o^{-1}]$ is a

minimal, u.e.s realization for the outer factor G_o , we have completed the proof that $G_o = G^{-1}U$.

Finally, note that the s-dim sequence of $[G^{-1}]$ is actually s-dim(T) which, in turn, equals s-dim(G_o^{-1}) (= s-dim(G_o)). Hence, we conclude that s-dim(G^{-1}) = s-dim(G) implying that $[G^{-1}]$ is a minimal realization. This completes the proof. ■

As we can see, the equation (3.33) stays for an additive minimal u.e.s realization $[G^{-1}]$ for the inverse of the causal operator G . Evaluation of local realizations for G^{-1} obviously does not involve solving any time-backward running equation and, as such gives rise to the algorithm stated in Figure 3.1.

1. **input:** sequence of minimal u.e.s realizations $[G]_k = \{A_k, B_k, C_k, D_k\}$, $k = k_{init}, k_{init} + 1, \dots$ starting from an initial time instant k_{init} for the boundedly invertible causal operator G admitting the inner/outer factorization $G = UG_o$ where s-dim(G) = s-dim(G_o).

Initial value for Y_k at the time k_{init} .

Set $\{X_i\}$ of bounded solutions for (3.31)

2. **output:** sequence of minimal, u.e.s additive realizations $[G^{-1}]_k = \{a_{c,k}, b_{c,k}, c_{c,k}, a_{a,k}, b_{a,k}, c_{a,k}, d_k\}$ for the mixed causality G^{-1} .
For $k = k_{init}, k_{init} + 1, \dots$ do
3. Find minimal realizations $[G_o]_k = \{A_k, B_{o,k}, C_k, D_{o,k}\}$ and $[U]_k = \{A_{U,k}, B_{U,k}, C_{U,k}, D_{U,k}\}$ for the outer G_o and the inner U by means of the square-root equation (3.3).
4. Find invertible R_k , $k, k+1$ and isometric Q_k , $k, k+1$ for which $Y_k = R_k Q_{1,k}$, $k, k+1$ is an RQ-decomposition for Y_k , $k, k+1$.
5. Find isometries $Q_{2,k}$, $k, k+1$ for which $\begin{bmatrix} Q_{1,k} \\ Q_{2,k} \end{bmatrix}$, $k, k+1$ are unitary matrices.
6. Set $T_{2,k} = Q_{2,k} - X_k Y_k$, $k, k+1$
7. Define

$$a_{c,k} := T_{2,k} \Delta_{o,k} T_{2,k+1}^*$$

$$b_{c,k} := D_{o,k}^{-1} B_{o,k} T_{2,k+1}^*$$

$$c_{c,k} := T_{2,k} \{(-C_k D_{o,k}^{-1}) D_{U,k}^* - \Delta_{o,k} Y_{k+1}^\dagger B_{U,k}^*\},$$

$$a_{a,k} := A_{U,k}^*$$

$$b_{a,k} := D_{o,k}^{-1} (C_{U,k}^* - B_{o,k} Y_{k+1}^\dagger A_{U,k}^*),$$

$$c_{a,k} := B_{U,k}^*$$

$$d_k := D_{o,k}^* (D_{U,k}^* - B_{o,k} Y_{k+1}^\dagger B_{U,k}^*).$$

8. End.

Fig. 3.1. Algorithm for obtaining a mixed causality inverse for a given causal boundedly invertible operator.

3.4 The general case of a causal invertible operator

In this section we extend the closed formula for the inverse as stated in Theorem 3.12 to include an inner intrinsic factor in the calculations. Let us now consider a general case of a boundedly invertible and causal G' admitting the inner/outer factorization

$$G' = U'G_o$$

whereby $s\text{-dim}(G') \geq s\text{-dim}(G_o)$. As a consequence of this degree constraint, the inner factor U' can be written as $U' = U_i U$ where U_i is the maximal degree (left) intrinsic inner factor for G' and U is the inner factor that introduces cancellations with G_o as the theory from the previous section announces. Hence, the following degree constraint holds

$$s\text{-dim}(G') = s\text{-dim}(U_i) + s\text{-dim}(G_o). \quad (3.34)$$

Also from

$$G'^{-1} = (G_o^{-1}U^*)U_i^*$$

we note that U_i^* will be the right intrinsic factor for G'^{-1} . Hence,

$$s\text{-dim}(G'^{-1}) = s\text{-dim}(U_i^*) + s\text{-dim}(G_o) \quad (s\text{-dim}(G_o) = s\text{-dim}(G_o^{-1})). \quad (3.35)$$

In this section we construct a minimal, u.e.s realization

$$[G'] = \{A', B', C', D'\}$$

for G' and a minimal, u.e.s realization $[G'^{-1}]$ for its inverse G'^{-1} .

To evaluate $[G']$ is easy. Since U_i is intrinsic for G , a realization for G' follows from $U_i G$ straightforwardly as

$$[G'] = \left[\begin{array}{cc|c} A_{U_i} & C_{U_i}B & C_{U_i}D \\ & A & C \\ \hline B_{U_i} & D_{U_i}B & D_{U_i}D \end{array} \right]$$

where $[U_i] = \{A_{U_i}, B_{U_i}, C_{U_i}, D_{U_i}\}$ is a u.e.s unitary realization for U_i and $[G] = \{A, B, C, D\}$ is a minimal, u.e.s realization for G as introduced in the previous sections.

By Theorem 3.12 a minimal additive realization for G^{-1} may be regarded as known. Hence, a minimal additive realization for G'^{-1} follows from $G'^{-1} = G^{-1}U_i^*$ when the mixed causality term is decoupled by virtue of the mixed partial fraction lemma 2.17.

Theorem 3.13. *Let G' be a boundedly invertible causal operator admitting the inner/outer factorization $G' = U_i U G_o$ for the maximal degree intrinsic inner U_i , inner U_i an an outer G_o . Let u.e.s unitary $[U_i] = \{A_{U_i}, B_{U_i}, C_{U_i}, D_{U_i}\}$ and u.e.s minimal $[G] = \{A, B, C, D\}$ be realizations for U_i and $G := U G_o$. Let also unitary, u.e.s*

$[U] = \{A_U, B_U, C_U, D_U\}$ and u.e.s minimal $[G_o] = \{A, B_o, C, D_o\}$ be realizations for the inner U and the outer G_o respectively. Define $F \in \mathcal{D}$ to be the unique bounded solution for

$$(T_2 \Delta_o T_2^{\ddagger(-1)}) F^{(-1)} A_U^* + T_2 \left\{ (-C D_o^{-1}) D_U^* - \Delta_o Y^{\ddagger(-1)} B_U^* \right\} C_U^* = F. \quad (3.36)$$

Then, an additive, u.e.s realization for G'^{-1} can be obtained in the form

$$[G'^{-1}] = \left[\begin{array}{c|cc|c} T_2 \Delta_o T_2^{\ddagger(-1)} & & & c'_c \\ \hline & A_U^* & B_U^* C_{U_i}^* & B_U^* D_{U_i}^* \\ & & A_{U_i}^* & B_{U_i}^* \\ \hline D_o^{-1} B_o T_2^{\ddagger(-1)} & b_a & b'_a & d' \end{array} \right] \quad (3.37)$$

where

$$\begin{aligned} c'_c &:= c_c D_{U_i}^* + (T_2 \Delta_o T_2^{\ddagger(-1)}) F^{(-1)} B_{U_i}^*, \\ b'_a &:= d C_{U_i}^* + (D_o^{-1} B_o T_2^{\ddagger(-1)}) F^{(-1)} A_{U_i}^* \quad \text{and} \\ d' &:= d D_{U_i}^* + (D_o^{-1} B_o T_2^{\ddagger(-1)}) F^{(-1)} B_{U_i}^* \end{aligned}$$

and where $c_c, d \in \mathcal{D}$ are as stated in Theorem 3.12.

PROOF: The proof consists of evaluation of G'^{-1} through $G'^{-1} = G^{-1} U_i^*$ where G^{-1} is given by the minimal additive realization (3.33). Elimination of the mixed causality term from the latter product is done by means of K and the mixed partial fraction lemma 2.17. The remaining terms allow for putting in the realization $[G'^{-1}]$ which is minimal since by (3.35) it has the degree

$$s\text{-dim}(G^{-1}) + s\text{-dim}(U_i^*) = (s\text{-dim}(T_1) + s\text{-dim}(T_2)) + s\text{-dim}(U_i^*)$$

where $s\text{-dim}(T_2) = s\text{-dim}(U)$. This completes the proof. ■

3.5 Inversion via URV-decomposition

From this section on, we derive minimal quasi-separable (i.e. state-space) representations for the upper and lower parts of the inverse of an invertible but otherwise general operator T which itself is given by its upper and lower minimal quasi-separable representations by applying system-theoretic equivalent to classical URV-decomposition. We show that if the original representation is given in an adequate normal form, then the computation of the representation of the inverse can be done in a single downward or upward pass, involving only small, local computations. The results are given

in closed form, provided one accepts the computation of a basis for a space and its orthogonal complement as numerically closed². The central workhorse is the classical square root algorithm utilized here in a generalized form.

The algorithmic procedure here follows largely the method originally presented in [vdV95] and later adopted by most authors, but it adds some essential components to it, namely the characterization of intrinsic inner factors and the state-space realization of the actual inverse, instead of intermediaries. This approach provides closed formulas for the state-space characterization of the inverse that are guaranteed to be minimal in state-space complexity.

We shall also make use of pseudo-inverses and summarize here the main properties we shall be using. Suppose that Y has independent rows (i.e. YY^* is non-singular), then the Moore-Penrose pseudo-inverse of Y is given by

$$Y^M := Y^*(YY^*)^{-1}. \quad (3.38)$$

The Moore-Penrose inverse solves the least squares problem in the Euclidean operator norm

$$x = \min_{\|x\|_2} \left(\operatorname{argmin}_x \|xY - b\|_2 \right)$$

for given Y and b as $x = bY^M$. A more general pseudo-inverse for Y is any conformal matrix Y^\dagger that satisfies $YY^\dagger = I$.

The following property holds:

Lemma 3.14. *Let the rows of Y^\perp form an orthonormal basis for the orthogonal complement of the space generated by the rows of Y , and let Y^M be the Moore-Penrose inverse of Y , then any pseudo-inverse for Y is found as*

$$Y^\dagger = Y^M + Y^\perp X \quad (3.39)$$

in which X is a conformal but otherwise arbitrary matrix.

Next, we look at embeddings. We have the following lemma.

Lemma 3.15. *Suppose that Y_1 and Y_2 are matrices with rows of the same dimension and suppose that the row stack*

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

has independent rows. Then

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^M = [Y_1^M \ Y_2^M] \quad (3.40)$$

if and only if the row spans of Y_1 and Y_2 are orthogonal.

Proofs of the two lemmas are elementary.

²those are QR-type factorizations, common in 'array computations'

3.6 System inversion and the inner/outer square-root algorithm

Our first main result concerns the minimal realization of the inverse of a bounded and block upper matrix (or operator) T ;

$$T = D + BZ(I - AZ)^{-1}C,$$

given by the minimal u.e.s realization

$$[T] = \{A, B, C, D\}.$$

In this and the next section we assume that the inverse T^{-1} of T exists as a bounded, but not necessarily block upper operator. Our goal will be to derive efficient and backward stable quasi-separable representations for this inverse. Our main workhorse will be the square-root algorithm for inner/outer factorization, which we now introduce. We follow the notation of [DvdV98] throughout and refer to it for further information on the relation between semi-separable matrices and time-varying systems.

It is known from the work cited that the operator T can be factored as $T = UT_o$ in which U is a block upper unitary operator and T_o is a block upper operator with block upper inverse, a so-called 'outer operator' or 'minimal-phase'. Both U and T_o possess state-space realizations that are at most of the same local degree (the local dimension of the state) as the degree of T . The factorization is unique except for a trivial unitary block diagonal operator to right of U with conjugate left of T_o . These state-space representations are found by the famous square-root algorithm for inner/outer factorization [DvdV00]. This square root algorithm involves an intermediary block diagonal matrix $Y = \text{diag}(Y_k)$ and produces the state-space realization matrices

$$[U] = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix} \quad \text{and} \quad [T_o] = \begin{bmatrix} A & C \\ B_o & D_o \end{bmatrix}$$

as the solution of the forward recursion of block QL-factorizations:

$$\begin{bmatrix} Y_k A_k & Y_k C_k \\ B_k & D_k \end{bmatrix} = \begin{bmatrix} A_{Uk} & C_{Uk} \\ B_{Uk} & D_{Uk} \end{bmatrix} \begin{bmatrix} Y_{k+1} & 0 \\ B_{ok} & D_{ok} \end{bmatrix} \quad k = \dots - 1, 0, 1, \dots \quad (3.41)$$

In these equations the initial Y_{k_o} at some time k_o is assumed known throughout. Its determination depends on the type of the time-varying system involved. In case the system starts at a given point k_o , the initial T_{k_o} can be taken empty. This will be the case when a singly infinite system is considered. Another interesting case is the IV case, or invariant-variant case, where the initial value is found as a fixed point solution for the square-root equation [DvdV00]. In [DvdV98] it is shown that even the more general case will always yield a bounded fixed point solution provided the original realization is u.e.s. It is not hard to see that there is a related Riccati equation for which Y^*Y is the positive definite solution, but in view of the fact that the solution

can be found directly from the square-root equations, it is not advisable to attempt a solution of the more complex Riccati equation, which moreover is often ill-conditioned (while the square root system is well conditioned [DvdV00]).

Lemma 3.16. *If the original realization $\{A, B, C, D\}$ is in input normal form, then each Y_k in (3.41) has to be contractive.*

PROOF: Premultiplying (3.41) with $\begin{bmatrix} A_U^* & B_U^* \\ C_U^* & D_U^* \end{bmatrix}$ and taking the (1,1) entry produces

$$A_{Uk}^* Y_k A_k + B_{Uk}^* B_k = Y_{k+1}.$$

Solving for the fixed point (see [DvdV98]) gives $Y = \mathbf{P}_0(\mathcal{R}_U \mathcal{R}^*)$, in which \mathcal{R}_U is an orthonormal sliced basis for the reachability space of U , and \mathcal{R} likewise for T , since the realization for T was assumed in input normal form. Hence Y is obtained as the projection of one orthonormal basis on another and has to be contractive. ■

The system will contain an intrinsic inner factor, i.e. we will be able to write $T = T'V$ for some inner V such that $s\text{-dim}(T) = s\text{-dim}(T') + s\text{-dim}(V)$ if and only if some Y_k contain an isometric part. We shall assume that if this is the case, then the intrinsic part V has been extracted, leaving a reduced T' for further processing. The inverse will then also contain an intrinsic anti-causal part V^* which can be handled separately. Hence we will be allowed to assume that the local Y_k is strictly contractive and no intrinsic part is present after extraction, which is handled by the following theorem.

Theorem 3.17. *Let T be a causal locally finite operator with u.e.s. realization*

$$\{A, B, C, D\}$$

in input normal form, and assume that Y is the solution of the square root algorithm given by (3.41). Then: (1) each Y_k can be written as

$$Y_k = \begin{bmatrix} Y_{1,k} \\ Y_{2,k} \end{bmatrix}$$

in which $Y_{1,k}$ is isometric and $Y_{2,k}$ is strictly contractive; (2) the coisometric factor U can be factored as $U = U_1 U_2$ with U_1 inner and such that $T = U_1 \hat{T}$ with $s\text{-dim}(T) = s\text{-dim}(U_1) + s\text{-dim}(\hat{T})$ and the square root equation

$$\begin{bmatrix} Y_{1,k} A_k & Y_{1,k} C_k \\ B_k & D_k \end{bmatrix} = \begin{bmatrix} A_{U_1,k} & C_{U_1,k} \\ B_{U_1,k} & D_{U_1,k} \end{bmatrix} \begin{bmatrix} Y_{1,k+1} & 0 \\ \hat{B}_k & \hat{D}_k \end{bmatrix} \quad (3.42)$$

for some \hat{B}_k and \hat{D}_k belonging to a minimal realization for \hat{T} , and a unitary realization for U_1 .

Comments

The following comments are in order here:

- Y_k and any of its subdivisions $Y_{1,k}$ and $Y_{2,k}$ can be empty.
- The original square-root algorithm given in (3.41) produces a Y_k that has maximal dimension under the condition that $\ker(\cdot Y_k) = 0$. In that case the companion factor will define an outer operator. Non-maximal square-root equations are possible, but the companion factor will then correspond to an operator which is not outer.

PROOF: Let

$$Y_k = P_k \left[\begin{array}{c|c} I_{Y,k} & 0 \\ \hline & \sigma_k \end{array} \right] \begin{bmatrix} Q_{k,1} \\ Q_{k,2} \\ Q_{k,3} \end{bmatrix}$$

be an SVD-decomposition of Y_k . Since Y_k is contractive, it may have unit singular values (represented by the matrix $I_{Y,k}$), and the singular values in the remainder σ_k will be strictly less than one. There will be no singular values zero because of the kernel condition on Y_k . Hence also

$$P_k^* Y_k = \begin{bmatrix} Q_{k,1} \\ \sigma_k Q_{k,2} \end{bmatrix}$$

and hence $Y_{1,k} =: Q_{k,1}$ is isometric and $Y_{2,k} =: \sigma_k Q_{k,2}$ strictly contractive. Next, we perform state transformations on the data used in the square-root algorithm, as follows:

$$\begin{bmatrix} A'_k & C'_k \\ B'_k & D'_k \end{bmatrix} = \begin{bmatrix} Q_k A_k Q_{k+1}^* & Q_k C_k \\ B_k Q_{k+1}^* & D_k \end{bmatrix}$$

and

$$\begin{bmatrix} A'_{U,k} & C'_{U,k} \\ B'_{U,k} & D'_{U,k} \end{bmatrix} = \begin{bmatrix} P_k^* A_{U,k} P_{k+1} & P_k^* C_k \\ B_{U,k} P_{k+1} & D_{U,k} \end{bmatrix}.$$

These are orthonormal state-space transformations that will preserve normal forms. In the new representation, the square-root algorithm transforms³ to

$$\left[\begin{array}{ccc|c} I_{Y,k} & 0 & 0 & \\ 0 & \sigma_k & 0 & \\ \hline & & & I \end{array} \right] \begin{bmatrix} A'_{k,11} & A'_{k,12} & A'_{k,13} & C'_{k,1} \\ A'_{k,21} & A'_{k,22} & A'_{k,23} & C'_{k,2} \\ A'_{k,31} & A'_{k,32} & A'_{k,33} & C'_{k,3} \\ \hline B'_{k,1} & B'_{k,2} & B'_{k,3} & D'_k \end{bmatrix} = [U]'_k \begin{bmatrix} I_{Y,k+1} & 0 & 0 & \\ 0 & \sigma_{k+1} & 0 & \\ \hline B_{o,k,1} & B_{o,k,2} & B_{o,k,3} & D'_{o,k} \end{bmatrix}$$

in which

$$[U]'_k = \begin{bmatrix} A'_{U,k,11} & A'_{U,k,12} & C'_{U,k,1} \\ A'_{U,k,21} & A'_{U,k,22} & C'_{U,k,2} \\ \hline B'_{U,k,1} & B'_{U,k,2} & D'_{U,k} \end{bmatrix}.$$

³Using conformal decompositions of the matrices.

Working the product out, and taking the first columns from the both sides, one obtains

$$\left[\begin{array}{c} A'_{k,11} \\ \sigma_k A'_{k,21} \\ 0 \cdot A'_{k,31} \\ \hline B'_{k,1} \end{array} \right] = [U]'_k \left[\begin{array}{c} I_{Y,k+1} \\ 0 \\ \hline B'_{o,k,1} \end{array} \right].$$

Premultiplying both sides with their adjoints and using the fact that $\left[\frac{A'_{k,1}}{B'_{k,1}} \right]$ is coisometric, because the prime realization for T is in input normal form, we obtain

$$A'^*_{k,21}(I - \sigma_k^* \sigma_k)A'_{k,21} + A'_{k,31}A_{k,31} = -B'^*_{o,k,1}B_{o,k,1}$$

It follows immediately that $A'_{k,21} = A'_{U,k,21} = 0$, $A_{k,31} = 0$ and $B_{o,k,1} = 0$, because σ_k is contractive, the first member of the equation must hence be positive definite and the second negative definite, and hence both have to be zero. It now remains to put the pieces back together to obtain the claims of the theorem. The coisometric realization for U factors as

$$\left[\begin{array}{c|c} A_{U_1,k} & C_{U_1,k} \\ \hline I & D_{U_1,k} \end{array} \right] \left[\begin{array}{c|c} I & \\ \hline A_{U_2,k} & C_{U_2,k} \\ B_{U_2,k} & D_{U_2,k} \end{array} \right]$$

defining two factors $U = U_1 U_2$ (realizations derived from the product), and the primed square-root equation becomes, after dropping the second row:

$$\left[\begin{array}{c|c} I_{Y,k} & \\ \hline 0 & \end{array} \right] \left[\begin{array}{c|c} A'_k & C'_k \\ \hline B'_k & D'_k \end{array} \right] \\ = \left[\begin{array}{c|c} A_{U_1,k} & C_{U_1,k} \\ \hline B_{U_1,k} & D_{U_1,k} \end{array} \right] \left[\begin{array}{c|c|c|c} I_{Y,k+1} & 0 & 0 & 0 \\ \hline 0 & B_{U_2,k}\sigma_{k+1} + D_{U_2,k}B'_{o,2,k} & D_{U_2,k}B'_{o,3,k} & D_{U_2,k}D'_{o,k} \end{array} \right].$$

Converting back to the original non-primed realization and using the definition for $Y_{1,k}$, the partial square-root expression (3.42) is obtained. The fact that U_1 is an intrinsic factor follows from the realization for \hat{T} , which in the primed realization used earlier is given by⁴

$$\left[\begin{array}{c|c|c|c} A'_{11} & A'_{12} & A'_{13} & C'_1 \\ 0 & A'_{22} & A'_{23} & C'_2 \\ 0 & A'_{32} & A'_{33} & C'_3 \\ \hline 0 & B'_{o,2} & B'_{o,3} & D'_o \end{array} \right].$$

The realization is obviously nonminimal. A minimal realization is obtained by dropping the first row and the first column, corresponding to states that are unreachable. The resulting realization has degree $s\text{-dim}(T) - s\text{-dim}(U_1)$ and minimality follows directly from the fact that the degree of a product cannot be larger than the sum of the degrees of the factors. ■

⁴By dropping the dependence on k .

The converse of the theorem is true also:

Theorem 3.18. *Let T be a locally finite causal operator with u.e.s. realization*

$$\{A, B, C, D\}$$

in input normal form and such that $T = U\hat{T}$ with U isometric and $s\text{-dim}(T) = s\text{-dim}(U) + s\text{-dim}(\hat{T})$, and let Y be the bounded solution of the square-root equation (3.41), then $Y_k = \begin{bmatrix} Y_{k,1} \\ Y_{k,2} \end{bmatrix}$ with $Y_{k,1}$ isometric of dimension $s\text{-dim}(U)$ and $Y_{k,2}$ contractive.

The proof is straightforward. It retraces some of the steps of the previous theorem in a backward direction.

Assuming now that intrinsic factors have been dealt with and that each Y_k resulting from the square-root algorithm is contractive, our goal is now to find a closed and minimal mixed representation for T^{-1} . To that aim, we explore the properties of $\Delta_o = A - C_o D_o^{-1} B$ further. We refer to the square-root algorithm given and the diagonal sequence of matrices Y_k defined therein (represented in the block diagonal operator $\text{diag}(Y_k)$). Let Y_k^\perp form a row basis for the orthogonal complement of the space spanned by the rows of Y_k at each point k , and let $\Delta_{ok} = A_k - C_k D_{ok}^{-1} B_{ok}$ as before.

We state the following result.

Lemma 3.19. *Let Y_k^\perp be a matrix whose rows form an orthonormal basis for the orthogonal complement of the space generated by the rows of Y_k , then for all k*

$$Y_k \Delta_{ok} Y_{k+1}^{\perp*} = 0. \tag{3.43}$$

Lemma 3.19 allows for block triangularization of Δ_o . Let

$$R_k = \begin{bmatrix} Y_k^\perp \\ Y_k \end{bmatrix}$$

be a non singular state transformation matrix of size $(m_k^\perp + m_k) \times \delta_k$ and applicable on the original state representation for T . Consider the transformed representation

$$\begin{bmatrix} A'_k & C'_k \\ B'_k & D'_k \end{bmatrix} = \begin{bmatrix} R_k A_k R_{k+1}^{-1} & R_k C_k \\ B_k R_{k+1}^{-1} & D_k \end{bmatrix}, \tag{3.44}$$

then the new representation will yield a realization of the inverse for which the state transition matrix is in block triangular form. To see this, it is instructive to detail the square-root algorithm to the prime representation. We state it in the following lemma.

Lemma 3.20. *The square-root algorithm corresponding to the prime representation given in the equation (3.44) is given by*

$$\begin{bmatrix} [0 \ I_{m_k}] A'_k & [0 \ I_{m_k}] C'_k \\ B'_k & D'_k \end{bmatrix} = \begin{bmatrix} A_{U_k} & C_{U_k} \\ B_{U_k} & D_{U_k} \end{bmatrix} \begin{bmatrix} [0 \ I_{m_{k+1}}] & 0 \\ B_{o_k} R_{k+1}^{-1} & D_{o_k} \end{bmatrix}. \quad (3.45)$$

This leads to our first result.

Proposition 3.21. *The state transition matrix Δ_o for the inverse of T_o satisfies*

$$\Delta_{o_k} = R_k^{-1} \begin{bmatrix} \Delta'_{o,11,k} & \Delta'_{o,12,k} \\ 0 & A_{U_{2,k}} \end{bmatrix} R_{k+1} \quad (3.46)$$

where $\Delta'_{o,11,k} = Y_k^\perp \Delta_{o_k} Y_{k+1}^{\perp*}$ and $\Delta'_{o,12,k} = Y_k^\perp \Delta_{o_k} Y_{k+1}^M$.

Hence we find a conformal primed representation

$$[\Delta'_{o_k} | C'_{o_k} D_o^{-1}] = \begin{bmatrix} \Delta'_{o,11,k} & \Delta'_{o,12,k} \\ 0 & A_{U_k} \end{bmatrix} \begin{bmatrix} C'_{o,1,k} D_o^{-1} \\ C_{U_k} \end{bmatrix}$$

in which the second block row is isometric. The first block row however is not necessarily isometric.

Our next goal is to find a closed form, minimal representation for the mixed operator T^{-1} . This can be obtained in a straightforward way from the primed representations, which we relabel for simplicity of notation as the ongoing representation for T_o :

$$T_o = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & C_1 D_o^{-1} \\ & A_U & C_U \\ \hline -D_o^{-1} B'_{o1} & -D_o^{-1} B'_{o2} & D_o^{-1} \end{array} \right], \quad U = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix}.$$

We notice that the representation for U has not changed in the primed version due to the last lemma, but the other quantities have been transformed.

Theorem 3.22. *Let m be the diagonal operator solution of the Lyapunov-Stein equation*

$$m_{k-1} = \Delta_{11,k} m_k A_{U,k}^* + C_{1,k} D_{o,k}^{-1} D_{U,k}^* + \Delta_{12,k} A_{U,k}^*$$

then a minimal realization for the inverse of T is given by

$$\begin{aligned} T^{-1} = D_o^{-1} \{ & (C_U^* - B_{o1} m A_U^* - B_{o2} A_U^*) (I - Z^* A_U^*)^{-1} Z^* B_U^* \\ & + D_U^* - B_{o1} m B_U^* - B_{o2} B_U^* \\ & - B_{o1} Z (I - \Delta_{11} Z)^{-1} (C_1 D_o^{-1} D_U^* + \Delta_{11} m B_U^* + \Delta_{12}) \}. \end{aligned} \quad (3.47)$$

The proof follows by direct computation. The fact that the representation is minimal follows directly from the fact that the overall degree of T^{-1} equals the original degree of T .

3.7 Minimal inverse of a mixed invertible operator

In this section we derive a minimal quasi-separable (or time-varying) representation of a mixed, boundedly invertible operator. We are given

$$T = B_a Z^* (I - A_a Z^*)^{-1} C_a + D + B_c Z (I - A_c Z)^{-1} C_c$$

as a minimal realization for the operator in additive form. Just as for the unilateral case we will write realizations for mixed forms as

$$[T] = \left[\begin{array}{c|c} A_a & C_a \\ \hline B_a & D \end{array} \right]$$

corresponding to the mixed state-space equations

$$\left[\begin{array}{c} x_a \\ x_c \end{array} \right]^{(-1)} | y] = \left[\begin{array}{c} x_a^{(-1)} \\ x_c \end{array} \right] | u] \left[\begin{array}{c|c} A_a & C_a \\ \hline B_a & D \end{array} \right],$$

with x_a the upward propagating state and x_c the downward. For details see [DvdV98]. We are asked to produce a compact or even closed form for the realization of the inverse operator T^{-1} . In this section we shall provide for such a form, by exploitation of the properties of minimal external and inner/outer factorizations. The strategy will be to follow the general inversion plan of [DvdV98] in the form proposed in [ChandDvGuPalsVdV04], and to reduce the system complexity of the realizations so that they become minimal.

It consists of the following steps:

- (1) produce the minimal external factorization

$$UT = T' \in \mathcal{U},$$

- (2) produce the (automatically minimal) outer/inner factorization

$$T' = T'_o V.$$

At this point realizations have been obtained for the factors in

$$T = U^* T'_o V.$$

A similar factorization for the inverse is found by direct inversion

$$T^{-1} = V^* T'^{-1} U.$$

and the final step will consist in

(3) reduce the factorization of the inverse to a minimal form.

The factorization of the inverse has exactly the same form as the factorization of the original, with the roles of U and V reversed. The final reduction will parallel the initial construction steps. In the present case we assume existence of all the inverses, the inverse of T_o will also be outer.

Important in the subsequent derivation is the observation that $T_o'^{-1}U$ is a minimal outer/inner factorization in \mathcal{U} and $V^*(T_o'^{-1}U)$ a minimal external factorization. We derive minimal realizations for the subsequent factors, beginning with T' . Assuming the anti-causal part in the original realization in *input normal Form* (INF) - i.e. $A_a^*A_a + B_a^*B_a = I$, which we can always assume, we find:

$$[T'] = \left[\begin{array}{cc|c} A_a^* & B_a^*B_c & A_a^*C_a + B_a^*D \\ \hline & A_c & C_c \\ \hline B_U & D_UB_c & C_UC_a + D_UD \end{array} \right] \quad (3.48)$$

PROOF: Since the anti-causal part is given in INF form, a realization for U is obtained as [DvdV98]

$$U = D_U + C_U Z(I - A_a^*Z)^{-1}B_a^*$$

in which

$$[U] = \left[\begin{array}{cc} A_a^* & B_a^* \\ \hline B_U & D_U \end{array} \right]$$

is a unitary matrix. The realization follows then from working out the product $T' = UT$ (the computation is standard in time-varying system theory.) ■

Next, the outer/inner factorization $T' = T_o'V$ is performed in the same way as in the previous section (we skip details for the time being). As stated before, T_o' inherits the reachability data from T' and we obtain

$$[T_o'] = \left[\begin{array}{cc|c} A_a^* & B_a^*B_c & C_{o1} \\ \hline & A_c & C_{o2} \\ \hline B_U & D_UB_c & D_o \end{array} \right]$$

in which the data marked with the subscript 'o' is new data obtained by the outer/inner realization. We assume that intrinsic factors have been removed as in the previous section, and hence that the realization is minimal.

In the next step, we endeavor to find a minimal realization for the product U^*T_o' . It turns out that this product is easy to evaluate, many terms cancel, and collecting the others produces

$$[U^*T_o'] = \left[\begin{array}{c|c} A_a & B_U^*D_o + A_aC_{o1} \\ \hline A_c & C_{o2} \\ \hline B_a & B_c | D_U^*D_o + B_aC_{o1} \end{array} \right]. \quad (3.49)$$

Now we know that $U^*T'_o$ has the causal inverse $T'_o{}^{-1}U$, this fact allows us to determine a realization for it. The technique to find this realization is straightforward but a little involved, we document the important steps (notice the redefinition of terms for simplicity).

Lemma 3.23. *Suppose*

$$[T] = \left[\begin{array}{cc|c} A_a & & C_a \\ & A_c & C_c \\ \hline B_a & B_c & D \end{array} \right]$$

is a minimal realization of a mixed causality invertible system, the inverse of which is causal and contains no intrinsic inner factors. Let the anti-causal part be in input normal form, and let B_U and D_U form the unitary completion

$$\left[\begin{array}{cc} A_a^* & B_a^* \\ \hline B_U & D_U \end{array} \right].$$

Define $\delta := B_U C_a + D_U D$.

Then δ is invertible and

$$[T^{-1}] = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & B_a^* - (A_a^* C_a + B_a^* D)\delta^{-1} D_U \\ \Delta_{21} & \Delta_{22} & -C_c \delta^{-1} D_U \\ \hline \delta^{-1} B_U & \delta^{-1} D_U B_c & \delta^{-1} D_U \end{array} \right].$$

where

$$\Delta := \left[\begin{array}{cc} A_a^* & B_a^* B_c \\ & A_c \end{array} \right] - \left[\begin{array}{c} A_a^* C_a + B_a^* D \\ C_c \end{array} \right] \delta^{-1} [B_U \ D_U B_c]$$

is a minimal, u.e.s realization for T^{-1} .

PROOF: Let $U_a = D_U + B_U Z(I - A_a^* Z)^{-1} B_a^*$. As before, the external factorization produces

$$[U_a T] = \left[\begin{array}{cc|c} A_a^* & B_a^* B_c & A_a^* C_a + B_a^* D \\ & A_c & C_c \\ \hline B_U & D_U B_c & B_U C_a + D_U D \end{array} \right].$$

The causality assumption for T^{-1} has as a consequence that $U_a T$ is actually outer. Hence, $B_U C_a + D_U D (= \delta)$ is invertible. Formally, $T^{-1}U_a^*$ then has the realization

$$[T^{-1}U_a^*] = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & -(A_a^* C_a + B_a^* D)\delta^{-1} \\ \Delta_{21} & \Delta_{22} & -C_c \delta^{-1} \\ \hline \delta^{-1} B_U & \delta^{-1} D_U B_c & \delta^{-1} \end{array} \right]$$

with Δ and δ as defined in the statement of the theorem. By the assumption that the causal T^{-1} contains no inner intrinsic factors, the realization $[T^{-1}U_a^*]$ will be minimal. The state transition operator Δ will actually be u.e.s. This fact follows from [DvdV98], (see there Proposition 13.2). The realization for T^{-1} is now easily deduced by reducing the direct realization for the product $[T^{-1}U_a^*]U_a$

$$\left[\begin{array}{ccc|c} \Delta_{11} & \Delta_{12} & -(A_a^* C_a + B_a^* D) \delta^{-1} B_U & -(A_a^* C_a + B_a^* D) \delta^{-1} D_U \\ \Delta_{21} & \Delta_{22} & -C_c \delta^{-1} B_U & -C_c \delta^{-1} D_U \\ 0 & 0 & A_a^* & B_a^* \\ \hline \delta^{-1} B_U & \delta^{-1} D_U B_c & \delta^{-1} B_U & \delta^{-1} D_U \end{array} \right].$$

Applying the state-space transformation $R := \begin{bmatrix} I & -I \\ & I \\ & & I \end{bmatrix}$ on the A,B and C terms from the previous realization, respectively $R^{-1}(\cdot)R^{(-1)}$, $(\cdot)R^{(-1)}$, $R^{-1}(\cdot)$ produces the alternative nonminimal realization

$$\left[\begin{array}{ccc|c} \Delta_{11} & \Delta_{12} & 0 & B_a^* - (A_a^* C_a + B_a^* D) \delta^{-1} D_U \\ \Delta_{21} & \Delta_{22} & 0 & -C_c \delta^{-1} D_U \\ 0 & 0 & A_a^* & B_a^* \\ \hline \delta^{-1} B_U & \delta^{-1} D_U B_c & 0 & \delta^{-1} D_U \end{array} \right].$$

Leaving out the third column- and the third row block yields a minimal realization $[T^{-1}]$ and completes the proof. \blacksquare

The remainder of the determination hinges on the following lemma.

Lemma 3.24. *Suppose $T = V^* T''$ is a minimal external factorization of a mixed causality T and $\begin{bmatrix} A_V & C_V \\ B_V & D_V \end{bmatrix}$ is a unitary realization for V while $\begin{bmatrix} A'' & C'' \\ B'' & D'' \end{bmatrix}$ is one for T'' , then there exists a state-space transformation $R \cdots R^{(-1)}$ on $[T'']$ such that*

$$R A'' R^{(-1)} = \begin{bmatrix} A_V & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B'' R^{(-1)} = [B_V \quad B_2]$$

for adequate minimal A_{12}, A_{22}, B_2 .

PROOF: The proof makes use of the property of reachability spaces as detailed in [DvdV98], whose notation we use here without further explanations. The assumptions immediately produce the containment

$$\mathcal{D}_2[B_V Z(I - A_V Z)^{-1}]^* \subset \mathcal{D}_2[B'' Z(I - A'' Z)^{-1}]^*$$

Hence we may choose a sliced reachability basis for T'' which first consists of a reachability basis for V and then complete it with an orthogonal complement to produce a sliced basis for the reachability space of T'' . Let \mathbf{F} be the basis so obtained and \mathbf{F}'' the original reachability basis of T'' , then, because of minimality, there will exist a state-space transformation R such that

$$\mathbf{F}'' = R^{(-1)*} \mathbf{F}.$$

The containment subdivides \mathbf{F} as

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_V \\ \mathbf{F}_2 \end{bmatrix}.$$

In this basis the A- and B-matrices are defined by

$$\mathbf{P}'\mathbf{Z}\mathbf{F} = \begin{bmatrix} A_V^* & 0 \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} \mathbf{F}_V \\ \mathbf{F}_2 \end{bmatrix}$$

and

$$\mathbf{P}_0\mathbf{Z}\mathbf{F} = \begin{bmatrix} B_V^* \\ B_2^* \end{bmatrix}.$$

It follows that there exists a state-space transformation as in the statement of the lemma. ■

The lemma admits a converse as well, in case the containment is satisfied a partial factorization will follow. The remainder of the determination consists in computing R and making the final result explicit.

Going back to our original case and utilizing the inversion result of Lemma 3.23 on the data of the equation (3.49), we obtain

$$[T'_oU] = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & B_a^* - C_{o1}D_o^{-1}D_U \\ \Delta_{21} & \Delta_{22} & -C_{o2}D_o^{-1}D_U \\ \hline D_o^{-1}B_U & D_o^{-1}D_UB_c & D_o^{-1}D_U \end{array} \right] \quad (3.50)$$

in which Δ is now given by

$$\Delta = \begin{bmatrix} A_a^* & B_a^*B_c \\ & A_c \end{bmatrix} - \begin{bmatrix} C_{o1} \\ C_{o2} \end{bmatrix} D_o^{-1} [B_U \ D_UB_c]$$

whereby the term that was previously denoted by δ now reduces to D_o . Returning to the original outer/inner factorization of T' (we had postponed the discussion), it defines the inner operator V , the C_{o1}, C_{o2}, D_o matrices and a connecting recursive diagonal Y through the square-root recursion, now filled with the quantities that are relevant here. Let

$$[T'] = \begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix}$$

with

$$A' = \begin{bmatrix} A_a^* & B_a^*B_c \\ & A_c \end{bmatrix}, B' = [B_U \ D_UB_c], C' = \begin{bmatrix} A_a^*C_a + B_a^*D \\ & C_c \end{bmatrix},$$

$D' = C_UC_a + D_UD$ and $C_o = \begin{bmatrix} C_{o1} \\ C_{o2} \end{bmatrix}$, then the square-root algorithm for outer/inner factorization produces a diagonal Y_1 and the realization of V by the recursion

$$\begin{bmatrix} A'Y_1 & C' \\ B'Y_1 & D' \end{bmatrix} = \begin{bmatrix} Y_1^{(1)} & C_o \\ & D_o \end{bmatrix} \begin{bmatrix} A_V & C_V \\ B_V & D_V \end{bmatrix},$$

with Y_1 left invertible and D_o invertible. Since $\Delta = A' - C_o D_o^{-1} B'$ and taking $B'' = \begin{bmatrix} B_a^* - C_{o1} D_o^{-1} D_U \\ -C_{o2} D_o^{-1} D_U \end{bmatrix}$, we also have that

$$[T_o'^{-1}U] = \begin{bmatrix} \Delta & B'' \\ D_o^{-1} B' & D_o^{-1} D_U \end{bmatrix}.$$

The transformation needed to put this realization in block triangular form so as to satisfy Lemma 3.24 now follows from Lemma 3.19. Let

$$Y = [Y_1 \ Y_2]$$

be an embedding of Y_1 into a fully invertible operator with $Y_2 = Y_1^{-1}$, and choose for the R in lemma 3.24

$$R = (Y^{(1)})^{-1}.$$

Because of the orthogonality property of Y_1 we can write

$$Y^{-1} = \begin{bmatrix} Y_1^M \\ Y_2^M \end{bmatrix}$$

where the subscript ' M ' indicates the Moore-Penrose inverse. Applying the state-space transformation $Y^{-1} \dots Y$ produces the realization

$$[T_o^{-1}U]_Y = \left[\begin{array}{cc|c} A_V & A_{12} & C_1 \\ 0 & A_{22} & C_2 \\ \hline B_V & B_2 & D_o^{-1} D_U \end{array} \right]$$

in which

$$\begin{aligned} B_2 &= [D_o^{-1} B_U \ D_o^{-1} B_U B_c] Y_2 \\ A_{12} &= Y_1^{(1)M} \Delta Y_2 \\ A_{22} &= Y_2^{(1)M} \Delta Y_2 \\ \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= Y^{(1)M} \begin{bmatrix} B_a^* - C_{o1} D_o^{-1} D_U \\ -C_{o2} D_o^{-1} D_U \end{bmatrix}, \end{aligned}$$

and, in addition, are all known quantities. The promised minimal realization for T^{-1} is now found by simply working out the product $T^{-1} = V^*(T_o^{-1}U)$. Since minimal realizations are known for the factors and since they are in a proper form to assure the necessary cancellations, we obtain by straight computation

$$\begin{aligned} T^{-1} &= C_V^* Z^* (I - A_V^* Z^*)^{-1} (A_V^* C_1 + B_V^* D_o^{-1} D_U) \\ &\quad + C_V^* Z^* (I - A_V^* Z^*)^{-1} (A_V^* A_{12} + B_V^* B_2) Z (I - A_{22} Z)^{-1} C_2 \\ &\quad + C_V^* C_1 + D_V^* D_o^{-1} D_U + (C_V^* A_{12} + D_V^* B_2) Z (I - A_{22} Z)^{-1} C_2. \end{aligned}$$

This form already corresponds to a minimal, mixed realization given by

$$[T^{-1}] = \left[\begin{array}{cc|c} A_{22} & & C_2 \\ \hline A_V^* A_{12} + B_V^* D_o^{-1} D_U & A_V^* & A_V^* C_1 + B_V^* D_o^{-1} D_U \\ C_V^* A_{12} + D_V^* B_2 & C_V^* & C_V^* C_1 + D_V^* D_o^{-1} D_U \end{array} \right]. \quad (3.51)$$

A realization without mixed term is obtained by splitting it. This can only be done by solving an extra Lyapunov-Stein equation, which unfortunately runs in the opposite direction of the recursion for Y :

$$m = (A_V^* A_{12} + B_V^* B_2) + [A_V^* m A_{22}]^{(1)}.$$

Our final expression for an additive realization for T^{-1} becomes

$$[T^{-1}]_{\text{add.}} = \left[\begin{array}{c|c} A_{22} & C_2 \\ & A_V^* \begin{array}{l} A_V^* (C_1 + m C_2) \\ + B_V^* D_o^{-1} D_U \end{array} \\ \hline \begin{array}{l} C_V^* (A_{12} + m A_{22}) \\ + D_V^* B_2 \end{array} & C_V^* \begin{array}{l} C_V^* (C_1 + m C_2) \\ + D_V^* D_o^{-1} D_U \end{array} \end{array} \right]. \quad (3.52)$$

3.8 Conclusions

In the first part of the chapter, we have considered the inversion problem for a boundedly invertible locally finite operator. The problem was tackled by an extensive use of external and inner/outer characterization of both the operator and its inverse. Some important relationships between them are stated and used in an algebraic characterization of the cancellation property. Square-root equations for obtaining: (1) the inner and outer terms for the operator and (2) the external and the inner term in characterization of the inverse were the key in deriving a minimal realization for the inverse and an algorithm for obtaining the minimal realization is formulated. It appears that it is possible to find the closed formula for the inverse in the form of a minimal, u.e.s additive realization. The formula however implies solving two Lyapunov-Stein equations running in opposite time directions and, in general, has no one-pass character what, from the practical point of view, is the only drawback.

The second part of the chapter (from Section 3.5 on) approaches the inversion problem by looking for a URV-decomposition of an invertible mixed causality operator. As it is known [ChandDvGuPalsVdV04] this decomposition is possible to accomplish algorithmically in either the time-forward or the time-backward fashion and hence exhibits an one-pass character. Moreover, it appears that this representation provides us with the possibility to find the inverse of the operator in the one-pass fashion too independently on eventual intrinsic inner factors that preferably are extracted before the inversion take place. In the discourse of this approach we have shown, by the equation (3.51), that a minimal realization for the inverse of a mixed causality operator (assuming the existence of the inverse) can be obtained simply by computing a left external factor and a right inner factor. These operations can be done in a one-pass backward algorithm, such as those from [ChandDvGuPalsVdV04], where direct elimination algorithms are given without realizations. Given the result of these two recursions, a minimal realization is hence obtained in a one-pass backward

recursion combining the two. In the case a split representation is desired, an extra Lyapunov-Stein recursion has to be executed that runs in the opposite direction. It can be shown by sensitivity arguments that this further step is unavoidable, because a complete decomposition necessarily involves the computation of the square root of a Gramian for which the Lyapunov-Stein equation runs in the opposite direction - but this step is unnecessary if only minimal realizations are desired. Of course, forward recursions are just as well possible, but then the order of factors has to be reversed, external to right and inner to the left. These results generalize to the Moore-Penrose case, when the operator T is not invertible, at little extra cost.

4

MAXIMAL-PHASE OPERATORS

4.1 Introduction

A common approach in tackling operator inversion problems uses both the algorithms for external and the inner/outer factorization (see [DvdV98]). If the operator is causal but not outer, its inverse is of mixed causality as we have already seen in Chapter 3. The question (1) which part of the state transition matrix does give rise to the causal part of the inverse and (2) which to the anti-causal one, hinges on the dichotomy of the state transition matrix involved and, in general, can not be straightforwardly answered unless the inner and the outer factors are found first.

In this chapter, the operator inversion problems are tackled from a different point of view. We still do the external factorization first and then we find the inner/outer decomposition for the (invertible) external factor. But before we start to assemble the inverse, we investigate which completion of the inner factor in the inner/outer factorization for the external factor is needed to produce a, so called, *maximal-phase operator*. Unlike outer operators having causal inverses, maximal-phase operators are causal ones having anti-causal inverses. In particular, inner operators and invertible block diagonals form a special class of maximal-phase operators. This defining property of a maximal-phase operator is extensively exploited in this chapter. One benefit is that it is possible to obtain an alternative minimal realization for the inverse that provides a nicer state-space description to that obtained in Chapter 3 where the inversion problem was approached by means of an outer factor. The other one is that the

work with products between certain operators of the same causality can be avoided if the external factor is adjusted as explained.

4.2 Construction of a minimal degree maximal-phase operator

In this section we show how to construct a minimal degree maximal-phase operator T_m corresponding to a given bounded causal invertible operator T . Since we focus on invertible operators, we assume ab initio that T_m is boundedly invertible. This invertibility aim however can be relaxed and replaced by the partial invertibility aim as we do in the following definition.

Definition 4.1. *Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a causal left (resp. right) invertible operator. T is said to be a maximal-phase operator if there is a bounded left inverse T_l^\dagger (resp. bounded right inverse T_r^\dagger) which is anti-causal.*

The outcomes of external- and outer/inner factorization algorithms provide us with realizations for outer and inner terms. According to Definition 4.1, the inner terms are maximal-phase operators. But the class of maximal-phase operators goes beyond of this. Perhaps the easiest way to find a maximal-phase operator is to consider an outer factor as we do in the following example.

Example 4.2. For complex numbers a_1, a_2 and b from the interior of the unit disc the function

$$t_o(z) := \frac{1 - bz}{1 - a_1z}$$

is an outer function. The function

$$u(z) := \frac{(z - \bar{a}_1)(z - \bar{a}_2)}{(1 - a_1z)(1 - a_2z)}$$

is then an inner function for which

$$t(z) := u(z)t_o^{-1}(z) = \frac{(z - \bar{a}_1)(z - \bar{a}_2)}{(1 - bz)(1 - a_2z)}$$

is a causal function having an anti-causal inverse. By Definition 4.1, $t(z)$ is of maximal phase. But, in order to construct it, it however suffices to take the inner factor of a smaller degree. That is, if we take $u_1(z) := \frac{z - \bar{a}_1}{1 - a_1z}$ the function

$$t_m(z) := u_1(z)t_o^{-1}(z)$$

is still a maximal-phase function.

In order to obtain a maximal-phase function from an outer function, we introduced an inner function depending entirely on the outer one. Moreover, as we have seen, the inner function is not unique, such that many maximal-phase functions can be associated to an outer factor. A similar situation holds in the time-varying context as well were the study of 'pole-zero' cancellations is replaced with the study of the spaces associated to the Hankel maps of operators in question. The non-uniqueness of a maximal-phase operator associated to a left- or right invertible causal operator is then generated by the arbitrariness of the inner factors involved. The operator to which a maximal-phase operator is associated will be an outer operator.

An important issue that is often encountered in operator theory is that the operators do not commute in general. This enforces us to differ between the left- and the right maximal-phase operators associated to the outer operator and is the reason for introducing the following two definitions.

Definition 4.3. Let $T_o \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a left- or a right outer operator with bounded causal partial inverse. Let for some space \mathcal{M}' , an inner $U \in \mathcal{U}(\mathcal{M}', \mathcal{M})$ be the minimal degree inner factor for which $T_{ml} := UT_o$ is a maximal-phase operator. Then, T_{ml} is called a minimal degree left maximal-phase operator associated to T_o .

Definition 4.4. Let $T_o \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a left- or a right outer operator with bounded causal partial inverse. Let for some space \mathcal{N}' , an inner $V \in \mathcal{U}(\mathcal{N}, \mathcal{N}')$ be the minimal degree inner factors for which $T_{mr} := T_oV$ is a maximal-phase operator. Then, T_{mr} is called a minimal degree right maximal-phase operator associated to T_o .

In the sequel we restrict our attention to the evaluation of the left inverse of a boundedly left invertible causal operator. The dual statements can be made by considering causal boundedly right invertible operators. For simplicity, the latter case will be left out of consideration.

The following lemma reveals the connection between a maximal-phase operator and an appropriate external factorization.

Lemma 4.5. Let $T_{or} \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a locally finite right outer operator having a minimal, u.e.s realization $[T_{or}] = \{A, B, C, D_{or}\}$. Define $\Delta_{or} = A - CD_{or}^\dagger B$ where D_{or}^\dagger is a right inverse of D_{or} and assume $\ell_{\Delta_{or}} < 1$. Define an inner U by identifying it through

$$\mathcal{H}_o(U) \supseteq \mathcal{D}_2(I - \Delta_{or}Z)^{-1}(-CD_{or}^\dagger)$$

Then the operator

$$T := UT_{or}$$

is a left maximal-phase operator.

In particular, if $\mathcal{H}_o(U) = \mathcal{D}_2(I - \Delta_{or}Z)^{-1}(-CD_{or}^\dagger)$ then $T_{ml} := T$ is a minimal degree left maximal-phase operator associated to T_{or} .

PROOF: The operator T_{or} has a causal bounded right inverse T_{or}^\dagger . A minimal realization for T_{or}^\dagger can be directly evaluated from $[T_{or}]$. By straightforward inspection it shows that $[T_{or}^\dagger] := \{\Delta_{or}, D_{or}^\dagger B, -CD_{or}^\dagger, D_{or}^\dagger\}$ is a minimal, u.e.s realization for T_{or}^\dagger . Then,

$$\mathcal{H}_o(T_{or}^\dagger) = \mathcal{D}_2(I - \Delta_{or}Z)^{-1}(-CD_{or}^\dagger)$$

is the output state-space for $\mathcal{H}_o(T_{or}^\dagger)$. Since U is defined such that

$$\mathcal{H}_o(U) \supseteq \mathcal{H}_o(T_{or}^\dagger)$$

by virtue of the external factorization lemma it holds that

$$T_{or}^\dagger U^* \in \mathcal{L}$$

as we needed to show. Since the s-dim sequence for U is defined to be the s-dim sequence of $\mathcal{H}_o(U)$, an inner U of the smallest possible degree for which $T_{or}^\dagger U^*$ is still anti-causal is those for which

$$\mathcal{H}_o(U) = \mathcal{H}_o(T_{or}^\dagger).$$

By Definition 4.3, this choice for U implies that $T_{rm} := UT_{or}$ is the minimal degree left maximal-phase operator associated to T_{or} . This completes the proof. ■

A similar statement holds in the case we start from a left outer T_{ol} that has a bounded causal left inverse. For simplicity reasons we do not provide the proof.

Lemma 4.6. *Let $T_{ol} \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a locally finite left outer operator having a minimal, u.e.s realization $[T_{ol}] = \{A, B, C, D_{ol}\}$. Define $\Delta_{ol} = A - CD_{ol}^\dagger B$ where D_{ol}^\dagger is a left inverse of D_{ol} and assume $\ell_{\Delta_{ol}} < 1$. Define an inner U by identifying it through*

$$\mathcal{H}_o(U) \supseteq \mathcal{D}_2(I - \Delta_{ol}Z)^{-1}(-CD_{ol}^\dagger)$$

Then the operator

$$T := UT_{ol}$$

is a left maximal-phase operator.

In particular, if $\mathcal{H}_o(U) = \mathcal{D}_2(I - \Delta_{ol}Z)^{-1}(-CD_{ol}^\dagger)$ then $T_{ml} := T$ is a minimal degree left maximal-phase operator associated to T_{ol} .

Let us now state an explicit form for a minimal, realization for the anti-causal right inverse of T_{mr} as introduced in Lemma 4.5. Because $\ell_{\Delta_{or}} < 1$ as assumed in the lemma, the Lyapunov-Stein equation

$$\Delta_{or}\Omega^{(-1)}\Delta_{or}^* + (-C_oD_{or}^\dagger)(-D_{or}^\dagger C_o^*) = \Omega.$$

has a unique boundedly invertible solution $\Omega \in \mathcal{D}$. For the inner U identified through $\mathcal{H}_o(U) = \mathcal{H}_o(T_{or}^\dagger)$, the realization

$$[U] := \begin{bmatrix} A - CD^\dagger B & -CD^\dagger \\ B_U^* & D_U^* \end{bmatrix}$$

is a minimal Ω -unitary realization for U . Now, a brute force evaluation of $T_{mr}^\dagger U^*$ in which the external factorization argument and a mixed partial fraction lemma 2.17 are used, leads to

$$\begin{aligned} T_{mr}^\dagger &= T_{or}^\dagger U^* \\ &= \{D_{or}^\dagger + D_{or}^\dagger B_o Z (I - \Delta_{or} Z)^{-1} (-C_o D_{or}^\dagger)\} \\ &\quad \times \{D_U^* + (-D_{or}^{\dagger*} C_o^*) Z^* (I - \Delta_{or}^* Z^*)^{-1} B_U^*\} \\ &= D_{or}^\dagger D_U^* + D_{or}^\dagger B_o \Omega^{(-1)} B_U^* \\ &\quad + \{-D_{or}^\dagger D_{or}^{\dagger*} C_o^* + D_{or}^\dagger B_o \Omega^{(-1)} \Delta_{or}^*\} Z^* (I - \Delta_{or}^* Z^*)^{-1} B_U^*. \end{aligned}$$

giving rise to the u.e.s realization

$$[T_{mr}^\dagger] = \left[\frac{\Delta_{or}^*}{D_{or}^\dagger (-D_{or}^{\dagger*} C_o^* + B_o \Omega^{(-1)} \Delta_{or}^*)} \middle| \frac{B_U^*}{D_{or}^\dagger (D_U^* + B_o \Omega^{(-1)} B_U^*)} \right]. \tag{4.1}$$

Because the s-dim sequence of T_{mr}^\dagger is the same as those of U which, in turn, is as minimal as possible, the realization $[T_{mr}^\dagger]$ is a minimal realization for the right inverse of the minimal degree left maximal-phase operator T_{mr} .

If either a left outer operator were our outgoings point or we were considering the construction of a right maximal-phase operators, the results are similar to the obtained. For simplicity reasons we do not consider these cases.

In the subsequent sections we shall deal with invertible maximal-phase operators. In the case T_m is such a operator and, in addition, is assumed of minimal degree, we are interested in a remarkable property of a minimal u.e.s realization $[T_m]$ for T_m namely in the fact that $[T_m]$ is invertible. We prove this in the following lemma.

Lemma 4.7. *Let $T_m \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a boundedly invertible maximal-phase operator of minimal degree and let $[T_m] = \{A_m, B_m, C_m, D_m\}$ be a u.e.s minimal realization for T_m . Then $[T_m]$ is invertible.*

PROOF: As T_m is causal, it admits an outer/inner factorization. That is there are some intermediate space \mathcal{M}' inner $U \in \mathcal{U}(\mathcal{M}', \mathcal{N})$ and outer $T_o \in \mathcal{U}(\mathcal{M}, \mathcal{M}')$ such that $T_m = T_o U$. When expressed in state-space terms, we have that for some left invertible $Y \in \mathcal{D}$ of maximal column rank at each time instant, there exist a u.e.s unitary realization $[U] := \{A_U, B_U, C_U, D_U\}$ and u.e.s $[T_o] := \{A, B, C_o, D_o\}$ with invertible D_o such that the right side of

$$\begin{bmatrix} AY^{(-1)} & C \\ BY^{(-1)} & D \end{bmatrix} = \begin{bmatrix} Y & C_o \\ & D_o \end{bmatrix} \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix}$$

is a QR-decomposition. As T_m is of minimal degree and $[T_m]$ is a minimal realization, Y is locally square. Since already left invertible we conclude then that Y is right invertible too. Clearly Y can be used as a state-space transformation to derive an alternative realization for T_m . We simply premultiply the above square-root equation with $\begin{bmatrix} Y^{-1} \\ I \end{bmatrix}$ and obtain

$$\begin{bmatrix} Y^{-1}AY^{(-1)} & Y^{-1}C \\ BY^{(-1)} & D \end{bmatrix} = \begin{bmatrix} I & Y^{-1}C_o \\ & D_o \end{bmatrix} \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix}.$$

As the left side of the latter equation represents an alternative minimal, u.e.s realization for T , that is invertible due to the invertible matrices from the right side, we conclude that $[T_m]$ is invertible. This completes the proof. \blacksquare

Another useful property of a minimal realization $[T_m]$ for a minimal degree maximal-phase invertible operator T_m is that it can be chosen such that its inverse $[T_m]^{-1}$ is, in fact, a minimal realization for T_m^{-1} . We state this in the following theorem.

Theorem 4.8. *Let $T_m \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be as in Lemma 4.7 and let $[T_m]$ be a u.e.s, minimal realization for T_m . Then, the realization $[T_m^{inv}] = \{A^{inv}, B^{inv}, C^{inv}, D^{inv}\}$ whereby the s -dim sequence of A^{inv} matches those of A^* from $[T_m]$ and defined as $[T_m^{inv}] := [T_m]^{-1}$, is a minimal realization for $T_m^{-1} \in \mathcal{L}(\mathcal{N}, \mathcal{M})$.*

PROOF: As T_m admits the outer/inner factorization $T_m = T_o U$, let us work on the realization terms for the inner and the outer factor of T_m directly. Since there is a locally square invertible block diagonal Y of maximal rank there exist minimal, u.e.s realization $[T_o] := \{A, B, C_o, D_o\}$ with invertible D_o and unitary $[U] := \{A_U, B_U, C_U, D_U\}$ such that the right side of

$$\begin{bmatrix} AY^{(-1)} & C \\ BY^{(-1)} & D \end{bmatrix} = \begin{bmatrix} Y & C_o \\ & D_o \end{bmatrix} \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix}$$

is a QR-decomposition. As Y is invertible it can be used as a state-space transformation for $[T_m]$. We rewrite the latter equation equivalently as

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} I & C_o \\ & D_o \end{bmatrix} \begin{bmatrix} Y A_U Y^{(-1)} & Y C_U \\ B_U Y^{(-1)} & D_U \end{bmatrix}$$

where now $[U]_Y = \{Y A_U Y^{(-1)}, B_U Y^{(-1)}, Y C_U, D_U\}$ is an alternative u.e.s, minimal realization for the inner U . Straightforward inspection shows that $Y Y^*$ is the observability Gramian for $[U]_Y$. By virtue of the latter equation we then may express T_m through the inner/outer terms as

$$T_m = D_o D_U + D_o B_U Z \left(I - (A_U + Y^{-1} C_o B_U) Z \right)^{-1} (C_U + Y^{-1} C_o D_U).$$

Let us also evaluate $[T_m]^{-1}$ from the latter square-root equation. We have that

$$\begin{aligned} [T_m]^{-1} &= \begin{bmatrix} Y^{(-1)}A_U^*Y^{-1} & Y^{(-1)}B_U^* \\ C_U^*Y^{-1} & D_U^* \end{bmatrix} \begin{bmatrix} I - C_oD_o^{-1} \\ D_o^{-1} \end{bmatrix} \\ &= \begin{bmatrix} Y^{(-1)}A_U^*Y^{-1} & Y^{(-1)}(B_U^* - A_U^*Y^{-1}C_o)D_o^{-1} \\ C_U^*Y^{-1} & (D_U^* - C_U^*Y^{-1}C_o)D_o^{-1} \end{bmatrix} \\ &=: \begin{bmatrix} A^{\text{inv}} & C^{\text{inv}} \\ B^{\text{inv}} & D^{\text{inv}} \end{bmatrix}. \end{aligned}$$

Note that $\text{s-dim}(A^{\text{inv}}) = \text{s-dim}(A_U^*) = \text{s-dim}(A^*)$.

Introduce now the anti-causal operator

$$\begin{aligned} T_m^{\text{inv}} &:= D^{\text{inv}} + B^{\text{inv}}Z^*(I - A^{\text{inv}}Z^*)^{-1}C^{\text{inv}} \\ &= (D_U^* - C_U^*Y^{-1}C_o)D_o^{-1} + C_U^*Z^*(I - A_U^*Z^*)^{-1}(B_U^* - A_U^*Y^{-1}C_o)D_o^{-1}. \end{aligned}$$

The remainder of the proof consists of showing that $T_m T_m^{\text{inv}} = I_{\mathcal{M}}$. As splitting the product $T_m T_m^{\text{inv}}$ into terms of single causalities aims at use of the mixed partial fraction lemma 2.17, let us introduce $\Gamma \in \mathcal{D}$ to be the unique bounded solution for

$$(A_U + Y^{-1}C_oB_U)\Gamma^{(-1)}A_U^* + (C_U + Y^{-1}C_oD_U)C_U^* = \Gamma.$$

Since $[U]$ is an isometric realization, the latter Lyapunov-Stein equation simplifies after employing the identities $C_U C_U^* = I - A_U A_U^*$ and $D_U C_U^* = -B_U A_U^*$ into it. After rearranging its terms, the equation reduces to the trivial Lyapunov-Stein equation

$$(A_U + Y^{-1}C_oB_U)(\Gamma - I)^{(-1)}A_U^* = \Gamma - I.$$

Because T_m is boundedly invertible we have that $\ell_{A_U} < 1$ and $\ell_{(A_U + Y^{-1}C_oB_U)} < 1$. As a consequence, the latter equation has zero as its unique bounded solution. Hence, $\Gamma = I$. Now, brute force evaluation of $T_m T_m^{\text{inv}}$ when the mixed partial fraction lemma is used to split the mixed causality term from the product by means of the intermediate variable Γ , produces

$$\begin{aligned} T_m T_m^{\text{inv}} &= D_o \left(D_U + B_U Z (I - (A_U + Y^{-1}C_oB_U)Z)^{-1} (C_U + Y^{-1}C_oD_U) \right) \\ &\quad \times \left(D_U^* - C_U^* Y^{-1} C_o + C_U^* Z^* (I - A_U^* Z^*)^{-1} (B_U^* - A_U^* Y^{-1} C_o) \right) D_o^{-1} \\ &= D_o \left(\underline{(D_U C_U^* + B_U A_U^*) Z^* (I - A_U^* Z^*)^{-1} (B_U^* - A_U^* Y^{-1} C_o)} \right. \\ &\quad \left. + \underline{D_U (D_U^* - C_U^* Y^{-1} C_o) + B_U (B_U^* - A_U^* Y^{-1} C_o)} \right. \\ &\quad \left. + B_U Z (I - (A_U + Y^{-1}C_oB_U)Z)^{-1} \underline{x} \right) D_o^{-1} \end{aligned}$$

where

$$x := (C_U + Y^{-1}C_o D_U)(D_U^* - C_U^* Y^{-1}C_o) + (A_U + Y^{-1}C_o B_U)(B_U^* - A_U^* Y^{-1}C_o).$$

As a consequence of the fact that $[U]$ is isometric we have that the single and the doubly underlined terms from the above decomposition are respectively zero and identity. This leads immediately to the conclusion that $T_m T_m^{\text{inv}} = D_o D_o^{-1} = I_{\mathcal{M}}$ as we needed to show. This completes the proof of this theorem. ■

Based on Theorem 4.8 we state in Figure 4.1 an one-pass algorithm for finding a minimal realization for the inverse of a minimal degree maximal-phase operator. Its proof is provided by the theorem.

1. **input:** A sequence of minimal realizations $[T_m]_k = \{A_k, B_k, C_k, D_k\}$ starting from some initial time instant k_{init} for a boundedly invertible minimal degree maximal-phase operator T_m .

output: A sequence of minimal realizations $[T_m^{-1}]_k = \{A_k^{\text{inv}}, B_k^{\text{inv}}, C_k^{\text{inv}}, D_k^{\text{inv}}\}$ for the anti-causal T_m^{-1} .

For each $k = k_{\text{init}}, k_{\text{init}} + 1, \dots$ do

2.

$$\begin{bmatrix} A_k^{\text{inv}} & C_k^{\text{inv}} \\ B_k^{\text{inv}} & D_k^{\text{inv}} \end{bmatrix} := \begin{bmatrix} A_k & C_k \\ B_k & D_k \end{bmatrix}^{-1}$$

where the partitioning in the matrix on the left is such that the size of A_k^{inv} matches those of A_k^* .

3. End.

Fig. 4.1. One-pass construction of a minimal realization for the inverse of a minimal degree maximal-phase operator T_m .

The time direction used in the algorithm for the evaluation of T_m^{-1} is $k = k_{\text{init}}, k_{\text{init}} + 1, \dots$. Note that it can also be reversed which easily leads to an one-pass backward algorithm.

4.3 Evaluation of the inverse

Let us now consider the evaluation of the inverse of a causal boundedly invertible operator $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ given by a minimal, u.e.s realization $[T] = \{A, B, C, D\}$ and containing no intrinsic inner factors. For some boundedly right invertible $Y \in \mathcal{D}$ of maximal row rank the following square-root decomposition

$$\begin{bmatrix} Y A Y C \\ B & D \end{bmatrix} = \begin{bmatrix} A_{U_2} & C_{U_2} \\ B_{U_2} & D_{U_2} \end{bmatrix} \begin{bmatrix} Y^{(-1)} & \\ & B_o & D_o \end{bmatrix} \quad (4.2)$$

when applied on $[T]$, produces: minimal, u.e.s realization $[T_o] = \{A, C, B_o, D_o\}$ for the outer T_o and a minimal, u.e.s realization $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ for the inner U_2 satisfying $T = U_2 T_o$.

Brute force calculation of the inverse for T involves the evaluation of the product $T_o^{-1} U_2^*$. As we encountered in Chapter 3, the latter product introduces possible cancellations between T_o^{-1} and U_2^* and hence aims at searching for a boundedly invertible state-space transformation which, when applied on $[T_o^{-1}]$, produces a minimal, u.e.s realization whose state-space transition operator has an appropriate block triangular form.

To avoid this we prefer not to work on $[T^{-1}]$ directly, but instead to turn T^{-1} into a maximal-phase operator of minimal degree. as a state transition operator of a u.e.s unitary realization for U_2^* . This motivates us to introduce an additional inner U_1 such that

$$T_m = U_1 U_2 T_o$$

is the minimal degree left maximal-phase operator related to T_o . The adjective 'left' will be left out of the consideration further on because we concentrate us on inner/outer factorizations. Associating a maximal-phase operator shall then be done at the left side of the outer factor.

The idea for inverting T can now be split into three steps:

- 1 Finding the inverse T_m^{-1} - this step depends on $[T_o] = \{A, B_o, C, D_o\}$ entirely.
- 2 Searching for the state-space transform which would bring the AC pair from a minimal realization $[T_m^{-1}]$ for T_m^{-1} - note this is also an AC- pair of the causal unitary $U = U_1 U_2$ - into the form whose state-space transition operator is in block lower form - the top row block of that realization is the $(A_{U_1}^*, B_{U_1}^*)$ -pair of $[U_1]$ obtained through the QR-decomposition (4.2).
- 3 The previous step gives rise to a minimal state-space extraction of U_1^* while obtaining a minimal, u.e.s realization for the mixed causal T^{-1} .

In what follows we evaluate the inverse T^{-1} of a boundedly invertible T by following the above recipe. We start from the minimal, u.e.s realization $[T] = \{A, B, C, D\}$ for T and, for some $Y^{(-1)}$, find the QR-decomposition (4.2), to obtain minimal u.e.s realizations for the inner U_2 and the outer T_o satisfying $T = U_2 T_o$. We also assume that $[T]$ is chosen such that the minimal, u.e.s realization for T_o^{-1} ;

$$[T_o^{-1}] = \begin{bmatrix} A - CD_o^{-1}B_o & -CD_o^{-1} \\ D_o^{-1}B_o & D_o^{-1} \end{bmatrix}$$

has the observability Gramian equal the identity block diagonal. Such a realization is said to be in *output normal form*. To see that this is possible, let $R \in \mathcal{D}$ be a boundedly invertible transformation of an appropriate s-dim sequence and let its action on $[T_o^{-1}]$

be defined by $R^{-1}(\cdot)R^{(-1)}$. Then viz. $R^{-1}(A - CD_o^{-1}B_o)R^{(-1)}$ we note that R acts on A , B_o and C via $R^{-1}AR^{(-1)}$, $B_oR^{(-1)}$ and $R^{-1}C_o$.

Define an inner U by identifying it through

$$\mathcal{H}_o(U) = \mathcal{D}_2(I - (A - CD_o^{-1}B_o)Z)^{-1}(-CD_o^{-1}).$$

Let block diagonals B_U and D_U be such that

$$[U] = \begin{bmatrix} A - CD_o^{-1}B_o & -CD_o^{-1} \\ B_U & D_U \end{bmatrix}$$

is a unitary, u.e.s realization for inner U . Due to the construction and by virtue of Lemma 4.5,

$$T_m = UT_o$$

is a minimal degree maximal-phase operator corresponding to T_o . Note also that U_2 is a factor of U and U_1 defined as

$$U_1 := UU_2^* \tag{4.3}$$

is inner. Moreover, it is the minimal degree inner factor for which U_1T ($= T_m$) is the maximal-phase operator from the consideration. Therefore, there will be some properly partitioned unitary state-space transformation $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \in \mathcal{D}^{2 \times 1}$ and a unitary, u.e.s realization $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ for an inner U_1 such that

$$[U]_Q := \left[\begin{array}{c|c} Q_1 & \\ \hline Q_2 & \\ \hline & I \end{array} \right] [U] \left[\begin{array}{c|c} Q_1^{*(-1)} & Q_2^{*(-1)} \\ \hline & I \end{array} \right]$$

$$\left(= \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline C_{U_1} B_{U_2} & C_{U_1} D_{U_2} \\ \hline D_{U_1} B_{U_2} & D_{U_1} D_{U_2} \end{array} \right] \right)$$

is also a unitary, u.e.s realization for U . The block lower structure of the state transition operator in this realization is induced by (4.3).

The next step is evaluation of the inverse T_m^{-1} . By following the same procedure as we did earlier to obtain (4.1) from the previous section, one finds that

$$[T_m^{-1}] = \left[\begin{array}{c|c} \Delta_o^* & B_U^* \\ \hline D_o^{-1}(-D_o^{-*}C^* + B_o\Delta_o^*) & D_o^{-1}(D_U^* + B_oB_U^*) \end{array} \right]$$

is the inverse of T_m where $\Delta_o := A - CD_o^{-1}B_o$ is the Schur complement of A with respect to the realization $[T_o]$.

Application of the unitary transformation Q on $[T_m^{-1}]^*$ and taking the transpose results in alternative minimal, u.e.s realization for T_m^{-1}

$$[T_m^{-1}]_Q = \left[\begin{array}{c|c} \begin{matrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{matrix} & \begin{matrix} B_{U_2}^* D_{U_1}^* \\ \hline B_{U_1}^* \end{matrix} \\ \hline \begin{matrix} b_1 & b_2 \end{matrix} & D_o^{-1}(D_U^* + B_o B_U^*) \end{array} \right]$$

in output normal form where

$$[b_1 \ b_2] := D_o^{-1}(-D_o^{-*} C^* + B_o \Delta_o^*) \left[Q_1^{*(-1)} \ Q_2^{*(-1)} \right].$$

It is straightforward to see from $T^{-1} = T_m^{-1} U_1$ that the evaluation of T^{-1} involves extraction of the inner U_1 from T_m^{-1} . Let us evaluate T^{-1} by working on realizations $[T_m^{-1}]_Q$ and $[U_1]$.

First, introduce a properly partitioned $\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \in \mathcal{D}^{2 \times 1}$ to be the bounded solution for

$$\begin{bmatrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} A_{U_1} + \begin{bmatrix} B_{U_2}^* D_{U_1}^* \\ \hline B_{U_1}^* \end{bmatrix} B_{U_1} = \begin{bmatrix} \Phi_1^{(-1)} \\ \Phi_2^{(-1)} \end{bmatrix}.$$

By straightforward substitution and using the fact that $[U_1]$ is coisometric, one finds that $\Phi = \begin{bmatrix} 0 \\ I \end{bmatrix}$. Finally, decoupling the mixed causality term from $T_m^* U_1$ by virtue of the mixed partial fraction lemma 2.17 where the quantity Φ is used, results in

$$\begin{aligned} & \left(\begin{bmatrix} I & \\ & I \end{bmatrix} - \begin{bmatrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{bmatrix} \begin{bmatrix} Z^* & \\ & Z^* \end{bmatrix} \right)^{-1} \begin{bmatrix} B_{U_2}^* D_{U_1}^* \\ \hline B_{U_1}^* \end{bmatrix} B_{U_1} Z (I - A_{U_1} Z)^{-1} \\ & = \left(\begin{bmatrix} I & \\ & I \end{bmatrix} - \begin{bmatrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{bmatrix} \begin{bmatrix} Z^* & \\ & Z^* \end{bmatrix} \right)^{-1} \begin{bmatrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{bmatrix} \Phi + \Phi \\ & + \Phi A_{U_1} Z (I - A_{U_1} Z)^{-1}. \end{aligned}$$

Using this decomposition, we find for the strictly anti-causal part of $T_m^{-1} U_1$ that

$$\begin{aligned} \mathbf{P}'(T_m^{-1} U_1) & = D_o^{-1}(-D_o^{-*} C^* + B_o \Delta_o^*) [Q_1^{*(-1)} \ Q_2^{*(-1)}] \\ & \quad \times \begin{bmatrix} Z^* & \\ & Z^* \end{bmatrix} \left(\begin{bmatrix} I & \\ & I \end{bmatrix} - \begin{bmatrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{bmatrix} \begin{bmatrix} Z^* & \\ & Z^* \end{bmatrix} \right)^{-1} \\ & \quad \times \left\{ \begin{bmatrix} B_{U_2}^* D_{U_1}^* \\ \hline B_{U_1}^* \end{bmatrix} D_{U_1} + \begin{bmatrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{bmatrix} \Phi C_{U_1} \right\} \\ & = D_o^{-1}(-D_o^{-*} C^* + B_o \Delta_o^*) [Q_1^{*(-1)} \ Q_2^{*(-1)}] \\ & \quad \times \begin{bmatrix} Z^* & \\ & Z^* \end{bmatrix} \left(\begin{bmatrix} I & \\ & I \end{bmatrix} - \begin{bmatrix} A_{U_2}^* & B_{U_2}^* C_{U_1}^* \\ \hline A_{U_1}^* \end{bmatrix} \begin{bmatrix} Z^* & \\ & Z^* \end{bmatrix} \right)^{-1} \begin{bmatrix} B_{U_2}^* \\ 0 \end{bmatrix} \\ & = D_o^{-1}(-D_o^{-*} C^* + B_o \Delta_o^*) Q_1^{*(-1)} Z^* (I - A_{U_2} Z^*)^{-1} B_{U_2}^*. \end{aligned}$$

In similar way, one finds for the causal operator part of $T_m^{-1}U_1$ that

$$\begin{aligned} \mathbf{P}(T_m^{-1}U_1) &= D_o^{-1} \left\{ (D_U^* + B_o B_U^*) D_{U_1} + (-D_o^{-*} C^* + B_o \Delta_o^*) Q_1^{*(-1)} C_{U_1} \right\} \\ &\quad + D_o^{-1} \left\{ (D_U^* + B_o B_U^*) B_{U_1} + (-D_o^{-*} C^* + B_o \Delta_o^*) Q_1^{*(-1)} A_{U_1} \right\} \\ &\quad \times Z(I - A_{U_1} Z)^{-1} C_{U_1}. \end{aligned}$$

Note that neither $\mathbf{P}'(T_m^{-1}U_1)$ nor $\mathbf{P}(T_m^{-1}U_1)$ depend on Q_2 . On the other side, we see from the realization $[U]_Q$ that unitary Q must be such that

$$A_{U_2} = Q_1 \Delta_o Q_1^{*(-1)} \text{ and } C_{U_2} = Q_1 (-C D_o^{-1}).$$

After postmultiplying the both sides of (4.2) with

$$\begin{bmatrix} Y^{\dagger(-1)} \\ -D_o^{-1} B_o Y^{\dagger(-1)} \quad D_o^{-1} \end{bmatrix},$$

we find that $A_{U_2} = Y \Delta_o Y^{\dagger(-1)}$ and $C_{U_2} = Y C D_o^{-1}$.

Recall that $[T]$ is chosen such that $(\Delta_o, -C D_o^{-1})$ is in output normal form. Hence, Y is, modulo a diagonal unitary constant, equivalent to Q_1 . Since we are interested in at least one Q_1 , we take $Q_1 = Y$.

Finally, an additive minimal, u.e.s realization for T^{-1} can be assembled from $\mathbf{P}'(T_m^{-1}U_1)$ and $\mathbf{P}(T_m^{-1}U_1)$ as

$$[T^{-1}] = \left[\begin{array}{c|c} A_{U_1} & C_{U_1} \\ \hline A_{U_2}^* & B_{U_2}^* \\ \beta_c & \beta_a \quad \delta \end{array} \right] \quad (4.4)$$

where

$$\begin{aligned} \beta_c &:= D_o^{-1} \left\{ (D_U^* + B_o B_U^*) B_{U_1} + (-D_o^{-*} C^* + B_o \Delta_o^*) Y^{*(-1)} A_{U_1} \right\}, \\ \beta_a &:= D_o^{-1} (-D_o^{-*} C^* + B_o \Delta_o^*) Y^{*(-1)} \text{ and} \\ \delta &:= D_o^{-1} \left\{ (D_U^* + B_o B_U^*) D_{U_1} + (-D_o^{-*} C^* + B_o \Delta_o^*) Y^{*(-1)} C_{U_1} \right\}. \end{aligned}$$

The previous consideration provides the proof for the following proposition.

Proposition 4.9. *Assume a causal bounded, locally finite invertible operator T is given by a minimal u.e.s realization $[T] = \{A, B, C, D\}$. Let the minimal u.e.s realizations $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and $[T_o] = \{A, B_o, C, D_o\}$ for respectively the inner U_2 and the outer T_o admitting $T = U_2 T_o$ be obtained from equation (4.2) in which $Y \in \mathcal{D}$ is boundedly right invertible and of maximal row rank. Assume $[T]$ is chosen such that for some B_U and D_U the realization*

$$[U] = \left[\begin{array}{c|c} A - CD_o^{-1}B_o & -CD_o^{-1} \\ \hline B_U & D_U \end{array} \right]$$

is a unitary realization for a causal unitary U . Define U_1 to be the causal unitary complement of U_2 in U i.e. $U_1 = UU_2^*$ and let $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ be a unitary realization for U_1 .

Then,

(1) The causal unitary U is such that UT_o is a maximal-phase operator.

(2) The realization (4.4) is a minimal, additive realization for T^{-1} .

The following lemma shall be important in the next section for the purpose of expressing the inner and the outer factors in the inner/outer decomposition for a causal operator through the inner and the outer factors from its outer/inner decomposition

Lemma 4.10. Let T be a locally finite maximal-phase operator given by a minimal, u.e.s realization $[T] = \{A, B, C, D\}$ and let it admit the inner/outer factorization $T_m = UT_o$ for some inner U and the outer T_o whose minimal, u.e.s realizations; $[U] = \{A_U, B_U, C_U, D_U\}$ and $[T_o] = \{A, B_o, C, D_o\}$ are obtained from the QL-decomposition

$$\begin{bmatrix} Y & A & Y & C \\ B & D & & \end{bmatrix} = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix} \begin{bmatrix} Y^{(-1)} \\ B_o & D_o \end{bmatrix} \quad (4.5)$$

in which $Y \in \mathcal{D}$ is a square boundedly invertible block diagonal operator. Assume that $L_Y \in \mathcal{D}$ and $Q_Y \in \mathcal{D}$ are respectively boundedly invertible block upper and unitary factors from the QL-factorization

$$Y = Q_Y L_Y.$$

for Y .

Then, L_Y is a boundedly invertible state-space transformation which, when applied on the realization $[T]$, brings the minimal u.e.s realization

$$[T_o^{-1}] = \{A - CD_o^{-1}B_o, D_o^{-1}B_o, -CD_o^{-1}, D_o^{-1}\}$$

for T_o^{-1} into output¹ normal form.

PROOF: Consider the square-root equation (4.5) and postmultiply its both sides with

$$\begin{bmatrix} Y^{(-1)} \\ -D_o^{-1}B_oY^{(-1)} & D_o^{-1} \end{bmatrix}.$$

After recalling that $\Delta_o = A - CD_o^{-1}B_o$, we find that

¹i.e. in a form for which the causal AC-pair $(A - CD_o^{-1}B_o, -CD_o^{-1})$ is isometric

$$\begin{bmatrix} (-Y)\Delta_o(-Y^{-(-1)}) & (-Y)(-CD_o^{-1}) \\ (-B + D_o^{-1}B_o)(-Y^{-(-1)}) & DD_o^{-1} \end{bmatrix} = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix}.$$

Since (A_U, C_U) pair is isometric, it also holds that

$$\Delta_o(Y^{-1}Y^{-*})^{(-1)}\Delta_o^* + (-CD_o^{-1})(-D_o^{-*}C^*) = Y^{-1}Y^{-*}.$$

When expressed through the QL-factorization for Y

$$Y = Q_Y L_Y$$

the previous Lyapunov-Stein equation reads

$$\Delta_o(L_Y^* L_Y)^{-(-1)}\Delta_o^* + (-CD_o^{-1})(-D_o^{-*}C^*) = (L_Y^* L_Y)^{-1}.$$

Now, it is obvious that if we want to have the pair $(\Delta_o, -CD_o^{-1})$ isometric, then the realization $[T]$ must be chosen such that L_Y is unitary. To find this alternative realization $[T]_L$ for T let us pre- and postmultiply the equation (4.5) with

$$\begin{bmatrix} Q_Y^* \\ I \end{bmatrix} \text{ and } \begin{bmatrix} L_Y^{-(-1)} \\ I \end{bmatrix}$$

respectively. This produces

$$\begin{bmatrix} L_Y A L_Y^{-(-1)} & L_Y C \\ B L_Y^{-1} & D \end{bmatrix} = \begin{bmatrix} Q_Y^* A_U Q_Y^{(-1)} & Q_Y^* C_U \\ B_U Q_Y^{(-1)} & D_U \end{bmatrix} \begin{bmatrix} I \\ B_o L_Y^{-(-1)} & D_o \end{bmatrix}$$

which is still a square-root type equation although not applicable in this form. From this equation we find that, in order to have the pair $(\Delta_o, -CD_o^{-1})$ isometric the realization $[T]$ for T must be brought into the form

$$[T]_L = \{L_Y A L_Y^{-(-1)}, B L_Y^{-1}, L_Y C, D\}$$

first. This completes the proof. ■

4.4 Outer/inner factorization of maximal-phase operators through the inner/outer factorization terms

In the first section of this chapter (see the lemmas 4.5 and 4.6), we obtained that the maximal-phase operator T_m is completely determined by the outer factor from either the inner/outer or the outer/inner factorization of T_m . Numerically speaking, the inner/outer decomposition can be found by applying a QR-decomposition in the form (4.2) on a minimal, u.e.s realization $[T_m]$ for T_m . For the outer/inner decomposition,

4.4. Outer/inner factorization of maximal-phase operators through ... 83

the dual statement holds true and, numerically speaking, in finding the decomposition we start from the dual form to 4.5 and apply $[T_m]$ to it.

The goal of this section is to find the relationship between the terms from the inner/outer and the outer/inner factorization of a minimal degree maximal-phase operator.

Let T_m be a minimal degree maximal-phase operator given by a minimal, u.e.s realization $[T_m] = \{A, B, C, D\}$. Let for some inner V and an outer G_o the representation $T_m = G_o V$ be the outer/inner representation for T_m . From the previous section, in Lemma 4.10 we have introduced the inner U and the outer T_o satisfying $T_m = UT_o$ minimal realizations of which: $[U]$ and $[T_o] = \{A, B_o, C, D_o\}$ are obtained through the QR-decomposition (4.5). According to Lemma 4.10, it is possible to chose a minimal, u.e.s realization for T_m such that Y from the equation (4.5) is the identity block diagonal. As the consequence of such a chosen realization we will have that the pair $(A - CD_o^{-1}B_o, -CD_o^{-1})$ is isometric.

Proposition 4.11. *Let $T_m \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a locally finite maximal-phase operator. Assume that T_m is given by a minimal, u.e.s realization $[T_m]$ and let minimal, u.e.s realizations $[T_o] = \{A, B_o, C, D_o\}$ and unitary $[U] = \{A_U, B_U, C_U, D_U\}$ be realizations for the inner U and outer T_o satisfying $T_m = UT_o$. Define $A^\times = A - CD_o^{-1}B_o$ and let $(A^\times, -CD_o^{-1})$ be an isometric pair. Let also $r \in \mathcal{U}$ be a Cholesky factor of the boundedly invertible solution A , $rr^* = A$ for*

$$A^\times A^{(-1)} A^{\times*} + (-CD_o^{-1} + A^\times B_o^*) D_o^{-*} D_o^{-1} (-D_o^{-*} C^* + B_o A^{\times*}) = A.$$

Then, there exists a minimal degree inner V having, for some B'_V and D'_V , u.e.s, unitary realization

$$[V] = \left[\begin{array}{c|c} r^{-*} A^\times r^{(-1)} & r^{-*} (-CD_o^{-1} + A^\times B_o^*) D_o^{-*} \\ \hline B'_V r^{(-1)} & D'_V \end{array} \right]$$

such that $G_o := (UT_o)V^$ is outer having a minimal, u.e.s. realization*

$$[G_o] = \left[\begin{array}{c|c} A | CD'_V{}^* - AB'_V{}^* \\ \hline B | DD'_V{}^* - BB'_V{}^* \end{array} \right].$$

PROOF: Let $[T_m] = \{A, B, C, D\}$ be a minimal, u.e.s realization for T_m . Let $[T_o] = \{A, B_o, C, D_o\}$ and unitary $[U] = \{A_U, B_U, C_U, D_U\}$ be minimal, u.e.s realizations for respectively the outer T_o and the inner U satisfying $T_m = UT_o$. Let us now construct a minimal degree inner V such that VT_m^{-1} is causal. We have already discovered in the previous section that a minimal u.e.s realization for T_m^{-1} in output normal form is

$$[T_m^{-1}] = \left[\begin{array}{c|c} A^{\times*} & B_U^* \\ \hline D_o^{-1} (-D_o^{-*} C^* + B_o A^{\times*}) & D_o^{-1} (D_U^* + B_o B_U^*) \end{array} \right].$$

Hence, there exist some B'_V, D'_V such that, for the boundedly invertible state-space transformation r as defined in the statement of this proposition, the realization

$$[V] = \left[\begin{array}{c|c} r^{-*} A^\times r^{(-1)} & r^{-*} (-CD_o^{-1} + A^\times B_o^*) D_o^{-*} \\ \hline B'_V r^{(-1)} & D'_V \end{array} \right]$$

is a unitary, u.e.s realization for a minimal degree inner V satisfying $VT_m^{-1} \in \mathcal{U}$. To complete the proof, we need to show that V is indeed such that $T_m V^*$ is causal. In that case $T_m V^*$ is an outer factor.

Since $T_m V^*$ is the product of two mixed causality terms whose realizations are u.e.s, we may define a block diagonal Γ to be the bounded fixed point solution of the equation

$$A\Gamma^{(-1)}r^{(-1)}A^{\times*}r^{-1} + CD_o^{-1}(-D_o^{-*}C^* + B_oA^{\times*})r^{-1} = \Gamma.$$

Define $\tilde{\Gamma} = \Gamma r$, then the previous equation becomes

$$A\tilde{\Gamma}^{(-1)}A^{\times*} + CD_o^{-1}(-D_o^{-*}C^* + B_oA^{\times*}) = \tilde{\Gamma}.$$

Let us now evaluate $\tilde{\Gamma}$. Since $(A^\times, -CD_o^{-1})$ is an isometric pair it holds that $A^\times A^{\times*} + (-CD_o^{-1})(-D_o^{-*}C^*) = I$ which, when substituted into the previous equation yields that

$$(A^\times + CD_o^{-1}B_o)\tilde{\Gamma}^{(-1)}A^{\times*} - CD_o^{-1}D_o^{-*}C^* + CD_o^{-1}B_oA^{\times*} = \tilde{\Gamma}$$

and, after a little algebra, yields

$$A^\times \tilde{\Gamma}^{(-1)} A^{\times*} + CD_o^{-1} B_o \tilde{\Gamma}^{(-1)} A^{\times*} + A^\times A^{\times*} - I + CD_o^{-1} B_o A^{\times*} = \tilde{\Gamma}.$$

The previous equation can be written as

$$A^\times (\tilde{\Gamma} + I)^{(-1)} A^{\times*} + CD_o^{-1} B_o (\tilde{\Gamma} + I)^{(-1)} A^{\times*} = \tilde{\Gamma} + I$$

or, even more simply, as $A^\times (\tilde{\Gamma} + I)^{(-1)} A^{\times*} = \tilde{\Gamma} + I$. This equation, because of $\ell_A < 1$, has the unique bounded solution $\tilde{\Gamma} = -I$. Let us employ $\tilde{\Gamma}$ and the mixed partial fraction lemma 2.17 to decouple the mixed causality term from $T_m V^*$. We have that

$$\begin{aligned} T_m V^* &= (D + BZ(I - AZ)^{-1}C) \\ &\quad \times \left(D'_V{}^* + D_o^{-1}(-D_o^{-*}C^* + B_oA^{\times*})r^{-1} \right. \\ &\quad \left. \times Z^*(I - r^{*(-1)}A^{\times*}r^{-1}Z^*)^{-1}r^{*(-1)}B'_V{}^* \right) \\ &= DD'_V{}^* - BB'_V{}^* \\ &\quad + \left\{ DD_o^{-1}(-D_o^{-*}C^* + B_oA^{\times*}) - BA^{\times*} \right\} Z^*(I - A^{\times*}Z^*)^{-1}B'_V{}^* \\ &\quad + BZ(I - AZ)^{-1} \left\{ CD'_V{}^* - AB'_V{}^* \right\}. \end{aligned}$$

Consider now the term

$$\mathbf{P}'(T_m V^*) = \left\{ DD_o^{-1}(-D_o^{-*}C^* + B_o A^{\times*}) - BA^{\times*} \right\} Z^*(I - A^{\times*}Z^*)^{-1} B'_V{}^*$$

We show that this term is actually equal to zero. By Lemma 4.10, in the same basis where T_m^{-1} has the above introduced realization $[T_m^{-1}]$, the realization $[T]$ admits the QL-factorization

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix} \begin{bmatrix} I \\ B_o & D_o \end{bmatrix}.$$

Let us first postmultiply the previous equation with

$$\begin{bmatrix} I \\ -D_o^{-1}B_o & D_o^{-1} \end{bmatrix}$$

and then with its transpose. After recalling that the realization $[U]$ is isometric, we obtain that

$$\begin{bmatrix} A^\times & CD_o^{-1} \\ B - DD_o^{-1}B_o & DD_o^{-1} \end{bmatrix} \begin{bmatrix} A^{\times*} & B^* - B_o^*D_o^{-*}D^* \\ D_o^{-*}C^* & D_o^{-*}D^* \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix}.$$

Extraction of the (2,1)-block then yields

$$(B - DD_o^{-1}B_o)A^{\times*} + DD_o^{-1}D_o^{-*}C^* = 0.$$

But this block is exactly equal to the 'B'-term in $\mathbf{P}'(T_m V^*)$. Hence $\mathbf{P}'(T_m V^*) = 0$ which completes the proof that $G_o := T_m V^*$ is outer being, by construction, such that

$$T_m = G_o V$$

with a minimal, u.e.s realization $[G_o]$ as mentioned in the statement of this proposition and straightforwardly obtainable from the evaluation of $T_m V^*$. This completes the proof. ■

The following two lemmas tell us how some external factorizations of inverses of mixed causality invertible operators introduce outer and maximal-phase factors in the operators themselves.

Lemma 4.12. *Let $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a causal, invertible operator given by additive, minimal, u.e.s realization*

$$[G^{-1}] = \left[\begin{array}{c|c} A_c & C_c \\ \hline A_a & C_a \\ \hline B_c & B_a \\ \hline & D \end{array} \right] \tag{4.6}$$

for its inverse $G^{-1} \in \mathcal{X}(\mathcal{N}, \mathcal{M})$.

If V is a minimal degree inner factor for which VG^{-1} is causal then, for some outer factor G_o , the representation

$$G = G_o V$$

is the outer/inner factorization for G .

PROOF: Since G is invertible, it admits an outer/inner factorization $G = G_o W$ for some outer G_o and inner W . Let V be inner and of minimal degree such that VG^{-1} is causal. Then, the causality of $VW^*G_o^{-1}$ implies that VW^* is inner. But then W must be a right factor of V and we conclude that $\#(V) \geq \#(W)$. But G^{-1} is by assumption of mixed causality and such that $\#(\mathbf{P}(G^{-1})) = \#(V)$. From $G^{-1} = W^*G_o^{-1}$ we conclude that only W^* in the latter representation introduces the anti-causality. This means that $\#(W) = \#(V)$. Because W is a (right) factor of V the previous degree constraint implies that $W = V$. Hence, $G = G_o V$ is a outer/inner representation for G as we needed to show. ■

Lemma 4.13. *Let $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be an invertible operator given by an additive minimal, u.e.s realization $[G^{-1}]$ of its inverse G^{-1} in the form (4.6) as introduced in Lemma 4.12.*

*If V is the minimal degree inner factor for which $G_m^{-1} := V^*G^{-1}$ is anti-causal then, there is a maximal-phase factor G_m for which G has the representation*

$$G = G_m V^*.$$

PROOF: Let V be a minimal degree inner factor for which V^*G^{-1} is anti-causal. Define $\Delta^* := V^*G^{-1}$. Then, the causal Δ is invertible because both G and V are invertible. We find that $G = \Delta^{-*}V^*$. Since for some inner U and outer T_o it holds that $G = UT_o$ is the inner/outer representation for G , we also find that

$$UT_o = \Delta^{-*}V^*.$$

Hence, $\Delta^{-*} = UT_o V$ and we conclude that Δ^{-*} is causal. Since Δ is causal too, Δ must be a maximal-phase factor. Define $G_m := \Delta^{-*}$. Then, we have that $G = G_m V^*$. This completes the proof. ■

4.5 Interchanging inner and outer factors

In the previous section, we have seen how to express the factors from an inner/outer factorization of a minimal degree maximal-phase factor through the terms from its outer/inner factorization. In this section we no longer restrict ourselves on maximal-phase operators but on any causal boundedly invertible operator and study the same problem.

Let us be given a minimal, u.e.s realization $[T] = \{A, B, C, D\}$ for a boundedly invertible and causal operator T . Let T admit the inner/outer factorization $T = U_2 T_o$ for some inner U_2 and outer T_o whose minimal u.e.s realizations

$$[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\} \quad \text{and} \quad [T_o] = \{A, B_o, C, D_o\}$$

are obtained from the QL-decomposition (4.2) in which $Y \in \mathcal{D}$ is a boundedly right invertible and of maximal row rank at each time instant. We consider here finding another outer G'_o and inner V'_2 satisfying

$$T = G'_o V'_2.$$

In other words, we want to find G'_o and V'_2 for which $U_2 T_o$ commutes, that is

$$U_2 T_o = G'_o V'_2. \tag{4.7}$$

These factors are guaranteed to exist because of the assumption that T^{-1} is bounded.

If we assume that minimal, u.e.s realizations for the inner U_2 and the outer T_o satisfying $T = U_2 T_o$ are a priori known and obtained from the QL-decomposition (4.2), we may apply the results from the previous section and construct the maximal-phase operator T_m corresponding to T_o . That is, we can find another inner U_1 such that $T_m = U_1 U_2 T_o$ is a maximal-phase operator. Define $U = U_1 U_2$, then it is possible to find the outer G_o and the inner V such that $T_m = G_o V$ where the G_o and V can completely be expressed in minimal realization terms of T_o and U . Then, it holds that

$$T = U_2 T_o = U_1^* G_o V$$

and since T admits the outer/inner factorization $T = G'_o V'_2$, we have that for some inner V_1 and V_2 being such that $V = V_1 V_2$, it holds that

$$T = (U_1^* G_o V_1) V_2$$

is also an outer/inner factorization for T . Because of the uniqueness of the inner (the outer) factor up to diagonal unitary constant on its right (left), we have that $G'_o = U_1^* G_o V_1$ and $V'_2 = V_2$.

Proposition 4.14. *Let a locally finite and boundedly invertible, causal operator $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s realization $[T] = \{A, B, C, D\}$. Let T admit inner/outer factorization $T = U_2 T_o$ for some inner U_2 and outer T_o whose minimal u.e.s realizations $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and $[T_o] = \{A, B_o, C, D_o\}$ are obtained from the equation (4.2). Assume also that the realization $[T]$ is chosen such that $(A^\times, -CD_o^{-1})$ is an isometric pair. Let the inner U be such that UT_o is the minimal degree maximal-phase operator related to T_o and let the inner U_1 be the complement of U_2 in U (i.e. $U_1 = UU_2^*$). Let $\Phi \in \mathcal{D}$ be the unique bounded fixed point solution for*

$$\begin{aligned} \Phi - A_{U_2} \Phi^{(-1)} A_{U_2}^* \\ = Y^{(-1)} (-CD_o^{-1} + A^\times B_o^*) D_o^{-*} D_o^{-1} (-D_o^{-*} C^* + B_o A^{\times*}) Y^{*(-1)}. \end{aligned}$$

Then,

(1) For some block diagonals B'_{V_1} , D_{V_1} and an invertible Cholesky factor $K \in \mathcal{D}$ of Φ , $\Phi = KK^* \in \mathcal{D}$, realizations

$$[V_1]_K = \left[\begin{array}{c|c} \frac{K^{-1}A_{U_2}K^{(-1)}}{B'_{V_1}K^{(-1)}} & \frac{K^{-1}Y^{(-1)}(-CD_o^{-1} + A^\times B_o^*)D_o^{-*}}{D_{V_1}} \end{array} \right]$$

and

$$[G_o^{-1}] = \left[\begin{array}{c|c} A_{U_1} & C_{U_1} \\ \hline a_{21} & c_2 \\ \hline b_1 & d \end{array} \right]$$

where

$$\begin{aligned} a_{21} &:= Y^{(-1)}(-CD_o^{-1} + A^\times B_o^*)D_o^{-*}D_o^{-1}(D_U^* \\ &\quad + B_o B_U^*)B_{U_1} + (\Phi - A_{U_2}\Phi^{(-1)}A_{U_2}^*)A_{U_1}, \\ c_1 &:= Y^{(-1)}(-CD_o^{-1} + A^\times B_o^*)D_o^{-*}D_o^{-1}(D_U^* + B_o B_U^*)D_{U_1} \\ &\quad + (\Phi - A_{U_2}\Phi^{(-1)}A_{U_2}^*)C_{U_1} + A_{U_2}\Phi^{(-1)}B_{U_2}^*, \\ b_1 &:= D_{V_1}D_o^{-1}\{(D_U^* + B_o B_U^*)B_{U_1} + (-D_o^{-*}C^* + B_o A^{\times*})Y^{*(-1)}A_{U_1}, \\ d &:= (D_{V_1} + B'_{V_1}\Phi^{(-1)})D_o^{-1}\{(D_U^* + B_o B_U^*)D_{U_1} \\ &\quad + (-D_o^{-*}C^* + B_o A^{\times*})Y^{*(-1)}C_{U_1}\} \end{aligned}$$

are respectively a u.e.s, unitary and a u.e.s, minimal realization for inner V_1 and a causal G_o^{-1} for which the factorization

$$T = G_o V_1$$

is the outer/inner factorization for T .

(2) G_o characterizes as

$$G_o = U_2 T_o V_1^*.$$

PROOF: Let us start from a minimal, u.e.s and canonical realization $[T^{-1}]$ given by (4.4) for $T^{-1} = T_o^{-1}U_2^*$. Therein, $[B_U \ D_U]$ is an isometric block diagonal coming from a unitary realization for $U = U_1 U_2$. Alternatively to realization $[T^{-1}]$, we may find another minimal, u.e.s canonical realization for T^{-1} starting from the outer/inner representation $T = G_o V_1$ for T for some outer G_o and inner V_1 . When only for our consideration important terms are evaluated, we obtain a minimal realization for T^{-1} in the form

$$[T^{-1}]_{alt.} = \left[\begin{array}{c|c} A_{V_2} & * \\ \hline A_{V_1}^* & * \\ \hline B_{V_2} & C_{V_1}^* \\ \hline & * \end{array} \right]$$

where (A_{V_1}, C_{V_1}) and (A_{V_2}, B_{V_2}) are AC- and AB-pair from unitary realizations for inner V_1 and V_2 respectively for which $(TV_2 =) G_o V_1 V_2$ is the minimal degree maximal-phase operator associated to outer G_o .

By comparing the causal and the anti-causal data from $[T^{-1}]$ and $[T^{-1}]_{alt}$ separately, we conclude from $\ell_{A_{U_1}} < 1$ and $\ell_{A_{U_2}} < 1$, that they are related via boundedly invertible state-space transformations. This fact implies that V_1 has a unitary, u.e.s realization. Equivalently said, the inner U_2 forces V_1 to be inner too. Since $[T^{-1}]$ is a minimal realization, data referring to the anti-causal operator part it realizes define a minimal u.e.s realization too. Hence, the Lyapunov-Stein equation from the statement has the boundedly invertible fixed point $\Phi \in \mathcal{D}$ expressible in the terms from the right side of equation (4.2).

Then, for some B'_{V_1} , D_{V_1} and a boundedly invertible state-space transformation $K \in \mathcal{D}$ being a Cholesky factor of Φ , $\Phi = KK^*$, realization

$$[V_1]_K = \left[\begin{array}{c|c} K^{-1}A_{U_2}K^{(-1)} & K^{-1}Y^{(-1)}(-CD_o^{-1} + A^\times B_o^*)D_o^{-*} \\ \hline B'_{V_1}K^{(-1)} & D_{V_1} \end{array} \right]$$

is a unitary, u.e.s realization for the inner V_1 for which $T = G_o V_1$. A brute force evaluation of $G_o^{-1} = V_1 T^{-1}$ whereby V_1 comes from the realization $[V_1]_K$ and T^{-1} from the realization $[T^{-1}]$ produces the u.e.s realization

$$[G_o^{-1}] := \left[\begin{array}{c|c} A_{U_1} & C_{U_1} \\ \hline a_{21} \ A_{U_2} & c_2 \\ \hline b_1 \ B'_{V_1} & d \end{array} \right]$$

for G_o^{-1} where

$$\begin{aligned} a_{21} &:= Y^{(-1)}(-CD_o^{-1} + A^\times B_o^*)D_o^{-*}D_o^{-1}\{(D_U^* \\ &\quad + B_o B_U^*)B_{U_1} + (-D_o^{-*}C^* + B_o A^{\times*})Y^{*(-1)}A_{U_1}\}, \\ c_2 &:= Y^{(-1)}(-CD_o^{-1} + A^\times B_o^*)D_o^{-*}D_o^{-1}\{(D_U^* + B_o B_U^*)D_{U_1} \\ &\quad + (-D_o^{-*}C^* + B_o A^{\times*})Y^{*(-1)}C_{U_1}\} + A_{U_2}\Phi^{(-1)}B_{U_2}^*, \\ b_1 &:= D_{V_1}D_o^{-1}\{(D_U^* + B_o B_U^*)B_{U_1} \\ &\quad + (-D_o^{-*}C^* + B_o A^{\times*})Y^{*(-1)}A_{U_1}\}, \\ d &:= (D_{V_1} + B'_{V_1}\Phi^{(-1)})D_o^{-1}\{(D_U^* + B_o B_U^*)D_{U_1} \\ &\quad + (-D_o^{-*}C^* + B_o A^{\times*})Y^{*(-1)}C_{U_1}\}. \end{aligned}$$

When the above Lyapunov-Stein equation is employed in a_{21} , c_2 , b_1 and d one obtains the realization $[G_o^{-1}]$ given in the statement. To show that $[G_o^{-1}]$ is a minimal realization, recall that for inner U_2 , V_1 and outer T_o and G_o for which $T (= U_2 T_o = G_o V_1)$ has no inner intrinsic factors at both sides, it holds that minimal realizations for G_o and T_o share, modulo a boundedly invertible state-space transformation, the same state transition operator with T . Hence,

$$\#(G_o^{-1}) = \#(A_{U_2}) + \#(A_{U_1}) = \#(T^{-1}) = \#(T)$$

from which we conclude that $[G_o^{-1}]$ as given in the statement is a minimal, u.e.s realization for G_o^{-1} . This proves the first assertion.

The proof for the second assertion follows immediately from the identity $U_2 T_o = G_o V_1$. This completes the proof. \blacksquare

Lemma 4.15. *Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a locally finite boundedly invertible operator given by a minimal, u.e.s realization $[T] = \{A, B, C, D\}$ and let minimal, u.e.s realizations $[T_o] = \{A, B_o, C, D_o\}$ and unitary u.e.s $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ be realizations for the inner U and outer T_o satisfying $T = U_2 T_o$ both obtained from the equation (4.2). Let $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ be u.e.s unitary realization for the inner U_1 for which $U_1 U_2 T_o$ is a maximal-phase operator. Let $[U] = \{A_U, B_U, C_U, D_U\}$ be a unitary realization for the causal unitary $U = U_1 U_2$ and define $A^\times := A - CD_o^{-1} B_o$. Let also $\Gamma \in \mathcal{D}$ be the unique bounded solution for*

$$A\Gamma^{(-1)}A_{U_2}^* + CD_o^{-1}(-D_o^{-*}C^* + B_oA^{\times*})Y^{*(-1)} = \Gamma.$$

Then, for some block diagonals B'_{V_1}, D_{V_1} , realizations

$$[G_o] = \left[\begin{array}{c|c} A & CD_{V_1} + A\Gamma^{(-1)}B'_{V_1} \\ \hline B & DD_{V_1} + B\Gamma^{(-1)}B'_{V_1} \end{array} \right]$$

and unitary

$$[V_1]_K = \left[\begin{array}{c|c} K^{-1}A_{U_2}K^{(-1)} & K^{-1}Y^{(-1)}(-CD_o^{-1} + A^\times B_o^*)D_o^{-*} \\ \hline B'_{V_1}K^{(-1)} & D_{V_1} \end{array} \right]$$

are minimal, u.e.s realizations for the outer G_o and the inner V_1 for which

$$T = G_o V_1.$$

PROOF: Since T is assumed boundedly invertible, it admits the outer/inner factorization. That is, for some outer G_o and the inner V_1 it holds that $T = G_o V_1$. Then,

$$TV_1^* = G_o$$

or equivalently,

$$V_1 T^{-1} = G_o^{-1}.$$

Hence, the inner V_1 must be such that the previous equations define respectively a causal operator and its causal inverse. In Proposition 4.9 a minimal, u.e.s canonical realization for T^{-1} has been evaluated in the form

$$[T^{-1}] = \left[\begin{array}{c|c} A_{U_1} & C_{U_1} \\ \hline \beta_c & \beta_a \end{array} \left| \begin{array}{c} A_{U_2}^* \\ B_{U_2}^* \\ \delta \end{array} \right. \right] \quad (4.8)$$

where

$$\begin{aligned} \beta_c &:= D_o^{-1} \left\{ (D_U^* + B_o B_U^*) B_{U_1} + (-D_o^{-*} C^* + B_o A^{\times*}) Y^{*(-1)} A_{U_1} \right\}, \\ \beta_a &:= D_o^{-1} (-D_o^{-*} C^* + B_o A^{\times*}) Y^{*(-1)} \text{ and} \\ \delta &:= D_o^{-1} \left\{ (D_U^* + B_o B_U^*) D_{U_1} + (-D_o^{-*} C^* + B_o A^{\times*}) Y^{*(-1)} C_{U_1} \right\}. \end{aligned}$$

Hence, for some $B'_{V_1}, D_{V_1} \in \mathcal{D}$, and a boundedly invertible state-space transformation $K \in \mathcal{D}$ the realization

$$[V_1]_K = \left[\begin{array}{c|c} K^{-1} A_{U_2} K^{(-1)} & K^{-1} Y^{(-1)} (-C D_o^{-1} + A^{\times} B_o^*) D_o^{-*} \\ \hline B'_{V_2} K^{(-1)} & D_{V_1} \end{array} \right]$$

is a unitary, u.e.s realization for the minimal degree inner V_1 satisfying $V_1 T^{-1} \in \mathcal{U}$. According to Lemma 4.12, this choice for V_1 guarantees the outerness of G_o^{-1} a minimal, u.e.s realization of which can be obtained by discarding the anti-causal operator part from the expression $T V_1^*$. For Γ as defined in the statement of this lemma, by applying the mixed partial fraction lemma 2.17, we decouple the mixed causality term in $T V_1^*$ which, after taking only $\mathbf{P}(T V_1^*)$ into account, results in a u.e.s realization $[G_o]$ for the outer G_o as given in the statement. $[G_o]$ is minimal since $\#(G_o) = \#(A)$ and, by construction, G_o is such that $T = G_o V_1$. This completes the proof. ■

Corollary 4.16. *Let the conditions of Lemma 4.15 hold for the boundedly invertible and locally finite operator $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ admitting respectively the inner/outer and the outer/inner factorizations*

$$\begin{aligned} T &= U T_o \\ T &= G_o V. \end{aligned}$$

Then,

$$\#(T) = \#(T_o) \Leftrightarrow \#(T) = \#(G_o).$$

If $\#(T) = \#(T_o)$ then $\#(U) = \#(V)$.

Let us now consider the problem of interchanging outer and causal unitary factors in an inner/outer factorization of a causal boundedly invertible operator T whose causal unitary factor is intrinsic with respect to the original operator. More formally, let for some inner U_i and some outer T_o , the locally finite operator T admit the inner/outer factorization

$$T = U_i T_o$$

where the following degree constraint holds

$$\#(T) = \#(U_i) + \#(T_o).$$

If minimal, u.e.s realization $[T_o] = \{A_o, B_o, C_o, D_o\}$ for T_o and $[U_i] = \{A_{U_i}, B_{U_i}, C_{U_i}, D_{U_i}\}$ for U_i are known then, a minimal u.e.s canonical realization for the inverse T^{-1} can easily be evaluated as

$$[T^{-1}] = \left[\begin{array}{c|c} A_{\times,o} & (-C_o D_o^{-1}) D_{U_i}^* + A_{\times,o} \Gamma_i^{(-1)} B_{U_i}^* \\ \hline A_{U_i}^* & B_{U_i}^* \\ \hline D_o^{-1} B_o D_o^{-1} \{C_{U_i}^* + B_o \Gamma_i^{(-1)} A_{U_i}^*\} & D_o^{-1} \{D_{U_i}^* + B_o \Gamma_i^{(-1)} B_{U_i}^*\} \end{array} \right] \quad (4.9)$$

where

$$A_{\times,o} := A_o - C_o D_o^{-1} C_o$$

and Γ_i is the unique fixed point bounded solution for

$$A_{\times,o} \Gamma_i^{(-1)} A_{U_i}^* + (-C_o D_o^{-1}) C_{U_i}^* = \Gamma_i \quad (4.10)$$

and is guaranteed to exist since $\ell_{A_{\times,o}} < 1$. For simplicity, in the calculations that follow, we will not assume the unitarity of the realization $[U]$.

To find another causal unitary V_i and outer G_o , for which $T = G_o V_i$, we need first to apply the external factorization argument on T^{-1} . Hence, let for some B_{V_i}, D_{V_i} and a boundedly invertible state-space transformation K the realization

$$[V_i] = \left[\begin{array}{c|c} K^{-1} A_{U_i} K^{(-1)} & K^{-1} \{C_{U_i} + A_{U_i} \Gamma_i^{*(-1)} B_o^*\} D_o^{-*} \\ \hline B_{V_i} & D_{V_i} \end{array} \right]$$

be a unitary realization for the causal unitary V_i which is such that

$$G_o^{-1} := V_i T^{-1}$$

is a causal operator. In the following proposition we prove that G_o is actually outer by showing that the operator

$$T V_i^*$$

is causal too. Since evaluation of the product $T V_i^*$ introduces mixed causality terms, to decouple these terms from each other we introduce the bounded block diagonal quantities: P, L and N being respectively unique solutions for the following Lyapunov-Stein equations;

$$A_o P^{(-1)} A_{U_i}^* + C_o D_o^{-1} (C_{U_i}^* + B_o \Gamma_i^{(-1)} A_{U_i}^*) = P, \quad (4.11)$$

$$A_o L^{(-1)} A_{U_i}^* + C_{U_i} (C_{U_i}^* + B_o \Gamma_i^{(-1)} A_{U_i}^*) = L \text{ and} \quad (4.12)$$

$$A_{U_i} N^{(-1)} A_{U_i}^* + C_{U_i} B_o P^{(-1)} A_{U_i}^* = N. \quad (4.13)$$

Proposition 4.17. *Let a locally finite and boundedly invertible operator $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be defined through its inner/outer factorization terms*

$$T = U_i T_o$$

where minimal, u.e.s realizations $[U_i] = \{A_{U_i}, B_{U_i}, C_{U_i}, D_{U_i}\}$ and $[T_o] = \{A_o, B_o, C_o, D_o\}$ for respectively the causal unitary U_i and the outer T_o are given and the following degree constraint holds

$$\#(T) = \#(U_i) + \#(T_o).$$

Let block diagonals Γ_i and bounded P and N be solutions for respectively (4.10), (4.11) and (4.13).

Then, for some block diagonals B_{V_i} and D_{V_i} realizations

$$[V_i] = \left[\begin{array}{c|c} A_{U_i} \{C_{U_i} + A_{U_i} \Gamma_i^{*(-1)} B_o^*\} D_o^{-*} & \\ \hline B_{V_i} & D_{V_i} \end{array} \right] \quad (4.14)$$

and

$$[G_o]_{n,m} = \left[\begin{array}{c|c} A_{U_i} \ C_{U_i} B_o & \{A_{U_i} (L + N)^{(-1)} B_{V_i}^* \\ + C_{U_i} (D_o D_{V_i}^* - B_o \Gamma_i^{(-1)} B_{V_i}^*)\} \\ \hline A_o & -A_o \Gamma_i^{(-1)} B_{V_i}^* + C_o D_{V_i}^* \\ \hline B_{U_i} \ D_{U_i} B_o & \{B_{U_i} (L + N)^{(-1)} B_{V_i}^* \\ + D_{U_i} (D_o D_{V_i}^* - B_o \Gamma_i^{(-1)} B_{V_i}^*)\} \end{array} \right] \quad (4.15)$$

are unitary and a non-minimal realization for respectively the inner V_i and the outer G_o satisfying

$$T = G_o V_i$$

whereby

$$\#(T) = \#(G_o) + \#(V_i).$$

PROOF: Since no cancellations between U_i and T_o are assumed, we evaluate a minimal, u.e.s realization for T by deriving a state-space representation for the product $T = U_i T_o$ and obtain

$$[T] = \left[\begin{array}{c|c} A_{U_i} \ C_{U_i} B_o & C_{U_i} D_o \\ \hline A_o & C_o \\ \hline B_{U_i} \ D_{U_i} B_o & D_{U_i} D_o \end{array} \right].$$

Since we have not constrained $[U]$ to be a unitary realization, without loss of generality, we may assume that the minimal realization

$$[T_o^{-1}] = \{A_o - C_o D_o^{-1} B_o, D_o^{-1} B_o, -C_o D_o^{-1}, D_o^{-1}\}$$

for T_o^{-1} is in output normal form. Let us prove that for the inner V_i given by the unitary realization $[V_i]$ from the statement, the expression $T V_i^*$ is a causal operator. The brute force evaluation of the product $T V_i^*$ yields

$$\begin{aligned}
TV_i^* &= \{D_{U_i}D_o \\
&+ [B_{U_i} \ D_{U_i}B_o] \begin{bmatrix} Z \\ Z \end{bmatrix} \left(\begin{bmatrix} I \\ I \end{bmatrix} - \begin{bmatrix} A_{U_i} & C_{U_i}B_o \\ & A_o \end{bmatrix} \begin{bmatrix} Z \\ Z \end{bmatrix} \right)^{-1} \begin{bmatrix} C_{U_i}D_o \\ C_o \end{bmatrix} \} \\
&\times \{D_{V_i}^* + D_o^{-1}(C_{U_i}^* + B_o\Gamma_i^{(-1)}A_{U_i}^*)Z^*(I - A_{U_i}^*Z^*)^{-1}B_{V_i}^*\} \\
&= \{D_{U_i}D_o + B_{U_i}Z(I - A_{U_i}Z)^{-1}C_{U_i}D_o \\
&\quad - B_{U_i}Z(I - A_{U_i}Z)^{-1}C_{U_i}B_oZ(I - A_oZ)^{-1}C_o \\
&\quad + D_{U_i}B_oZ(I - A_oZ)^{-1}C_o\} \\
&\times \{D_{V_i}^* + D_o^{-1}(C_{U_i}^* + B_o\Gamma_i^{(-1)}A_{U_i}^*)Z^*(I - A_{U_i}^*Z^*)^{-1}B_{V_i}^*\}.
\end{aligned}$$

To be able to proceed with the previous evaluation, we shall employ Γ_i , P , L and N being solutions for (4.10) (4.11), (4.12), (4.13) respectively. However, summing up the Lyapunov-Stein equations in P and Γ_i results in

$$A_o(P + \Gamma_i)^{(-1)}A_{U_i}^* = P + \Gamma_i$$

whose the unique fixed point solution implies that $P = -\Gamma_i$. On the other side, by adding Lyapunov-Stein equations in L and N , we obtain

$$A_{U_i}(L + N)^{(-1)}A_{U_i}^* + C_{U_i}(C_{U_i}^* + B_o(\Gamma_i + P)^{(-1)}A_{U_i}) = L + N$$

which, because $P = -\Gamma_i$, implies that

$$A_{U_i}(L + N)^{(-1)}A_{U_i}^* + C_{U_i}C_{U_i}^* = L + N$$

and means that the minimal, u.e.s realization $[U_i]$ for the causal unitary U_i is a $(L + N)$ -unitary realization. Now, we evaluate that

$$\begin{aligned}
TV_i^* &= D_{U_i}D_oD_{V_i}^* + D_{U_i}(C_{U_i}^* + B_o\Gamma_i^{(-1)}A_{U_i}^*)Z^*(I - A_{U_i}^*Z^*)^{-1}B_{V_i}^* \\
&+ B_{U_i}Z(I - A_{U_i}Z)^{-1}C_{U_i}D_oD_{V_i}^* + B_{U_i}Z(I - A_{U_i}Z)^{-1}A_{U_i}L^{(-1)}B_{V_i}^* \\
&+ B_{U_i}L^{(-1)}B_{V_i}^* + B_{U_i}L^{(-1)}A_{U_i}^*Z^*(I - A_{U_i}^*Z^*)^{-1}B_{V_i}^* \\
&+ B_{U_i}Z(I - A_{U_i}Z)^{-1}C_{U_i}B_oZ(I - A_oZ)^{-1}C_oD_{V_i}^* \\
&+ B_{U_i}Z(I - A_{U_i}Z)^{-1}C_{U_i}B_oZ(I - A_oZ)^{-1}A_oP^{(-1)}B_{V_i}^* \\
&+ B_{U_i}Z(I - A_{U_i}Z)^{-1}C_{U_i}B_oP^{(-1)}B_{V_i}^* - B_{U_i}Z(I - A_{U_i}Z)^{-1}A_{U_i}N^{(-1)}B_{V_i}^* \\
&+ B_{U_i}N^{(-1)}B_{V_i}^* - B_{U_i}N^{(-1)}A_{U_i}^*Z^*(I - A_{U_i}^*Z^*)^{-1}B_{V_i}^* \\
&+ D_{U_i}B_oZ(I - A_oZ)^{-1}C_oD_{V_i}^* \\
&+ D_{U_i}B_oZ(I - A_oZ)^{-1}A_oP^{(-1)}B_{V_i}^* + D_{U_i}B_oP^{(-1)}B_{V_i}^* \\
&+ D_{U_i}B_oP^{(-1)}A_{U_i}^*Z^*(I - A_{U_i}^*Z^*)^{-1}B_{V_i}^*
\end{aligned}$$

$$\begin{aligned}
&= B_{U_i}(L+N)^{(-1)}B_{V_i}^* + D_{U_i}(D_oD_{V_i}^* + B_oP^{(-1)}B_{V_i}^*) \\
&\quad + B_{U_i}Z(I - A_{U_i}Z)^{-1}\left\{A_{U_i}(L+N)^{(-1)}B_{V_i}^* + C_{U_i}(D_oD_{V_i}^* + B_oP^{(-1)}B_{V_i}^*)\right\} \\
&\quad + \left\{D_{U_i}(C_{U_i}^* + B_o\Gamma_i^{(-1)}A_{U_i}^*) + B_{U_i}(L+N)^{(-1)}A_{U_i}^* + D_{U_i}B_oP^{(-1)}A_{U_i}^*\right\} \\
&\quad \times Z^*(I - A_{U_i}^*Z^*)^{-1}B_{V_i}^* \left. \right\} \\
&\quad + D_{U_i}B_oZ(I - A_oZ)^{-1}(C_oD_{V_i}^* + A_oP^{(-1)}B_{V_i}^*) \\
&\quad + B_{U_i}Z(I - A_{U_i}Z)^{-1}C_{U_i}B_oZ(I - A_oZ)^{-1}(C_oD_{V_i}^* + A_oP^{(-1)}B_{V_i}^*)
\end{aligned}$$

where, because of $P = -\Gamma_i$, we find that

$$\mathbf{P}'(TV_i^*) = B_{U_i}(L+N)^{(-1)}A_{U_i}^* + D_{U_i}C_{U_i}^*.$$

This block diagonal is obviously equal to the zero because $[U]$ is a $(L+N)$ -unitary realization. This completes the proof that TV_i^* is a causal operator. We still need to prove that $(TV_i^*)^{-1}$ is also causal. By invoking the minimal canonical realization $[T^{-1}]$ as given by (4.9), we easily note that V_i is a minimal degree causal unitary factor for which V_iT^{-1} ($= (TV_i^*)^{-1}$) is a causal operator. Equivalently, the operator $G_o = TV_i^*$ is an outer operator and it holds that

$$T = G_oV_i = U_iT_o.$$

The terms from the evaluation of TV_i^* can be used to assemble a realization for G_o ; $[G_o]_{n.m}$. Doing so, we obtain the realization (4.15). Let us prove, by reducing to contradiction, that $[G_o]_{n.m}$ is not minimal. Indeed if $[G_o]_{n.m}$ is a minimal realization for G_o , then T has a representation in the form

$$T = G_oV_i$$

where $\#(T) = \#(G_o)$ with V_i being an intrinsic factor of T . Since T admits also the outer/inner factorization, we have that for some inner U' and outer T'_o it holds that

$$G_oV_i = U'T'_o$$

where, by Corollary 4.16, we have that $\#(G_o) = \#(T'_o)$. But the inner/outer factorization for T is degree unique, i.e it must hold that

$$\#(T_o) = \#(T'_o)$$

because U_iT_o is also an inner/outer factorization.

Since $\#(T_o) < \#(T)$, we have achieved the contradiction and completed the proof that $[G_o]_{n.m}$ is a non-minimal realization. \blacksquare

Finally, we consider exchanging inner and outer terms from a general causal invertible operator admitting the inner/outer factorization. Let $T = U_2T_o$ be the inner/outer factorization of a causal T satisfying the degree condition $\#(T) = \#(T_o)$

and given by a minimal u.e.s realization $[T] = \{A, B, C, D\}$. Minimal u.e.s realizations: unitary $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and $[T_o] = \{A, B_o, C, D_o\}$ for respectively U_2 and T_o are assumed known from the square-root algorithm (4.2) in which $Y, Y \in \mathcal{D}$ is a boundedly right invertible block diagonal of maximal row rank.

Let U_i be an inner factor having a proper s-dim sequence, and given by a unitary u.e.s realization $[U_i] = \{A_{U_i}, B_{U_i}, C_{U_i}, D_{U_i}\}$. Define $T_g := U_i U_2 T_o$. In the sequel, we shall apply the following theorem in which a minimal, canonical realization for T^{-1} is evaluated from the data produced at each time instant by the square-root equation (4.2).

Theorem 4.18. *Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a boundedly invertible operator given by a minimal, u.e.s realization $[T] = \{A, B, C, D\}$. Let the inner U_2 and the outer T_o be given by $T = U_2 T_o$. Let also their respective realizations: unitary $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and minimal u.e.s $[T_o] = \{A, B_o, C, D_o\}$, be obtained from the equation (4.2). Let, for some locally square termwise upper triangular $R \in \mathcal{D}$ and isometric $Q_1 \in \mathcal{D}$, the representation $Y = RQ_1$ be a QR-decomposition for $Y \in \mathcal{D}$. Let Q_2 be any isometry for which $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \in \mathcal{D}^{2 \times 1}$ is a unitary. Define $A^\times := A - CD_o^{-1}B_o$ and let Y^\dagger be the Moore-Penrose right pseudo-inverse for Y .*

Then, a minimal u.e.s realization for the mixed causality T^{-1} is found in the following additive form

$$[T^{-1}] = \left[\begin{array}{c|c} Q_2 A^\times Q_2^{*(-1)} & \begin{matrix} c_c \\ B_{U_2}^* \end{matrix} \\ \hline D_o^{-1} B_o Q_2^{*(-1)} & \begin{matrix} b_a \\ d \end{matrix} \end{array} \right] \quad (4.16)$$

where

$$\begin{aligned} c_c &:= Q_2 \left\{ (-CD_o^{-1})D_{U_2}^* - A^\times Y^{\dagger(-1)} B_{U_2}^* \right\}, \\ b_a &:= D_o^{-1} \left\{ C_{U_2}^* - B_o Y^{\dagger(-1)} A_{U_2}^* \right\} \text{ and} \\ d &:= D_o^{-1} \left\{ D_{U_2}^* - B_o Y^{\dagger(-1)} B_{U_2}^* \right\}. \end{aligned}$$

The following theorem about the inverse of a general causal invertible operator holds.

Theorem 4.19. *Let for some inner U_2 and the outer T_o the representation $T = U_2 T_o$ be the inner/outer representation for a causal invertible operator $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ for which the degree constraint*

$$\#(T) = \#(T_o)$$

holds. Assume T is given by a minimal, u.e.s realization $[T] = \{A, B, C, D\}$ and let unitary $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and minimal u.e.s $[T_o] = \{A, B_o, C, D_o\}$ be respective realizations for the inner U_2 and the outer T_o and both obtained from the

equation (4.2) in which $Y \in \mathcal{D}$ is a locally right invertible and of maximal row rank at each of its snapshots. Let, for some locally square termwise upper triangular $R \in \mathcal{D}$ and isometric $Q_1 \in \mathcal{D}$, the representation $Y = RQ_1$ be a QR-decomposition for Y .

Let Q_2 be any isometry for which $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ is a unitary block diagonal.

Let further U_i be an arbitrary inner factor given by a unitary realization $[U_i] = \{A_{U_i}, B_{U_i}, C_{U_i}, D_{U_i}\}$ and having an appropriate s -dim sequence and such that for

$$T_g := U_i T$$

the degree condition $\#(T_g) = \#(U_i) + \#(T)$ holds. Assume a block diagonal E is the bounded solution for the Lyapunov-Stein equation

$$(Q_2 A^\times Q_2^{*(-1)}) E^{(-1)} A_{U_i}^* + Q_2 \left\{ (-CD_o^{-1}) D_{U_2}^* - A^\times Y^\dagger^{(-1)} B_{U_2}^* \right\} C_{U_i}^* = E. \quad (4.17)$$

Then, the inverse T_g^{-1} is given by the following minimal, u.e.s additive realization

$$[T_g^{-1}] = \left[\begin{array}{ccc|c} Q_2 A^\times Q_2^{*(-1)} & & & c_{gc} \\ & A_{U_i}^* & & B_{U_i}^* \\ & B_{U_2}^* C_{U_i}^* & A_{U_2}^* & B_{U_2}^* D_{U_i}^* \\ \hline D_o^{-1} B_o Q_2^{*(-1)} & b_{ga,1} & b_{ga,2} & d_g \end{array} \right]$$

where

$$\begin{aligned} c_{gc} &:= Q_2 \left((-CD_o^{-1}) D_{U_2}^* - A^\times Y^\dagger^{(-1)} B_{U_2}^* \right) D_{U_i}^* + (Q_2 A^\times Q_2^{*(-1)}) E^{(-1)} B_{U_i}^*, \\ b_{ga,1} &:= D_o^{-1} \left\{ \left(D_{U_2}^* - B_o Y^\dagger^{(-1)} B_{U_2}^* \right) C_{U_i}^* + B_o Q_2^{*(-1)} E^{(-1)} A_{U_i}^* \right\}, \\ b_{ga,2} &:= D_o^{-1} \left\{ C_{U_2}^* - B_o Y^\dagger^{(-1)} A_{U_2}^* \right\} \text{ and} \\ d_g &:= D_o^{-1} \left\{ \left(D_{U_2}^* - B_o Y^\dagger^{(-1)} B_{U_2}^* \right) D_{U_i}^* + B_o Q_2^{*(-1)} E^{(-1)} B_{U_i}^* \right\}. \end{aligned}$$

PROOF: For the causal unitary factor U_i as introduced in the statement of this theorem and T whose inverse is given by a minimal realization (4.16) from Theorem 4.18, we can evaluate the product $T^{-1}U_i^*$ by applying the mixed partial fraction lemma in which the bounded solution E of Lyapunov-Stein equation (4.17) is used to decouple the mixed causality term.

Since

$$T^{-1}U_i^* = \mathbf{P}(T^{-1})U_i^* + \mathbf{P}'(T^{-1})U_i^*$$

let us evaluate the terms from the right side separately. We find that

$$\begin{aligned}
\mathbf{P}(T^{-1})U_i^* &= D_o^{-1} \left\{ \left(D_{U_2}^* - B_o Y^{\dagger(-1)} B_{U_2}^* \right) D_{U_i}^* + B_o Q_2^{*(-1)} E^{(-1)} B_{U_i}^* \right\} \\
&+ D_o^{-1} \left\{ \left(D_{U_2}^* - B_o Y^{\dagger(-1)} B_{U_2}^* \right) C_{U_i}^* + B_o Q_2^{*(-1)} E^{(-1)} A_{U_i}^* \right\} Z^* (I - A_{U_i}^* Z^*)^{-1} B_{U_i}^* \\
&+ D_o^{-1} B_o Q_2^{*(-1)} Z (I - Q_2 A^\times Q_2^{*(-1)} Z)^{-1} \left\{ Q_2 \left((-C D_o^{-1}) D_{U_2}^* - A^\times Y^{\dagger(-1)} B_{U_2}^* \right) D_{U_i}^* \right. \\
&\left. + (Q_2 A^\times Q_2^{*(-1)}) E^{(-1)} B_{U_i}^* \right\}
\end{aligned}$$

and that

$$\begin{aligned}
\mathbf{P}'(T^{-1})U_i^* &= D_o^{-1} \left\{ C_{U_2}^* - B_o Y^{\dagger(-1)} A_{U_2}^* \right\} Z^* (I - A_{U_2}^* Z^*)^{-1} B_{U_2}^* D_{U_i}^* \\
&+ \left\{ D_o^{-1} \left\{ C_{U_2}^* - B_o Y^{\dagger(-1)} A_{U_2}^* \right\} \right. \\
&\left. \times Z^* (I - A_{U_2}^* Z^*)^{-1} B_{U_2}^* C_{U_i}^* Z^* (I - A_{U_i}^* Z^*)^{-1} B_{U_i}^* \right\}.
\end{aligned}$$

When put into a single canonical realization for the mixed causality operator, the previous calculations yield the realization $[T_g^{-1}]$ announced in the theorem. To prove that this realization is minimal, recall that $\#(A_{U_2}) = \#(Q_1)$ and that $\#(A) = \#(A^\times) = \#(Q_1) + \#(Q_2)$. Hence,

$$\#(T_g^{-1}) = \#(U_i) + \#(A) = \#(T_g)$$

showing that $[T_g^{-1}]$ is minimal as it is claimed. This completes the proof. \blacksquare

In the following theorem we exploit the statement from Lemma 4.12 to show that the inner V given by the minimal realization $[V]_K$ as defined in Theorem 4.19 is, in fact, the inner factor in an outer/inner factorization for T_g whose inverse T_g^{-1} is given by a minimal additive realization.

Theorem 4.20. *Assume the conditions from Theorem 4.19 hold for a causal $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ admitting inner/outer factorization $T = U_2 T_o$ with the degree constraint $\#(T) = \#(T_o)$ and given by a minimal, u.e.s realization $[T] = \{A, B, C, D\}$. Let $[U_i] = \{A_{U_i}, B_{U_i}, C_{U_i}, D_{U_i}\}$ be a unitary u.e.s realization for an inner factor U_i for which $\#(U_i T) = \#(U_i) + \#(T)$. Define $T_g = U_i T$. Assume that there is a block diagonal Hermitian invertible solution $\Pi \in \mathcal{D}$ for the Lyapunov-Stein equation*

$$\alpha_a \Pi^{(-1)} \alpha_a^* + \beta_a \beta_a^* = \Pi \quad (4.18)$$

where

$$\alpha_a^* := \left[\frac{A_{U_i}^*}{B_{U_2}^* C_{U_i}^* | A_{U_2}^*} \right],$$

$$\beta_a^* := \left[\begin{array}{c} D_o^{-1} \left\{ \left(D_{U_2}^* - B_o Y^{+(-1)} B_{U_2}^* \right) C_{U_i}^* \right. \\ \left. + B_o Q_2^{*(-1)} E^{(-1)} A_{U_i}^* \right\} \\ \left. D_o^{-1} \left\{ C_{U_2}^* - B_o Y^{+(-1)} A_{U_2}^* \right\} \right].$$

Let $F \in \mathcal{D}$ be a locally square invertible Cholesky factor for Π , i.e. $\Pi = FF^*$.

Then, for some outer G_o and for some block diagonals B'_V and D_V for which the realization

$$[V]_F := \left[\frac{F^{-1} \alpha_a F^{(-1)} | F^{-1} \beta_a}{B'_V F^{(-1)} | D_V} \right]$$

is a unitary realization for an inner V , the representation

$$T_g = G_o V$$

is the outer/inner representation for T_g .

PROOF: By Theorem 4.19 in which a minimal canonical realization for T_g^{-1} is expressed in realization terms of U_2 , U_i , T_o and the intermediate block diagonal locally left invertible variable Y , the causal unitary V given by a unitary realization $[V]_F$ provided (4.18) holds, implies the invertibility for $\Pi \in \mathcal{D}$ and is a minimal degree causal unitary factor for which VT_g^{-1} is causal. By Lemma 4.12, T_g has the outer/inner representation

$$T_g = G_o V$$

for some outer G_o . This completes the proof. \blacksquare

To find a minimal, u.e.s realization for G_o , it suffices to evaluate a minimal realization for G_o^{-1} through the product $(G_o^{-1} =) VT_g^{-1}$ where the additive minimal u.e.s realization for $[T_g^{-1}]$ for T_g is stated Theorem 4.19 and inner V is given by a unitary, u.e.s realization $[V]_F$ from the statement.

4.6 Completion of a causal operator to achieve maximal-phase

In this section, we consider again a boundedly invertible causal operator T , given by a minimal u.e.s realization $[T] = \{A, B, C, D\}$ and admitting the inner/outer factorization $T = U_2 T_o$ for the inner U_2 being non-intrinsic w.r.t T , $\#(T) = \#(T_o)$, and the outer T_o with their respective realizations: unitary $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and minimal, u.e.s $[T_o] = \{A, B_o, C, D_o\}$ both obtained from the square-root equation

$$\begin{bmatrix} Y' A Y' C \\ B \quad D \end{bmatrix} = \begin{bmatrix} A_{U_2} & C_{U_2} \\ B_{U_2} & D_{U_2} \end{bmatrix} \begin{bmatrix} Y'^{(-1)} \\ B_o \quad D_o \end{bmatrix} \quad (4.19)$$

in which the intermediate block diagonal Y' is boundedly right invertible and of maximal row rank at each time instant. We are interested in finding another inner U_1 of as minimal as possible degree for which $T_m := U_1 T$ is of a maximal phase. As we will see an extension of Y' , by an appropriate locally right invertible Y'' into a locally square invertible $\begin{bmatrix} Y' \\ Y'' \end{bmatrix}$ will be necessary in the construction of a minimal u.e.s realization for T_m .

Lemma 4.21. *Let T_m be a maximal-phase operator given by a minimal u.e.s regular realization $[T_m] = \{A, B_m, C, D_m\}$. Let the inner U and the outer T_o be such that $T_m = U T_o$ is the outer/inner representation for T_m . Let their respective realizations unitary $[U] = \{A_U, B_U, C_U, D_U\}$ and minimal u.e.s $[T_o] = \{A, B_o, C, D_o\}$ be obtained from the square-root equation*

$$\begin{bmatrix} Y A Y C \\ B_m \quad D_m \end{bmatrix} = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix} \begin{bmatrix} Y^{(-1)} \\ B_o \quad D_o \end{bmatrix} \quad (4.20)$$

in which Y is locally square invertible block diagonal matrix. Let $U = U_1 U_2$ be a factorization for U into two inner factors U_1 and U_2 whose respective unitary realizations $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ and $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ are obtained by means of a properly partitioned unitary $q = [q_1 \quad q_2]$ such that

$$U := \left[\begin{array}{c|c} q^* A_U q^{(-1)} & q^* C_U \\ \hline B_U q^{(-1)} & D_U \end{array} \right] = \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline C_{U_1} B_{U_2} & C_{U_1} D_{U_2} \\ \hline D_{U_1} B_{U_2} & D_{U_1} D_{U_2} \end{array} \right].$$

Then, for any locally square invertible block diagonal $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ partitioned such that $\#(X_1) = \#(U_2)$ and $\#(X_2) = \#(U_1)$ and sharing the same local rank with Y , there is a boundedly invertible state-space transformation K which, when applied on $[T_m]$, gives rise to the alternative minimal, u.e.s realization

$$\begin{aligned} [T_m]_{\text{alt.}} &= \{a_m, b_m, c_m, d_m\} \\ &:= \{K^{-1} A K^{(-1)}, B_m K^{(-1)}, K^{-1} C, D_m\} \end{aligned}$$

for T_m , for which the equation (4.20) has the form

$$\left[\begin{array}{c|c} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} a_m & \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} c_m \\ \hline b_m & d_m \end{array} \right] = \begin{bmatrix} A_{U_2} & C_{U_2} \\ \hline C_{U_1} B_{U_2} & C_{U_1} D_{U_2} \\ \hline D_{U_1} B_{U_2} & D_{U_1} D_{U_2} \end{bmatrix} \left[\begin{array}{c|c} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^{(-1)} & \\ \hline b_o & D_o \end{array} \right] \quad (4.21)$$

where $b_o := B_o K^{(-1)}$.

In addition,

$$K = Y^{-1} [q_1 \ q_2] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

PROOF: We start from equation (4.20) and apply the transformation K as defined in the statement of the lemma onto the realization $[T_m]$. By applying a little algebra, we can rearrange the terms q , X , and Y ; we apply the unitary q to the unitary realization $[U]$ and cancel Y , such that we obtain a new square-root equation. By invoking the definition for q , we immediately obtain equation (4.21) which defines the new square-root equation for obtaining minimal realizations for the inner U and the outer T_o but now expressed in the intermediate parameter X rather than in Y . This completes the proof. ■

Let us establish the link between Y' from (4.19) and a new 'Y' which comes up when we turn T into the maximal-phase one by premultiplying it with an appropriate inner factor of as smallest as possible degree. We again start from the factorization $T = U_2 T_o$ and its realization counterpart (4.19). Let, for some locally square invertible $R' \in \mathcal{D}$ and isometric $Q' \in \mathcal{D}$, the representation $Y' = R'Q'$ be an RQ-representation for Y' . Let also $Q = \begin{bmatrix} Q' \\ Q'' \end{bmatrix} \in \mathcal{D}^{2 \times 2}$ be a unitary embedding of Q' by an isometry Q'' . Let for some appropriately partitioned locally square block lower invertible R and an unitary Q

$$X := \begin{bmatrix} R_{11} \\ R_{21} \ R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

be a QR-decomposition for the arbitrary locally square invertible X as defined in Lemma 4.21.

Let $[U_1]$ be a unitary realization for the inner U_1 for which $T_m = U_1 U_2 T_o$ is a minimal degree maximal-phase operator with a minimal, u.e.s realization $\mathbf{T}_m = \{a_m, b_m, c_m, d_m\}$. After substitution for the above RQ-representation for X into the equation (4.20) from Lemma 4.21, we obtain that

$$\left[\begin{array}{c|c} \left(\begin{bmatrix} R_{11} \\ R_{21} \ R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \right) a_m & \left(\begin{bmatrix} R_{11} \\ R_{21} \ R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \right) c_m \\ \hline b_m & d_m \end{array} \right] \tag{4.22}$$

$$= \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ C_{U_1} B_{U_2} & A_{U_1} C_{U_1} D_{U_2} \\ \hline D_{U_1} B_{U_2} & B_{U_1} D_{U_1} D_{U_2} \end{array} \right] \left[\begin{array}{c|c} \begin{bmatrix} R_{11} \\ R_{21} \ R_{22} \end{bmatrix}^{(-1)} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{(-1)} & \\ \hline b_o & D_o \end{array} \right] \tag{4.23}$$

where $b_o := B_o K^{(-1)}$.

We release the unitary matrix on the right side of the previous equation from the unitary realization $[U_1]$ by shifting it to the left side. So doing, we obtain

$$\begin{aligned} & \left[\begin{array}{c|c} I & \\ \hline A_{U_1}^* & B_{U_2}^* \\ C_{U_2}^* & D_{U_2} \end{array} \right] \left[\begin{array}{c|c} \left(\begin{array}{c|c} R_{11} & Q_1 \\ \hline R_{21} & R_{22} \end{array} \right) a_m & \left(\begin{array}{c|c} R_{11} & Q_1 \\ \hline R_{21} & R_{22} \end{array} \right) c_m \\ \hline b_m & d_m \end{array} \right] \\ &= \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline B_{U_2} & I \\ D_{U_1} & \end{array} \right] \left[\begin{array}{c|c} \left(\begin{array}{c|c} R_{11} & Q_1 \\ \hline R_{21} & R_{22} \end{array} \right)^{(-1)} & \left(\begin{array}{c|c} Q_1 \\ \hline Q_2 \end{array} \right)^{(-1)} \\ \hline b_o & D_o \end{array} \right]. \end{aligned}$$

By deleting the central block row and the central block column from the unitary matrix on the right side of the previous equation we obtain

$$\begin{aligned} & \left[\begin{array}{c|c} I & \\ \hline C_{U_2}^* & D_{U_2} \end{array} \right] \left[\begin{array}{c|c} \left(\begin{array}{c|c} R_{11} & Q_1 \\ \hline R_{21} & R_{22} \end{array} \right) a_m & \left(\begin{array}{c|c} R_{11} & Q_1 \\ \hline R_{21} & R_{22} \end{array} \right) c_m \\ \hline b_m & d_m \end{array} \right] \\ &= \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline B_{U_2} & D_{U_1} \end{array} \right] \left[\begin{array}{c|c} R_{11}^{(-1)} & Q_1^{(-1)} \\ \hline b_o & D_o \end{array} \right]. \end{aligned}$$

Let us include the transformation $K = Y^{-1} [q_1 \ q_2] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, as introduced in the statement from Lemma 4.21, into equation (4.19). We have that

$$\left[\begin{array}{c|c} (Y'K)(K^{-1}AK^{(-1)}) & (Y'K)(K^{-1}C) \\ \hline BK^{(-1)} & D \end{array} \right] = \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline B_{U_2} & D_{U_2} \end{array} \right] \left[\begin{array}{c|c} (Y'K)^{(-1)} & \\ \hline b_o & D_o \end{array} \right]. \quad (4.24)$$

Recall that R and Q from $X = RQ$ are still arbitrary quantities. Let us fix them by making the following choice

$$R_{11}Q_1 = Y'K$$

and let $R_{11} = R'$. After recalling definitions for Y' and K , the previous constraint allows to be worked out further into

$$q_1^*Y = Y'$$

and implies that for some locally right invertible Y'' , Y can be written as

$$Y = [q_1 \ q_2] \begin{bmatrix} Y' \\ Y'' \end{bmatrix}.$$

But then, we conclude that $Y = [q_1 \ q_2] \begin{bmatrix} q_1^*Y \\ Y'' \end{bmatrix}$ holds. By recalling that q is isometric; $I - q_1q_1^* = q_2q_2^*$, we have that $q_2q_2^*Y = q_2Y''$ which, in turn, implies that

$$q_2^*Y = Y''.$$

Define $\Delta_o := A - CD_o^{-1}B_o$ and let block diagonal L' be the bounded solution for the Lyapunov-Stein equation

$$\Delta_o L'^{(-1)} A_{U_2}^* + (-CD_o^{-1}) C_{U_2}^* = L'. \quad (4.25)$$

The second result reveals the connection between Y' from the equation (4.19) and L' defined by the equation (4.25).

The above consideration provides the proof for the following proposition.

Proposition 4.22. *Let T be a causal boundedly invertible operator given by a regular, u.e.s minimal realization $[T] = \{A, B, C, D\}$. Let the inner U_2 and the outer T_o be such that $T = U_2 T_o$. Let also their respective realizations unitary $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and minimal u.e.s $[T_o] = \{A, B_o, C, D_o\}$ be obtained from the square-root algorithm (4.19) in which $Y \in \mathcal{D}$ is locally right invertible of maximal row rank at each of its snapshots.*

Then L' , defined to be the bounded solution for (4.25), admits the representation

$$L' = -Y'^{\dagger}$$

where Y'^{\dagger} is the Moore-Penrose right pseudo-inverse of Y' .

In the particular case when a causal T_m admitting the inner/outer factorization $T_m = UT_o$ is the minimal degree maximal-phase operator, we have that $\#(T_m) = \#(T_o) = \#(U)$ and as the consequence, the square-root equation corresponding to the minimal, u.e.s realization $[T_m] = \{A_m, B_m, C_m, D_m\}$ will deal with the intermediate variable Y which is square invertible and produce realizations; unitary $[U] = \{A_U, B_U, C_U, D_U\}$ and u.e.s, minimal $[T_o] = \{A, B_o, C, D_o\}$ for respectively the inner U and the T_o . On the other side, the operator T_m^{-1} is anti-causal and the external factorization of T_m^{-1} implies finding the bounded solution L for the Lyapunov-Stein equation

$$\Delta_o L^{(-1)} A_U^* + (-CD_o^{-1}) C_U^* = L \quad (4.26)$$

which, because $\#(T) = \#(U)$, is a locally square block diagonal. Then, by Proposition 4.22, we have that

$$L = -Y^{-1}.$$

The following result will be useful in the sequel as well for purpose of interchanging of inner and outer factors from an inner/outer representation for an invertible causal operator .

Theorem 4.23. *Let T be a causal boundedly invertible operator given by a regular, u.e.s minimal realization $[T] = \{A, B, C, D\}$. Let the inner U_2 and the outer T_o be such that $T = U_2 T_o$. Let also their respective realizations: unitary $[U_2] = \{A_{U_2}, B_{u_2}, C_{U_2}, D_{U_2}\}$ and minimal u.e.s $[T_o] = \{A, B_o, C, D_o\}$, be obtained from the*

square-root algorithm (4.19) in which Y is a locally right invertible quantity of maximal row rank at each of its snapshots. Let, for some locally square termwise upper triangular R_{11} and isometric Q' , the representation $Y' = R_{11}Q'$ be a QR-decomposition for Y' . Let Q'' be any isometry for which $\begin{bmatrix} Q' \\ Q'' \end{bmatrix}$ is unitary block diagonal.

Then, a u.e.s and minimal realization for the mixed causality T^{-1} is given by the following additive form

$$[T^{-1}] = \left[\begin{array}{c|c} Q'' \Delta_o Q''^{*(-1)} & \begin{array}{c} c_c \\ B_{U_2}^* \end{array} \\ \hline D_o^{-1} B_o Q''^{*(-1)} & D_o^{-1} \{ C_{U_2}^* - B_o Y'^{\dagger(-1)} A_{U_2}^* \} \\ \hline & d \end{array} \right] \quad (4.27)$$

where Y'^{\dagger} is the Moore-Penrose right pseudo-inverse for Y' and

$$c_c := Q'' \left\{ (-CD_o^{-1}) D_{U_2}^* - \Delta_o Y'^{\dagger(-1)} B_{U_2}^* \right\},$$

$$d := D_o^{-1} \left\{ D_{U_2}^* - B_o Y'^{\dagger(-1)} B_{U_2}^* \right\}.$$

Now, we state the following result.

Proposition 4.24. Let T , a causal, boundedly invertible locally finite operator containing no intrinsic inner factors, be given by a regular, minimal, u.e.s realization $[T] = \{A, B, C, D\}$. Let the inner U_2 and the outer T_o be such that

$$T = U_2 T_o$$

and let their respective realizations; unitary $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and minimal u.e.s $[T_o] = \{A, B_o, C, D_o\}$, be obtained from the algorithm (4.19) in which Y' is a locally right invertible block diagonal with maximal row rank at each time instant. Define Ω to be an invertible Hermitian and the bounded solution for

$$\Delta_o \Omega^{(-1)} \Delta_o^* + (-CD_o^{-1})(-CD_o^{-1})^* = \Omega \quad (4.28)$$

where $\Delta_o := A - CD_o^{-1}B_o$ and let X_Ω be a locally square invertible Cholesky factor of Ω , i.e. $X_\Omega X_\Omega^* = \Omega$ for which the matrix

$$\left[\begin{array}{c|c} A_{U,11} & C_{U,1} \\ \hline A_{U,21} & A_{U,22} \end{array} \middle| \begin{array}{c} C_{U,1} \\ C_{U,2} \end{array} \right] = \left[X_\Omega^{-1} \Delta_o X_\Omega^{(-1)} \middle| X_\Omega^{-1} (-CD_o^{-1}) \right] \quad (4.29)$$

is an isometric AC-pair with the state transition operator in the block lower triangular form whereby $\#(A_{U,11}) = \#(A_{U_2})$. Let also, for some $B_{U,1}$, $B_{U,2}$ and D_U , the realization

$$\left[\begin{array}{c|c} A_{U,11} & C_{U,1} \\ \hline A_{U,21} & A_{U,22} \\ \hline B_{U,1} & B_{U,2} \end{array} \middle| \begin{array}{c} C_{U,1} \\ C_{U,2} \\ \hline D_U \end{array} \right]$$

be unitary. Define ψ and $L = [L' \ L'']$ to be respectively the unique unitary solution for

$$A_{U,11}\psi^{(-1)}A_{U_2}^* + C_{U,1}C_{U_2}^* = \psi \tag{4.30}$$

and the bounded solution for

$$\Delta_o[L' \ L'']^{(-1)} \begin{bmatrix} A_{U_2}^* & \psi^{(-1)}A_{U,21}^* \\ & A_{U,22}^* \end{bmatrix} + (-CD_o^{-1})[C_{U_2}^* \ C_{U,2}^*] = [L' \ L'']. \tag{4.31}$$

Then, L is a locally square invertible block diagonal quantity and realizations

$$[U_1] = \left[\begin{array}{c|c} A_{U_1} & C_{U_1} \\ \hline B_{U_1} & D_{U_1} \end{array} \right] := \left[\begin{array}{c|c} A_{U,21}\psi^{(-1)} & A_{U,22}C_{U,2} \\ \hline B_{U,1}\psi^{(-1)} & B_{U,2}D_{U,2} \end{array} \right] \begin{bmatrix} I & B_{U_2}^* \\ \hline & D_{U_2}^* \end{bmatrix} \tag{4.32}$$

and

$$[T_m] := \left[\begin{array}{c|c} A & C \\ \hline B_m & D_m \end{array} \right] \tag{4.33}$$

$$= \left(\left[\begin{array}{c|c} L & \\ \hline & I \end{array} \right] \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline A_{U,21}\psi^{(-1)} & A_{U,22}C_{U,2} \\ \hline B_{U,1}\psi^{(-1)} & B_{U,2}D_{U,2} \end{array} \right] \left[\begin{array}{c|c} L^{(-1)} & \\ \hline & I \end{array} \right] \right) \left[\begin{array}{c|c} I & \\ \hline & D_o \end{array} \right] \tag{4.34}$$

are respectively a unitary realization for the inner U_1 and a minimal, u.e.s realization for a causal operator T_m which admits the representation $T_m = U_1U_2T_o$.

In addition, T_m is a minimal degree maximal-phase operator.

PROOF: Since T is boundedly invertible, because of $\ell_{\Delta_o} < 1$, the inverse T^{-1} exists as a bounded, possibly mixed causality operator. A u.e.s and non-minimal realization for $T_o^{-1}U_2^*$ can be obtained in the canonical form

$$[T_o^{-1}U_2^*] = \left[\begin{array}{c|c} \Delta_o & -CD_o^{-1}D_{U_2}^* + \Delta_oL'^{(-1)}B_{U_2}^* \\ \hline A_{U_2}^* & B_{U_2}^* \\ \hline D_o^{-1}B_o \ D_o^{-1}(C_{U_2}^* + B_oL'^{(-1)}A_{U_2}^*) & D_o^{-1}(D_{U_2}^* + B_oL'^{(-1)}B_{U_2}^*) \end{array} \right]$$

where, because of $\ell_{A_{U_2}} < 1$, the intermediate quantity L' is the unique bounded solution for

$$\Delta_oL'^{(-1)}A_{U_2}^* + (-CD_o^{-1})C_{U_2}^* = L'. \tag{4.35}$$

The non-minimal part of the previous realization lies in the AC-pair of the causal realization part.

According to Proposition 4.22 we have the following relationship between Y' and L' ;

$$L' = -Y'^{\dagger} \tag{4.36}$$

where Y'^{\dagger} is the Moore-Penrose right pseudo-inverse of Y' . Since T^{-1} is, in general, of a mixed causality, there exists a, modulo an unitary diagonal constant, unique

inner U_1 with a unitary realization $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ having possibly a nontrivial s-dim sequence of minimal degree, such that

$$T^{-1}U_1^*$$

is an anti-causal operator of the minimal degree. By recalling the definitions for Ω and X_Ω and ψ , we obtain that

$$\Delta_o X_\Omega^{(-1)} \begin{bmatrix} A_{U,11}^* & A_{U,21}^* \\ & A_{U,22}^* \end{bmatrix} + (-CD_o^{-1})[C_{U,1}^* \ C_{U,2}^*] = X_\Omega \tag{4.37}$$

where we conclude that for the unitary ψ it holds that

$$\begin{aligned} A_{U_2} &= \psi A_{U,11} \psi^{*(-1)}, \\ C_{U_2} &= \psi C_{U,1}. \end{aligned}$$

Hence, we have that

$$\Delta_o(X_\Omega \begin{bmatrix} \psi & \\ & I \end{bmatrix})^{(-1)} \begin{bmatrix} A_{U_2}^* & \psi^{*(-1)} A_{U,21}^* \\ & A_{U,22}^* \end{bmatrix} + (-CD_o^{-1})[C_{U_2}^* \ C_{U,2}^*] = X_\Omega \begin{bmatrix} \psi & \\ & I \end{bmatrix}. \tag{4.38}$$

The equation (4.38) tells us that L admits the representation

$$L = [L' \ L''] = X_\Omega \begin{bmatrix} \psi & \\ & I \end{bmatrix}$$

where $L' = Y'^\dagger$ and proves that L is locally square invertible. Since L is locally square invertible, the AC-pair

$$\left[\begin{bmatrix} A_{U_2} & \\ A_{U,21} \psi^{(-1)} & A_{U,22} \end{bmatrix} \middle| \begin{bmatrix} C_{U_2} & C_{U,2} \end{bmatrix} \right]$$

with the block lower triangularized A-matrix is a possible candidate for the AC-pair of an unitary realization for $U = U_1 U_2$ which is such that $T_o^{-1}U^*$ is anti-causal and of a minimal degree s-dim sequence. To construct Y being fixed by ψ recall that

$$Y = -L^{-1}.$$

Then, the equation (4.38) can be rewritten equivalently as

$$\left(\begin{bmatrix} Y' \\ Y'' \end{bmatrix} \Delta_o \begin{bmatrix} Y' \\ Y'' \end{bmatrix}^{(-1)} \right) \begin{bmatrix} A_{U_2}^* & \psi^{*(-1)} A_{U,21}^* \\ & A_{U,22}^* \end{bmatrix} + \left(\begin{bmatrix} Y' \\ Y'' \end{bmatrix} (-CD_o^{-1}) \right) [C_{U_2}^* \ C_{U,2}^*] = I \tag{4.39}$$

and obviously implies that

$$\begin{aligned} \begin{bmatrix} Y' \\ Y'' \end{bmatrix} (A - CD_o^{-1}B_o) \begin{bmatrix} Y' \\ Y'' \end{bmatrix}^{(-1)} &= \begin{bmatrix} A_{U_2} & \\ A_{U,21} \psi^{(-1)} & A_{U,22} \end{bmatrix} \\ \begin{bmatrix} Y' \\ Y'' \end{bmatrix} (-CD_o^{-1}) &= \begin{bmatrix} C_{U_2} \\ C_{U,2} \end{bmatrix}. \end{aligned}$$

After a little algebra, the previous two equations recast compactly into

$$\left[\begin{array}{c} Y' \\ Y'' \end{array} \right] A \left[\begin{array}{c} Y' \\ Y'' \end{array} \right] C = \left[\begin{array}{cc} A_{U_2} & C_{U_2} \\ A_{U,21}\psi^{(-1)} & A_{U,22} \end{array} \middle| \begin{array}{c} C_{U_2} \\ C_{U,2} \end{array} \right] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right]. \quad (4.40)$$

In order to obtain a unitary realization for U_1U_2 , let us extend the first matrix on the right hand side of the previous equation with the block row

$$\left[B_{U,1}\psi^{(-1)} \quad B_{U,2} \middle| D_U \right].$$

Since, for some C_{U_1} and D_{U_1} , it holds that

$$\left[\begin{array}{c|c} \psi & \\ \hline A_{U,22} & C_{U_1} \\ \hline B_{U,2} & D_{U_1} \end{array} \right] = \left[\begin{array}{cc|c} A_{U,11} & C_{U,1} & \\ A_{U,21} & A_{U,22} & C_{U,2} \\ \hline B_{U,1} & B_{U,2} & D_U \end{array} \right] \left[\begin{array}{c|c} \psi^{(-1)} & \\ \hline & I \\ \hline & I \end{array} \right] \left[\begin{array}{c|c} A_{U_2}^* & B_{U_2}^* \\ \hline C_{U_2}^* & D_{U_2}^* \end{array} \right]$$

we conclude that the realization $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ where $(A_{U_1}, B_{U_1}) := (A_{U,22}, B_{U,2})$ is, in fact, a unitary realization for U_1 .

Let us define block diagonals B_m and D_m as

$$[B_m | D_m] := \left[B_{U,1}\psi^{(-1)} \quad B_{U,2} \middle| D_U \right] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right] \quad (4.41)$$

and prove that the realization $[T_m] := \{A, B_m, C, D_m\}$ is minimal. It is, by the construction, uniformly observable, since $\{A, B_o, C, D_o\}$ is a minimal, u.e.s realization for the outer factor T_o . Hence, we need to show that the AB-pair; (A, B_m) is uniformly reachable. Therefore, define Φ to be the bounded solution for

$$A^*\Phi A + B_m^*B_m = \Phi^{(-1)}.$$

Also, define Ψ to be the boundedly invertible solution for

$$A^*\Psi A + B_o^*B_o = \Psi^{(-1)}.$$

From (4.41), we find that

$$A^*(Y^*Y)A + B_m^*B_m = (Y^*Y)^{(-1)} + B_o^*B_o.$$

Eliminating $B_o^*B_o$ and $B_m^*B_m$ from the previous three equations yields

$$A^*(Y^*Y - \Phi + \Psi)A = (Y^*Y - \Phi + \Psi)^{(-1)}$$

which, because of $\ell_A < 1$, has the fixed point $Y^*Y - \Phi + \Psi = 0$. Hence, the strict positive definiteness of Φ is induced by the invertibility of Y and follows from

$$\Phi = Y^*Y + \Psi.$$

This proves that the pair (A, B_m) is uniformly reachable and completes the proof that $[T_m]$ is a minimal, u.e.s realization.

Equations (4.40) and (4.41) can be written compactly as

$$\left[\begin{array}{c|c} \left[\begin{array}{c} Y' \\ Y'' \end{array} \right] & \left[\begin{array}{c} A \\ C \end{array} \right] \\ \hline B_m & D_m \end{array} \right] = \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline A_{U,21}\psi^{(-1)} & C_{U_1}D_{U_2} \\ B_{U,1}\psi^{(-1)} & B_{U,2} \\ \hline & D_U \end{array} \right] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right] \tag{4.42}$$

telling us that a causal operator T_m given by a minimal, u.e.s realization $[T_m]$ admits the inner/outer factorization $T_m = (U_1U_2)T_o$ for the minimal degree inner U_1 for which $T^{-1}U_1^*$ is anti-causal. By definition, such an operator is then a minimal degree maximal-phase operator.

When solved in the realization $[T_m] = \{A, B_m, C, D_m\}$, the equation (4.42) looks like as given by (4.33)-(4.34) in the statement of this proposition. This makes the proof complete. ■

By Proposition 4.24, a minimal degree maximal-phase operator $T_m = U_1U_2T_o$ can be constructed by extending the intermediate block diagonal Y' coming from the square-root equation (4.19) with an appropriate Y' .

Lemma 4.25. *Let T , a causal, boundedly invertible locally finite operator containing no intrinsic inner factors, be given by a regular, minimal, u.e.s realization $[T] = \{A, B, C, D\}$. Let the inner U_2 and the outer T_o be such that*

$$T = U_2T_o$$

and let their respective realizations:

$$[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\} \quad \text{and} \quad [T_o] = \{A, B_o, C, D_o\}$$

be obtained from the equation (4.19) in which Y' is a locally right invertible block diagonal of maximal row rank at each time instant. Let, viz. Proposition 4.24, for some locally right invertible block diagonal Y'' , block diagonals B_m and D_m and properly partitioned unitary realization $[U] = \{A_U, B_U, C_U, D_U\}$, the realization

$$\left[\begin{array}{c|c} A & C \\ \hline B_m & D_m \end{array} \right] = \left[\begin{array}{c} Y' \\ Y'' \\ \hline I \end{array} \right]^{-1} \left[\begin{array}{c|c} A_{U,11} & A_{U,12} \\ \hline A_{U,21} & A_{U,22} \\ B_{U,1} & B_{U,2} \\ \hline C_{U,1} & C_{U,2} \\ & D_U \end{array} \right] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right] \tag{4.43}$$

be a minimal u.e.s realization for a minimal degree maximal-phase operator T admitting, for some inner U_1 , the factorization $T_m = U_1T$.

Then,

$$[A_{U,11} \ A_{U,12} | C_{U,1}] = [A_{U_2} \ 0 | C_{U_2}] \tag{4.44}$$

and the AB -pair $(A_{U,22}, B_{U,2})$ is the AB -pair from a unitary realization for U_1 .

PROOF: We write (4.43) as

$$\left[\begin{array}{c|c} \begin{bmatrix} Y' \\ Y'' \end{bmatrix} & A \\ \hline B_m & D_m \end{array} \middle| \begin{array}{c} Y' \\ Y'' \\ C \end{array} \right] = \left[\begin{array}{c|c} A_{U,11} & A_{U,12} \\ A_{U,21} & A_{U,22} \\ \hline B_{U,1} & B_{U,2} \end{array} \middle| \begin{array}{c} C_{U,1} \\ C_{U,2} \\ D_U \end{array} \right] \left[\begin{array}{c|c} Y'^{(-1)} \\ \hline Y''^{(-1)} \\ B_o \\ \hline D_o \end{array} \right] \tag{4.45}$$

and compare the top block row with the top block row of (4.19). The top block row of the latter equation can equivalently be written as

$$[Y'A|Y'C] = [A_{U_2} \ 0|C_{U_2}] \left[\begin{array}{c|c} Y'^{(-1)} \\ \hline Y''^{(-1)} \\ B_o \\ \hline D_o \end{array} \right]$$

which, when compared with (4.45), yields equation (4.44). Since $T_m = U_1T, T = U_2T_o$, the equation (4.45) is a square-root equation for obtaining the unitary realization U for inner $U = U_1U_2$. Because of the appropriate block lower triangular form for A_U in $[U]$, U factorizes in the minimal way as $U = U_1U_2$. But then the AB-pair $(A_{U,22}, B_{U,2})$ must belong to the AB-pair from a unitary realization for U_1 . This completes the proof. ■

Corollary 4.26. *Assume the conditions from Lemma 4.25 hold for the causal boundedly invertible operator T containing no intrinsic inner factors. Let for some properly partitioned locally square block lower triangular R and a unitary $Q = \begin{bmatrix} Q' \\ Q'' \end{bmatrix}$, the block diagonal locally square invertible $Y = \begin{bmatrix} Y' \\ Y'' \end{bmatrix}$ as introduced in Lemma 4.25 admits the QR-decomposition*

$$\begin{bmatrix} Y' \\ Y'' \end{bmatrix} = \begin{bmatrix} R_{11} & \\ & R_{22} \end{bmatrix} \begin{bmatrix} Q' \\ Q'' \end{bmatrix}. \tag{4.46}$$

Then, Q is a unitary transform which when applied to the AC-pair

$$(A - CD_o^{-1}B_o, -CD_o^{-1})$$

produces

$$\left[\begin{array}{c|c} Q'(A - CD_o^{-1}B_o)Q'^{*(-1)} & 0 \\ \hline Q''(A - CD_o^{-1}B_o)Q''^{*(-1)} & Q''(A - CD_o^{-1}B_o)Q''^{*(-1)} \end{array} \middle| \begin{array}{c} Q'(-CD_o^{-1}) \\ \hline Q''(-CD_o^{-1}) \end{array} \right].$$

PROOF: Similarly as we did in the proof of Lemma 4.25, we write the top block row from (4.19) as

$$[Y'A|Y'C] = [A_{U_2} \ 0|C_{U_2}] \left[\begin{array}{c|c} Y'^{(-1)} \\ \hline Y''^{(-1)} \\ B_o \\ \hline D_o \end{array} \right]$$

and encounter the representation (4.46) for Y into it. Hence,

$$R_{11}Q' [A|C] = [A_{U_2} \ 0 | C_{U_2}] \left[\begin{array}{c|c} \left[\begin{array}{cc} R_{11} & \\ R_{21} & R_{22} \end{array} \right]^{(-1)} & \left[\begin{array}{c} Q' \\ Q'' \end{array} \right]^{(-1)} \\ \hline B_o & D_o \end{array} \right]$$

which, after a little algebra, leads to

$$\begin{aligned} R_{11} \left[Q'(A - CD_o^{-1}B_o)Q''^{*(-1)} \quad Q'(A - CD_o^{-1}B_o)Q''^{*(-1)} \middle| Q'(-CD_o^{-1}) \right] \\ = \left[A_{U_2}R_{11}^{(-1)} \quad 0 \middle| C_{U_2} \right]. \end{aligned}$$

Because of the invertibility of R_{11} we conclude that Q is such that

$$Q'(A - CD_o^{-1}B_o)Q''^{*(-1)} = \underline{0}$$

as we needed to prove. This completes the proof. ■

By following the recipe as stated in Proposition 4.24, one can construct the minimal degree maximal-phase operator corresponding to the outer factor arising from a non-intrinsic-inner/outer factorization of a locally finite boundedly invertible time-varying linear operator. As mentioned in the statement of this proposition, solving two Lyapunov-Stein equations is necessary for completions of both inner factor U_2 by an appropriate inner U_1 and the completion of the locally right invertible intermediate variable Y' by an appropriate Y'' to achieve the local squareness and invertibility. If the inner U_1 and Y'' are known then we can assemble the minimal u.e.s realization $[T_m]$ as determined in Proposition 4.24 in an interesting way as stated in the following theorem.

Theorem 4.27. *Let T , a causal, boundedly invertible locally finite operator containing no intrinsic inner factors, be given by a regular, minimal, u.e.s realization $[T] = \{A, B, C, D\}$. Let the inner U_2 and the outer T_o be such that*

$$T = U_2T_o$$

and let their respective realizations:

$$[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\} \quad \text{and} \quad [T_o] = \{A, B_o, C, D_o\}$$

be obtained from the square-root equation (4.19) in which Y' is a locally right invertible block diagonal of maximal row rank at each time instant. Let, for some termwise upper triangular R_{11} and isometric Q' , the decomposition

$$Y' = R_{11}Q'$$

be a QR-decomposition for Y' . And let Q'' be any isometry for which $Q := \begin{bmatrix} Q' \\ Q'' \end{bmatrix}$ is unitary. Let the boundedly invertible quantity $L = [L' \ L'']$ and the realization $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ for the inner U_1 be as stated in Proposition 4.24. Define

$$Y = \begin{bmatrix} Y' \\ Y'' \end{bmatrix} = [Y'^{\ddagger} \ Y''^{\ddagger}]^{-1} = -[L' \ L'']^{-1} \tag{4.47}$$

with the block row partitioning conformably to that of L^* . Define $\Delta_o := A - CD_o^{-1}B_o$. Then,

(1) For the minimal, u.e.s realization $[T_m]$ given by (4.33)-(4.34) it holds that

$$\left[\begin{array}{c|c} A & C \\ \hline B_m & D_m \end{array} \right] = \left[\begin{array}{c|c} \begin{bmatrix} Y' \\ Y'' \end{bmatrix}^{-1} \begin{bmatrix} Y' A \\ C_{U_1} B + A_{U_1} Y''^{(-1)} \end{bmatrix} & \begin{bmatrix} Y' \\ Y'' \end{bmatrix}^{-1} \begin{bmatrix} Y' C \\ C_{U_1} D \end{bmatrix} \\ \hline D_{U_1} B + B_{U_1} Y''^{(-1)} & D_{U_1} D \end{array} \right]. \tag{4.48}$$

(2) The AC-pair

$$C_{U_1} = Y''(\Delta_o Y'^{\ddagger(-1)} B_{U_2}^* + CD_o^{-1} D_{U_2}^*) \tag{4.49}$$

$$A_{U_1} = (Y'' A - C_{U_1} B) Y''^{\ddagger(-1)} \tag{4.50}$$

$$= Y'' \Delta_o Y''^{\ddagger(-1)} \tag{4.51}$$

is the isometric AC-pair from unitary realization $[U_1]$.

PROOF: First, note that saying that the outer factor T_o given by a minimal, u.e.s realization $[T_o] = \{A, B, C_o, D_o\}$ admits

$$T^* T = T_o^* T_o$$

is equivalent to saying that there exists a locally right invertible block diagonal Y' of maximal rank at each of its snapshot for which equations

$$A^* Y'^* Y' A + B^* B = (Y'^* Y')^{(-1)} + B_o^* B_o \tag{4.52}$$

$$A^* Y'^* Y' C + B^* D = B_o^* D_o \tag{4.53}$$

$$C^* Y'^* Y' C + D^* D = D_o^* D_o \tag{4.54}$$

hold. Then, for the embedding of Y' by the appropriate Y'' as announced in the statement of this theorem, into the locally square invertible $Y = \begin{bmatrix} Y' \\ Y'' \end{bmatrix}$ and for the unitary realization $[U_1] = \{A_{U_1}, B_{U_1}, C_{U_1}, D_{U_1}\}$ as stated in Proposition 4.24, the previous set of equations can be written as

$$\begin{aligned}
& A^*Y'^*Y'A + Y''^{*(-1)}\left(A_{U_1}^*C_{U_1} + B_{U_1}^*D_{U_1}\right)B + Y''^{*(-1)}\left(A_{U_1}^*A_{U_1} + B_{U_1}^*B_{U_1}\right)Y''^{*(-1)} \\
& + B^*\left(C_{U_1}^*A_{U_1} + D_{U_1}^*B_{U_1}\right)Y''^{*(-1)} + B^*\left(C_{U_1}^*C_{U_1} + D_{U_1}^*D_{U_1}\right)B \\
& = (Y'^*Y')^{(-1)} + (Y''^*Y'')^{(-1)} + B_o^*B_o \\
& A^*Y'^*Y'C + Y''^{*(-1)}\left(A_{U_1}^*C_{U_1} + B_{U_1}^*D_{U_1}\right)D + B^*\left(C_{U_1}^*C_{U_1} + D_{U_1}^*D_{U_1}\right)D \\
& = B_o^*D_oC^*Y'^*Y'C + D^*\left(C_{U_1}^*C_{U_1} + D_{U_1}^*D_{U_1}\right)D = D_o^*D_o
\end{aligned}$$

which, in turn, is again an identity equivalent to (4.52)-(4.53)-(4.54) and, after a little algebra can be written in a more compact form as

$$\begin{aligned}
& A^*Y'^*Y'A + (Y''^{*(-1)}A_{U_1}^* + B^*C_{U_1}^*)(C_{U_1}B + A_{U_1}Y''^{*(-1)}) \\
& \quad + (Y''^{*(-1)}B_{U_1}^* + B^*D_{U_1}^*)(D_{U_1}B + B_{U_1}Y''^{*(-1)}) \\
& \quad = (Y'^*Y')^{(-1)} + (Y''^*Y'')^{(-1)} + B_o^*B_o \\
& A^*Y'^*Y'C + (Y''^{*(-1)}A_{U_1}^* + B^*C_{U_1}^*)C_{U_1}D + (Y''^{*(-1)}B_{U_1}^* + B^*D_{U_1}^*)D_{U_1}D \\
& \quad = B_o^*D_o \\
& C^*Y'^*Y'C + D^*C_{U_1}^*C_{U_1}D + D^*D_{U_1}^*D_{U_1}D = D_o^*D_o
\end{aligned}$$

from which we recognize the following equation

$$\begin{aligned}
& \left[\frac{A^*Y'^* \quad B^*C_{U_1}^* + Y''^{*(-1)}A_{U_1}^*}{C^*Y'^*} \quad \left| \quad \frac{B^*D_{U_1}^* + Y''^{*(-1)}B_{U_1}^*}{D^*D_{U_1}^*} \right. \right] \left[\begin{array}{c|c} Y'A & Y'C \\ \hline C_{U_1}B + A_{U_1}Y''^{*(-1)} & C_{U_1}D \\ \hline D_{U_1}B + B_{U_1}Y''^{*(-1)} & D_{U_1}D \end{array} \right] \\
& = \left[\frac{Y'^{*(-1)} \quad Y''^{*(-1)} \quad \left| \quad B_o^* \right.}{D_o^*} \right] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right].
\end{aligned}$$

The previous equation implies that there exists some properly partitioned coisometric $[U]$ such that

$$\left[\begin{array}{c|c} Y'A & Y'C \\ \hline C_{U_1}B + A_{U_1}Y''^{(-1)} & C_{U_1}D \\ \hline D_{U_1}B + B_{U_1}Y''^{(-1)} & D_{U_1}D \end{array} \right] = [U] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right] \quad (4.55)$$

To evaluate $[U]$, let us have look at the equation (4.19) first. The latter equation is equivalent to the following equation

$$\left[\begin{array}{c|c} Y'A & Y'C \\ \hline Y''^{(-1)} & 0 \\ \hline B & D \end{array} \right] = \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline I & \\ \hline B_{U_2} & D_{U_2} \end{array} \right] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right]$$

from which, we find that

$$\left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right] = \left[\begin{array}{c|c} A_{U_2}^* & B_{U_2}^* \\ \hline I & \\ \hline C_{U_2}^* & D_{U_2}^* \end{array} \right] \left[\begin{array}{c|c} Y'A & Y'C \\ \hline Y''^{(-1)} & 0 \\ \hline B & D \end{array} \right].$$

Let us substitute the previous equation into (4.55). We obtain

$$\left[\begin{array}{c|c} Y'A & Y'C \\ \hline C_{U_1}B + A_{U_1}Y''^{(-1)} & C_{U_1}D \\ \hline D_{U_1}B + B_{U_1}Y''^{(-1)} & D_{U_1}D \end{array} \right] = [U] \left[\begin{array}{c|c} A_{U_2}^*Y'A + B_{U_2}^*B & A_{U_2}^*Y'C + B_{U_2}^*D \\ \hline Y''^{(-1)} & 0 \\ \hline C_{U_2}^*Y'A + D_{U_2}^*B & C_{U_2}^*Y'C + D_{U_2}^*D \end{array} \right] \quad (4.56)$$

where we have evaluated

$$\left[\begin{array}{c|c} A_{U_2}^* & B_{U_2}^* \\ \hline I & \\ \hline C_{U_2}^* & D_{U_2}^* \end{array} \right] \left[\begin{array}{c|c} Y'A & Y'C \\ \hline Y''^{(-1)} & 0 \\ \hline B & D \end{array} \right] = \left[\begin{array}{c|c} A_{U_2}^*Y'A + B_{U_2}^*B & A_{U_2}^*Y'C + B_{U_2}^*D \\ \hline Y''^{(-1)} & 0 \\ \hline C_{U_2}^*Y'A + D_{U_2}^*B & C_{U_2}^*Y'C + D_{U_2}^*D \end{array} \right].$$

Now, that

$$[U] = \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline C_{U_1}B_{U_2} & A_{U_1}C_{U_1}D_{U_2} \\ \hline D_{U_1}B_{U_2} & B_{U_1}D_{U_1}D_{U_2} \end{array} \right]$$

is the solution of (4.56), can be verified by straightforward substitution followed by the fact that $[U_1]$ and $[U_2]$ are isometric realizations. This also proves that $[U]$ is a unitary realization for the causal unitary U_1U_2 .

From the previous, we conclude that the equation (4.55) can be written as

$$\left[\begin{array}{c|c} Y' & \\ \hline Y'' & \\ \hline & I \end{array} \right] \mathbf{T}_m = \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline C_{U_1}B_{U_2} & A_{U_1}C_{U_1}D_{U_2} \\ \hline D_{U_1}B_{U_2} & B_{U_1}D_{U_1}D_{U_2} \end{array} \right] \left[\begin{array}{c|c} Y'^{(-1)} & \\ \hline Y''^{(-1)} & \\ \hline B_o & D_o \end{array} \right] \quad (4.57)$$

where we define \mathbf{T}_m as

$$\mathbf{T}_m = \left[\begin{array}{c|c} a_m & c_m \\ \hline b_m & d_m \end{array} \right] := \left[\begin{array}{c|c} [Y']^{-1} & \\ \hline Y'' & \\ \hline & I \end{array} \right] \left[\begin{array}{c|c} Y'A & Y'C \\ \hline C_{U_1}B + A_{U_1}Y''^{(-1)} & C_{U_1}D \\ \hline D_{U_1}B + B_{U_1}Y''^{(-1)} & D_{U_1}D \end{array} \right]. \quad (4.58)$$

By comparing the equations (4.33)-(4.34) from the statement of Proposition 4.24 and the equation (4.57), we conclude that

$$\mathbf{T}_m = [T_m].$$

This proves that the equality (4.48) holds and completes the proof for the first assertion.

To prove the second assertion, note that the equation (4.48) implies that

$$Y''A = C_{U_1}B + A_{U_1}Y''^{(-1)}, \text{ and} \quad (4.59)$$

$$Y''C = C_{U_1}D. \quad (4.60)$$

Note also that both matrices on the right side of the equation (4.19) are locally right invertible. So is then the matrix on the left side. Hence, its bottom block row $[B \ D]$ is locally right invertible. Define

$$[B \ D]_r^\ddagger := \begin{bmatrix} Y'^{\ddagger(-1)} & \\ -D_o^{-1}B_oY'^{\ddagger(-1)} & D_o^{-1} \end{bmatrix} \begin{bmatrix} B_{U_2}^* \\ D_{U_2}^* \end{bmatrix}$$

to be the right inverse of $[B \ D]$ corresponding to the right inverse Y'^{\ddagger} for Y' . Let us rewrite (4.59)-(4.60) compactly as

$$Y''[A \ C] = C_{U_1}[B \ D] + A_{U_1}[Y''^{(-1)} \ 0]$$

and postmultiply the previous equation with $[B \ D]_r^\ddagger$. We obtain that

$$\begin{aligned} C_{U_1} &= (Y''[A \ C] - A_{U_1}[Y''^{(-1)} \ 0]) \begin{bmatrix} Y'^{\ddagger(-1)} & \\ -D_o^{-1}B_oY'^{\ddagger(-1)} & D_o^{-1} \end{bmatrix} \begin{bmatrix} B_{U_2}^* \\ D_{U_2}^* \end{bmatrix} \\ &= [Y''A - (Y''A - C_{U_1}B)(Y''^{\ddagger}Y'')^{(-1)} \ Y''C] \begin{bmatrix} Y'^{\ddagger(-1)} & \\ -D_o^{-1}B_oY'^{\ddagger(-1)} & D_o^{-1} \end{bmatrix} \begin{bmatrix} B_{U_2}^* \\ D_{U_2}^* \end{bmatrix} \\ &= [Y''A - (Y''A - C_{U_1}B)(I - Y'^{\ddagger}Y')^{(-1)} \ Y''C] \begin{bmatrix} Y'^{\ddagger(-1)} & \\ -D_o^{-1}B_oY'^{\ddagger(-1)} & D_o^{-1} \end{bmatrix} \begin{bmatrix} B_{U_2}^* \\ D_{U_2}^* \end{bmatrix} \\ &= (Y''AY'^{\ddagger(-1)} - Y''A(I - Y'^{\ddagger}Y')^{(-1)}Y'^{\ddagger(-1)} + C_{U_1}B(I - Y'^{\ddagger}Y')^{(-1)}Y'^{\ddagger(-1)} \\ &\quad - Y''CD_o^{-1}B_oY''^{\ddagger(-1)})B_{U_2}^* + Y''CD_o^{-1}D_{U_2}^* \\ &= Y''(A - CD_o^{-1}B_o)Y'^{\ddagger(-1)}B_{U_2}^* + Y''CD_o^{-1}D_{U_2}^* \\ &= Y''(\Delta_oY'^{\ddagger(-1)}B_{U_2}^* + CD_o^{-1}D_{U_2}^*) \end{aligned}$$

where, in the second line, we used the fact that equation (4.59) solves in A_{U_1} as

$$A_{U_1} = (Y''A - C_{U_1}B)Y''^{\ddagger(-1)}, \quad (4.61)$$

in the third one the identity $Y''^{\ddagger}Y'' = I - Y'^{\ddagger}Y'$ and in the fourth one the identity $C_{U_1}B(I - Y'^{\ddagger}Y')^{(-1)}Y'^{\ddagger(-1)} = 0$. Hence, we have shown that (4.49) holds. The representation for A_{U_1} is given by equation (4.61).

To prove that the representation (4.51) holds, let us start from the equation (4.57) where we have already shown that $\mathbf{T}_m = [T_m]$ and solve it in $[U]$. We obtain

$$\left[\begin{array}{c|c} Y(A - CD_o^{-1}B_o)Y^{(-1)} & YCD_o^{-1} \\ \hline (B_m - D_mD_o^{-1}B_o)Y^{(-1)} & D_mD_o^{-1} \end{array} \right] = \left[\begin{array}{c|c} A_{U_2} & C_{U_2} \\ \hline C_{U_1}B_{U_2} \ A_{U_1} & C_{U_1}D_{U_2} \\ \hline D_{U_1}B_{U_2} \ B_{U_1} & D_{U_1}D_{U_2} \end{array} \right]$$

where we conclude that $A_{U_1} = Y''(A - CD_o^{-1}B_o)Y''^{\ddagger(-1)}$ as needed. This completes the proof. \blacksquare

Proposition 4.28. *Assume the conditions from Theorem 4.27 hold.*

Then, for a properly chosen locally square invertible R_{22} , by setting $Y'' = R_{22}Q''$ in both (4.48) and in the AC-pair (A_{U_1}, C_{U_1}) followed by finding any B_{U_1} and D_{U_1} for which $[U_1]$ is unitary, results in a minimal realization $[T_m]$ given by (4.48) for the minimal degree maximal-phase $T_m (= U_1U_2T_o)$.

PROOF: Viz. Lemma 4.25, we have that Y admits a QR-decomposition in the form

$$Y = \begin{bmatrix} Y' \\ Y'' \end{bmatrix} = \begin{bmatrix} R_{11} & \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q' \\ Q'' \end{bmatrix}$$

for some properly partitioned locally square invertible block lower triangular R and unitary Q . Define

$$[Y'^{\ddagger} \ Y''^{\ddagger}] = \begin{bmatrix} Y' \\ Y'' \end{bmatrix}^{-1}$$

to be the inverse of $\begin{bmatrix} Y' \\ Y'' \end{bmatrix}$ partitioned conformably to $[Y'^* \ Y''^*]$. We need to prove that, among all previous RQ-representations for Y , there is a RQ-representation with $R_{21} = 0$ giving rise to a minimal realization for T_m . Let us start from the equation (4.48) from Theorem 4.27 and rewrite it equivalently as

$$\begin{aligned} \left[\begin{array}{c|c} Q' & \\ \hline Q'' & \\ \hline I & \end{array} \right] & \left[\begin{array}{cc|c} \Delta_o Y'^{\ddagger(-1)} & \Delta_o Y''^{\ddagger(-1)} & CD_o^{-1} \\ * & * & * \end{array} \right] \left[\begin{array}{c|c} A_{U_2}^* & B_{U_2}^* \\ \hline I & \\ \hline C_{U_2}^* & D_{U_2}^* \end{array} \right] \\ & = \left[\begin{array}{ccc|c} R_{11}^{-1} & & & \\ \hline -R_{22}^{-1}R_{21}R_{11}^{-1} & R_{22}^{-1} & & \\ \hline & & & I \end{array} \right] \left[\begin{array}{c|c} I & \\ \hline A_{U_1} & C_{U_1} \\ \hline B_{U_1} & D_{U_1} \end{array} \right]. \end{aligned}$$

From this equation we find that

$$Q''(\Delta_o Y'^{\ddagger(-1)} A_{U_2}^* + CD_o^{-1} C_{U_2}^*) = -R_{22}^{-1} R_{21} R_{11}^{-1}.$$

Let us premultiply both sides of the latter equation with Q''^* and employ the fact that $Q''^* Q'' = I - Q'^* Q'$. We have that

$$(I - Q'^* Q')(\Delta_o Y'^{\ddagger(-1)} A_{U_2}^* + CD_o^{-1} C_{U_2}^*) = -Q''^* R_{22}^{-1} R_{21} R_{11}^{-1}.$$

Let us now premultiply the latter equation with Y' and employ facts that $Y' = R_{11}Q'$, $Q'Q''^* = 0$. Also, viz. the square-root equation (4.19) we employ the fact that

$$Y'(\Delta_o Y'^{\ddagger(-1)} A_{U_2}^* + CD_o^{-1} C_{U_2}^*) = I. \tag{4.62}$$

We obtain

$$I - Y'Q'^*Q'(\Delta_o Y'^{\ddagger(-1)} A_{U_2}^* + CD_o^{-1} C_{U_2}^*) = 0.$$

By substitution of the identity $Q'^*Q' = Q'^*R_{11}^{-1}R_{11}Q' (= Q'^*R_{11}^{-1}Y')$ into the previous equation we obtain

$$Y'Q'^*R_{11}^{-1}\left\{Y'(\Delta_o Y'^{\dagger(-1)})A_{U_2}^* + CD_o^{-1}C_{U_2}^*\right\} = I.$$

By employing the equation (4.62) again, we have that

$$Y'Q'^*R_{11}^{-1} = I$$

implying that

$$Q''^*(-R_{22}^{-1}R_{21}R_{11}^{-1}) = 0 \quad (4.63)$$

in the expression for Y''^{\dagger} . Applying the invertibility argument for R_{ii} , $i = 1, 2$ and the isometry argument for Q'' in the equation (4.63) leads to the conclusion that

$$R_{21} = 0.$$

Hence, Y necessary for construction of the maximal-phase operator $T_m = U_1U_2T_o$ admits the representation

$$Y = \begin{bmatrix} R_{11} & \\ & R_{22} \end{bmatrix} \begin{bmatrix} Q' \\ Q'' \end{bmatrix}.$$

This completes the proof. ■

Proposition 4.28 states that the construction of minimal u.e.s realization $[T_m]$ for the minimal degree maximal-phase $T_m = U_1U_2T_o$ is possible by making a choice for Y in the form

$$Y = \begin{bmatrix} R_{11} & \\ & R_{22} \end{bmatrix} \begin{bmatrix} Q' \\ Q'' \end{bmatrix} \quad (4.64)$$

for an appropriate, boundedly invertible locally square R_{22} . In the following theorem we evaluate the time-forward update $Y''^{(-1)}$ for Y'' .

Theorem 4.29. *Let T be a causal, boundedly invertible locally finite operator given by a regular, minimal and u.e.s realization $[T] = \{A, B, C, D\}$. Let the inner U_2 and the outer T_o be such that*

$$T = U_2T_o$$

is the inner/outer factorization for T whereby $\#(T) = \#(T_o)$ and let their respective realizations; $[U_2] = \{A_{U_2}, B_{U_2}, C_{U_2}, D_{U_2}\}$ and $[T_o] = \{A, B_o, C, D_o\}$ be obtained from the time-forward square-root equation (4.19) in which Y' is a locally right invertible block diagonal of maximal row rank at each time instant. Assume the conditions from Proposition 4.24 hold for T . Define $Y := \begin{bmatrix} Y' \\ Y'' \end{bmatrix} = -L^{-1}$. Let also block diagonals C_{U_1} and A_{U_1} be as stated in Theorem 4.27 by the equations (4.49) and (4.50) respectively. Let, further on, equation (4.64) represent an RQ-factorizations for Y giving rise to a

minimal u.e.s realization $[T_m]$ for a minimal degree maximal-phase T_m from Theorem 4.27.

Assume that, for some properly partitioned block diagonal unitary matrix $u = [u_1 \ u_2]$ and a termwise diagonal invertible σ , the representation

$$I - C_{U_1} C_{U_1}^* = [u_1 \ u_2] \begin{bmatrix} \sigma & \\ & \underline{0} \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}$$

whereby the rank of the underlined zero may disappear, is an SVD² for $I - C_{U_1} C_{U_1}^*$. Then,

(1) $\#(u_1^*) = \#(Q''^{(-1)})$.

(2) The block diagonal

$$h^{(-1)} := u_1^*(Y'' A - C_{U_1} B) Q''^{*(-1)}$$

is square invertible.

(3) The update $R_{22}^{(-1)}$ satisfies the equation

$$R_{22}^{(-1)} R_{22}^{*-(-1)} = h^{(-1)} \sigma h^{*-(-1)}. \tag{4.65}$$

PROOF: For the SVD-decomposition of $I - C_{U_1} C_{U_1}^*$ as presumed in the statement:

$$I - C_{U_1} C_{U_1}^* = [u_1 \ u_2] \begin{bmatrix} \sigma & \\ & 0 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix},$$

one can easily see that u_1 is such that

$$u_1^*(I - C_{U_1} C_{U_1}^*) u_1 = \sigma.$$

According to Theorem 4.27, for the proper choice for R_{22} in the RQ-representation

$$Y'' = R_{22} Q'' \tag{4.66}$$

for Y'' , block diagonal quantities

$$A_{U_1} = (Y'' A - C_{U_1} B) Y''^{\dagger(-1)} \tag{4.67}$$

$$C_{U_1} = Y'' (\Delta_o Y'^{\dagger(-1)} B_{U_2}^* + C D_o^{-1} D_{U_2}^*) \tag{4.68}$$

in which Y'^{\dagger} and Y''^{\dagger} are Moore-Penrose right pseudo-inverse for Y' and Y'' respectively, constitute the isometric AC-pair from a unitary realization for the minimal degree inner U_1 for which $T^{-1} U_1^*$ is anti-causal.

²singular value decomposition

Since, $A_{U_1} A_{U_1}^* = I - C_{U_1} C_{U_1}^*$, we conclude that u_1 is such that $u_1^* (A_{U_1} A_{U_1}^*) u_1 (= \sigma)$ is locally positive definite. When the equation (4.67) is taken into account, we obtain that

$$\left\{ u_1^* (Y'' A - C_{U_1} B) Q''^{*(-1)} R_{22}^{-(-1)} \right\} \left\{ \cdot \right\}^* = \sigma. \quad (4.69)$$

Note that $u_1^* A_{U_1}$ is a well defined product. Then, $u_1^{*(+1)} A_{U_1}^{(+1)}$ is a well defined product too. So is the product $A_{U_1}^{(+1)} u_1$. Since $Q'' \Delta_o Q''^{*(-1)}$ is a well defined product and has the same local rank as the matrix A_{U_1} , we conclude that $Q''^{(+1)} \Delta_o^{(+1)} Q''^*$ has the same local rank as $A_{U_1}^{(+1)}$. Hence,

$$(Q''^{(+1)} \Delta_o^{(+1)} Q''^*) u_1$$

is a well defined product implying the same property for the expression $Q''^* u_1$. But $\#(u_1) = \#(u_1^{*(+1)})$ and we conclude that the product $Q''^* u_1^{*(+1)}$ is also well defined. Applying the one-step forward shift onto the latter expression yields the conclusion that the expression $Q''^{*(-1)} u_1^*$ is well defined. Hence, it holds that

$$\#(u_1^*) = \#(Q''^{(-1)})$$

which completes the proof of the first assertion.

As the consequence of the previous degree constraint we conclude that

$$h^{(-1)} := u_1^* (Y'' A - C_{U_1} B) Q''^{*(-1)}$$

is a locally square matrix. Then, the equation (4.69) can be written as

$$h^{(-1)} R_{22}^{-(-1)} R_{22}^{-*(-1)} h^{*(-1)} = \sigma.$$

Since the left side of the previous equation consists of locally square matrices and because of the invertibility argument of σ , we conclude that $h^{(-1)}$ is a locally invertible block diagonal. This completes the proof of the second assertion.

As the consequence of the second assertion, we use the previous equation and evaluate the update $R_{22}^{(-1)}$, from the QR-representation for $(Y''^{(-1)}) = (R_{22} Q'')^{(-1)}$ given in the equation (4.66) simply by applying a Cholesky factorization on the right side of

$$R_{22}^{-(-1)} R_{22}^{-*(-1)} = h^{(-1)} \sigma h^{*(-1)}.$$

This completes the proof of the theorem. ■

Theorem 4.29 states that inverting the recursion in Lyapunov-Stein equations arising from the construction of minimal realizations for maximal-phase operators is possible although in a not necessarily stable way. The reason for this lies in the singular values for the expression $(A_{U_1} A_{U_1}^*) = I - C_{U_1} C_{U_1}^*$ also mentioned in the latter theorem. Because (A_{U_1}, C_{U_1}) forms an isometric pair, these values are smaller or equal to one

and therefore introduce the exponential accumulation of the error in the evaluation of $R_{22}^{(-1)}$ by means of the equation (4.65).

In obtaining the sequence $A_{U_1,k}$ where $k = k_0, k_0 + 1, \dots$ for some initial time instant k_0 , it was not necessary to invert the expression (4.65) since, for a Cholesky factor R_{22,k_0+1}^{-1} admitting (4.65), we have viz. Theorem 4.27 that

$$A_{U_1,k} = Y_k'' \Delta_{o,k} (Q_{k+1}'' * R_{22,k+1}^{-1})$$

is the state transition matrix in the k -th step from a local unitary realization $[U_1]_k$ for the to be constructed inner U_1 . In evaluation of the update $R_{22,k+1}^{-1}$, we have assumed that $C_{U_1,k}$ is known at the k -th. recursion step. This, in turn, is fulfilled if the value for Y_k'' is known in the k -th. step. Unlike the initial value $Y_{k_0}'' (= R_{11,k_0} Q'_{k_0})$ from the sequence of time-forward square-root equations specified by equation (4.19), the quantity Y_{k_0}'' is not arbitrary but it is constrained by the time-backward nature of the Lyapunov-Stein equation referring to the AC-pair of the causal realization part of T^{-1} whose minimal realization $[T^{-1}]$ is given by the equation (4.27). Once R_{22,k_0} is known, the quantity C_{U_1,k_0} and the first update R_{22,k_0+1} can be evaluated. Using the above equation for $A_{U_1,k}$, we embed $[A_{U_1,k} C_{U_1,k}]$ at each time instant by some $[B_{U_1,k} D_{U_1,k}]$ into sequence of unitary realization $[U_1]_k$ for U_1 and, viz. equation (4.48) from Theorem 4.27, obtain the sequence of minimal, u.e.s realizations $[T_m]_k$ for $T_m = U_1 U_2 T_o$ as claimed in the latter theorem.

Based on Theorems 4.27, 4.29 and the square-root equation (4.19), we state an one-pass algorithm for construction of a minimal u.e.s realization for the minimal degree maximal-phase operator associated to a given causal boundedly invertible operator T . See Figure 4.2.

4.7 Conclusions

In this chapter, two problems were considered: (1) evaluation of the inverse of a causal boundedly invertible locally finite operator T in an alternative way to that proposed in Chapter 3 and (2) expressing the inner and outer terms from an inner/outer factorization for T through the outer and the inner term in its outer/inner decomposition. Bearing in mind the operator inversion problem, the 'alternative way' consists of introducing a new type representation for the causal operator in question; a *maximal-phase/anti-inner* factorization. A maximal-phase boundedly invertible operator is a causal operator having anti-causal bounded inverse. Its inverse however is easy to find (see the realization (4.1)). The maximal-phase/anti-inner factorization is thus a causal/anti-causal factorization for T and inversion of its terms in the inverse order provides the causal/anti-causal factorization for T^{-1} . Because time backward Lyapunov-Stein equations always go together with causal/anti-causal products, we have that representing T and assembling T^{-1} depends on (bounded) solutions of these equations. The benefit of evaluation of a minimal realization for T^{-1} in this

1. **input:** A sequence of minimal realizations $[T]_k = \{A_k, B_k, C_k, D_k\}$ for a causal boundedly invertible operator T admitting the inner/outer factorization $T = U_2 T_o$ whereby $\#(T) = \#(T_o)$.

An initial value for Y'_{init} and a proper initial value for Y''_{init} .

output: A sequence of unitary realizations $[U_1]_k = \{A_{U_1,k}, B_{U_1,k}, C_{U_1,k}, D_{U_1,k}\}$ for the inner U_1 for which $T_m := U_1 T$ is the minimal degree maximal phase operator. Sequence of block diagonals $B_{m,k}$ and $D_{m,k}$ for which, by virtue of Theorem 4.27, the sequence $[T_m]_k = \{A_k, B_{m,k}, C_k, D_{m,k}\}$ is a sequence of minimal u.e.s realizations for T_m .

For each $k = k_{init}, k_{init} + 1, \dots$ do

2. Find square invertible $R_{11,k}$ and isometric Q'_k for which $Y'_k = R_{11,k} Q'_k$ is an RQ -factorization for Y'_k .
3. Find an isometry Q''_k for which $Q_k = \begin{bmatrix} Q'_k \\ Q''_k \end{bmatrix}$ is unitary.
4. Find minimal realizations $[U_2]_k = \{A_{U_2,k}, B_{U_2,k}, C_{U_2,k}, D_{U_2,k}\}$ and $[T_o]_k = \{A_k, B_{o,k}, C_k, D_{o,k}\}$ for the inner U_1 and the outer T_o via square-root equation (4.19).
5. Evaluate $C_{U_1,k}$ viz. Theorem 4.27 as

$$C_{U_1,k} = Y''_k (\Delta_{o,k} Y_k'^{\dagger(-1)} B_{U_2,k}^* + C_k D_{o,k}^{-1} D_{U_2,k}^*)$$

6. Find a unitary $u_k = [u_{1,k} \ u_{2,k}]$ and termwise diagonal positive σ_k for which

$$I - C_{U_1,k} C_{U_1,k}^* = [u_{1,k} \ u_{2,k}] \begin{bmatrix} \sigma_k & \\ & \underline{0} \end{bmatrix} \begin{bmatrix} u_{1,k}^* \\ u_{2,k}^* \end{bmatrix}$$

is an SVD-decomposition for $I - C_{U_1,k} C_{U_1,k}^*$ whereby the rank of the underlined zero might disappear.

7. Evaluate, viz. Theorem 4.29, the following matrix

$$h_{k+1} := u_{1,k}^* (Y''_k A_k - C_{U_1,k} B_k) Q_k''^{*(-1)}$$

8. Find a square invertible Cholesky factor $R_{22,k+1}$ for

$$R_{22,k+1}^{-1} R_{22,k+1}^{-*} = h_{k+1}^{-1} \sigma_k h_{k+1}^{-*}$$

9. Set $Y''_{k+1} = R_{22,k+1} Q''_{k+1}$, $A_{U_1,k} = Y''_k \Delta_{o,k} Y''_{k+1}^{\dagger}$ and find some $B_{U_1,k}$ and $D_{U_1,k}$ for which

$$[U_1]_k := \{A_{U_1,k}, B_{U_1,k}, C_{U_1,k}, D_{U_1,k}\}$$

is a unitary realization.

10. Set

$$[B_{m,k} \ D_{m,k}] = [D_{U_1,k} B_k + B_{U_1,k} Y''_{k+1} \ D_{U_1,k} D_k]$$

11. End.

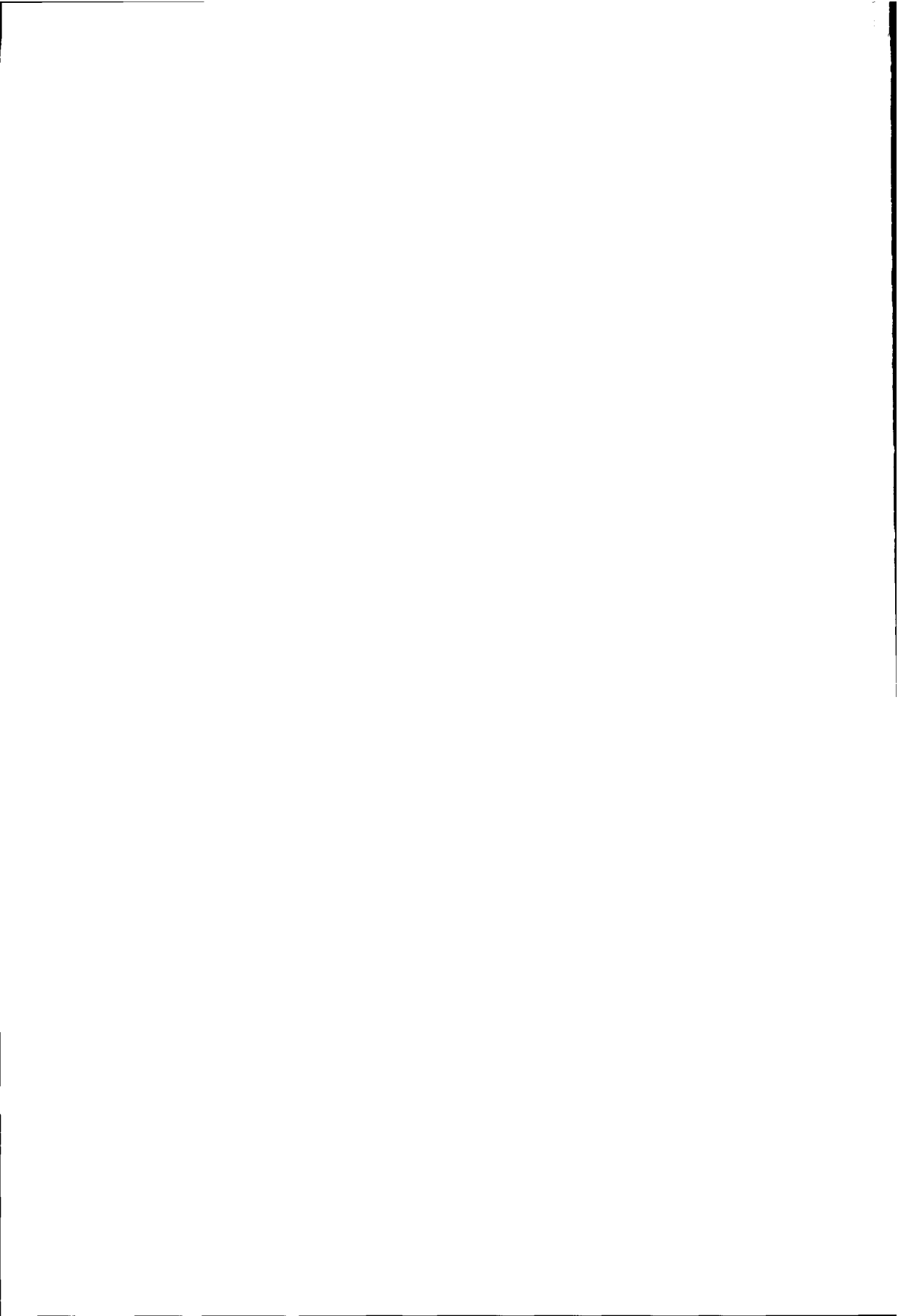
Fig. 4.2. One-pass construction of a minimal degree maximal-phase operator T_m associated to a given causal boundedly invertible T .

way is that more can be said about the geometry of the Hankel map associated to T than this was the case in Chapter 3. The problem of expressing the inner/outer decomposition through outer/inner decomposition terms is also tackled and closed formulas for minimal realizations are given. We approached this problem by considering how to interchange the inner and the outer terms from the inner/outer representation for the maximal-phase factor of minimal degree. Then, we investigated how to extend this consideration to a general causal boundedly invertible locally finite operator. Finally, the problem of one-pass construction of the maximal-phase factor was considered and we have stated an algorithm (see Figure 4.2) for obtaining a minimal realization for it.



Part III

FACTORIZATION



5

INNER/ANTI-INNER FACTORIZATIONS

In this chapter, we prove that every isometric, semi-separable operator Σ of mixed causality and given by a minimal, uniformly exponentially stable (u.e.s) realization $[\Sigma]$ admits a minimal isometric/anti-inner factorization. We also show that state-space properties of isometric Σ can be uniquely specified by single unitary state-space transformation up to a state-space isomorphism. These properties are worked out further to produce isometric, u.e.s realizations for Σ . Dual results concerning coisometric operators are stated too.

5.1 Introduction

Following two problems are considered throughout:

1. Given a linear operator Σ of a mixed causality by a minimal, u.e.s realization $[\Sigma]$, find necessary and sufficient conditions for Σ to be isometric (i.e. $\Sigma\Sigma^* = I$).
2. (The minimal factorization problem) For if Σ is isometric, find a causal isometric Σ_c and anti-inner Σ_a such that Σ admits the representation

$$\Sigma = \Sigma_c \Sigma_a$$

in which state-space complexities of minimal, u.e.s realizations $[\Sigma_c]$ and $[\Sigma_a]$ for respectively Σ_c and Σ_a add up to that of $[\Sigma]$.

Given a transfer function, the system properties are best understood if the transfer function admits a factorization into, at least two, easy-to-understand factors. As an example, solving systems of linear equations involves the inversion of system matrices and, to reduce calculations, one prefers to split the system matrix by means of the well-known RQ-factorization. The inverse is then expressed easily in terms of the inverses of R and Q. Extending QR-decomposition to infinite systems of equations is possible and goes into the realm of the operator theory [Yos71] [Er80] [Tak93] where, viz. Beurling-Lax theorem [Avr98] [DvdV98], the inner/outer factorization framework is used in describing a wide class of causal linear operators and their inverses. To approach the inversion problem for certain class of operators of a mixed causality, the external (or coprime [Alb03]) factorization is needed. By external factorization, it is possible [DvdV98] to write a mixed causality operator as the product of two terms one anti-inner and other causal admitting the inner/outer factorization. Equivalently, the original operator admits an anti-inner/inner/outer factorization. Since in general, the factors can cancel, it is preferable to know, at least from the algorithmic point of view, the conditions under which the computational effort can be reduced. This problem leads to the minimal inner/anti-inner factorization problem we consider throughout.

In the sequel we shall make heavy use of the mixed partial fraction lemma 2.17 stated in Chapter 2 below to split the products of two operators of opposite causalities¹ into the sum of operators of opposite causalities. The following lemma is standard and will be instrumental throughout. For the proof see [DvdV98] (p.344).

Lemma 5.1 (Schur complement/inversion formula). *Let X be a block-partitioned operator,*

$$X = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$$

where A , B and C are bounded operators on Hilbert spaces, and let A and C be self-adjoint (i.e. $A = A^*$, $C = C^*$).

Then,

$$X > 0 \quad \Leftrightarrow \quad \begin{cases} C > 0 \\ A - B^*C^{-1}B > 0 \end{cases}.$$

We conclude this section by giving definitions for Sylvester and Lyapunov-Stein equations.

Definition 5.2 (Sylvester equation). *Let \mathcal{N}_1 , \mathcal{N}_2 , \mathcal{B}_1 and \mathcal{B}_2 be spaces of non-uniform sequences and let $A_1 \in (\mathcal{B}_1, \mathcal{B}_1^{(-1)})$, $A_2 \in (\mathcal{B}_2^{(-1)}, \mathcal{B}_2)$, $C_1 \in \mathcal{D}(\mathcal{B}_1, \mathcal{N}_1)$ and $C_2 \in \mathcal{D}(\mathcal{B}_2^{(-1)}, \mathcal{N}_2)$ be block diagonal operators defined on former spaces. The equation $YA_2^* - A_1Y^{(-1)} = C_1C_2^*$ is called the Sylvester equation in $Y \in \mathcal{D}(\mathcal{B}_1, \mathcal{B}_2)$ corresponding to the AC-pairs (A_i, C_i) , $i = 1, 2$.*

¹one is causal the another one is anticausal

Definition 5.3 (Lyapunov equation). Let \mathcal{N} and \mathcal{B} be spaces of non-uniform sequences and let $A \in (\mathcal{B}, \mathcal{B}^{(-1)})$ and $C \in \mathcal{D}(\mathcal{B}, \mathcal{N})$ be block diagonal operators defined on former spaces.

The equation $AX^{(-1)}A^* + CC^* = X$ is called the Lyapunov-Stein equation in $X \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ corresponding to the AC-pair (A, C) .

We approach the problem as follows. In the beginning, we investigate and describe isometric and coisometric operators simultaneously through state-space terms of their corresponding minimal, u.e.s realizations. Then we show that the minimal factorization problem always has a solution. This fact is used later to produce an alternative isometric, u.e.s realization having its state transition operators block-triangularized. As we know, a certain class of unitary operators has J-unitary counterparts. The output and the input signatures of the J-unitaries introduce a natural partitioning in the unitary operators and minimal realizations for them. Hence, it is of interest to determine the Gramians for these realizations and their mutual dependencies.

5.2 Properties of isometric and coisometric operators

In this section, we investigate state-space properties of isometric and coisometric operators arising from their corresponding defining equations; $\Sigma\Sigma^* = I_{\mathcal{M}}$, $\Sigma^*\Sigma = I_{\mathcal{N}}$. The latter equations reveal the cancelling property - a feature that can be explained through the process of minimization of certain nonminimal realizations defined in terms in which Σ is given.

Given an operator $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ by a minimal, u.e.s realization $[\Sigma]$ defined as the map

$$[\Sigma] = \left[\begin{array}{c|c} A_c & C_c \\ \hline A_a & C_a \\ B_c & B_a \end{array} \middle| D \right] \quad (5.1)$$

$$\left[x_c^{(-1)} \ x_a \middle| y \right] = \left[x_c \ x_a^{(-1)} \middle| u \right] [\Sigma] \quad (5.2)$$

for some state-space vector $[x_c x_a]$, input u and output y , there are invertible Gramians; $A_c, A_a, A_{o,c}, A_{o,a} \in \mathcal{D}$, being unique bounded solutions for the following Lyapunov-Stein equations

$$A_c^* A_c A_c + B_c^* B_c = A_c^{(-1)}, \quad (5.3)$$

$$A_a^* A_a^{(-1)} A_a + B_a^* B_a = A_a, \quad (5.4)$$

$$A_c A_{o,c}^{(-1)} A_c^* + C_c C_c^* = A_{o,c}, \quad (5.5)$$

$$A_a A_{o,a} A_a^* + C_a C_a^* = A_{o,a}^{(-1)}. \quad (5.6)$$

We state the first result.

Proposition 5.4. $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ given by the minimal, u.e.s additive realization $[\Sigma]$ from (5.1) is isometric ($\Sigma\Sigma^* = I_{\mathcal{M}}$) if and only if realization

$$[\mathbf{P}(\Sigma\Sigma^*)] = \left[\begin{array}{c|c} \begin{array}{c} A_c \\ C_c C_a^* \\ A_a^* \end{array} & \begin{array}{c} A_c \Lambda_{o,c}^{(-1)} B_c^* + C_c D^* \\ B_a^* \end{array} \\ \hline \begin{array}{c} B_c B_a \Lambda_{o,a} A_a^* + D C_a^* \\ DD^* + B_c \Lambda_{o,c}^{(-1)} B_c^* + B_a \Lambda_{o,a} B_a^* \end{array} & \end{array} \right] \quad (5.7)$$

for causal operator $\mathbf{P}(\Sigma\Sigma^*)$ is completely nonminimal and

$$DD^* + B_c \Lambda_{o,c}^{(-1)} B_c^* + B_a \Lambda_{o,a} B_a^* = I_{\mathcal{M}}. \quad (5.8)$$

PROOF: We evaluate the product $\Sigma\Sigma^*$ where $\Sigma = D + B_c Z(I - AZ)^{-1} C_c + B_a Z^*(I - A_a Z^*)^{-1} C_a$ and use the mixed partial fraction lemma 2.17 and the equations (5.5)-(5.6) to decompose the mixed causality terms from the product. Since $\Sigma\Sigma^*$ is a Hermitian operator, it suffices to consider a realization for $\mathbf{P}(\Sigma\Sigma^*)$. When evaluated, such a realization is obtainable in the form (5.7). It is completely nonminimal and $\mathbf{P}(\Sigma\Sigma^*) = I_{\mathcal{M}}$ if and only if $\Sigma\Sigma^* = I_{\mathcal{M}}$. ■

The following proposition concerns a coisometric operator of a mixed causality and is the dual of Proposition 5.4.

Proposition 5.5. $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ given by the minimal, u.e.s additive realization $[\Sigma]$ from (5.1) is coisometric ($\Sigma^*\Sigma = I_{\mathcal{N}}$) if and only if realization

$$[\mathbf{P}_{\mathcal{L}}(\Sigma^*\Sigma)] = \left[\begin{array}{c|c} \begin{array}{c} A_a \\ B_c^* B_a \\ A_c^* \end{array} & \begin{array}{c} C_a \\ A_c^* A_c C_c + B_c^* D \end{array} \\ \hline \begin{array}{c} C_a^* \Lambda_a^{(-1)} A_a + D^* B_a \\ C_c^* \end{array} & \begin{array}{c} D^* D + C_c^* A_c C_c + C_a^* \Lambda_a^{(-1)} C_a \end{array} \end{array} \right] \quad (5.9)$$

for anti-causal operator $\mathbf{P}_{\mathcal{L}}(\Sigma^*\Sigma)$ is completely nonminimal and

$$D^* D + C_c^* A_c C_c + C_a^* \Lambda_a^{(-1)} C_a = I_{\mathcal{N}}. \quad (5.10)$$

In the rest of this chapter, for simplicity reasons, we drop out the indices from the unit operators: $I_{\mathcal{M}} \in \mathcal{D}$ and $I_{\mathcal{N}} \in \mathcal{D}$, whenever it is clear from the context what is being meant.

The following theorem states some important facts, we shall use in the sequel, concerning a minimal, u.e.s realization for an isometric operator $\Sigma \in \mathcal{X}$.

Theorem 5.6. Let $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be an isometric operator given by a minimal, u.e.s additive realization $[\Sigma]$ in which (A_c, B_c) and (A_a, C_a) are respectively a causal uniformly reachable and anti-causal uniformly observable pairs. Let $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in$

$\mathcal{D}^{2 \times 2}$ and $\hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \in \mathcal{D}^{2 \times 2}$ be properly partitioned unitary state-space transformations which bring the partially reachable, partially observable and completely nonminimal realization $[\mathbf{P}(\Sigma\Sigma^*)]$ from Proposition 5.4, into Kalman canonical forms

$$\left[\begin{array}{c|c} Q^* & \\ \hline & I \end{array} \right] [\mathbf{P}(\Sigma\Sigma^*)] \left[\begin{array}{c|c} Q^{(-1)} & \\ \hline & I \end{array} \right] = \left[\begin{array}{cc|c} a_{11} & & c_1 \\ a_{21} & a_{22} & c_2 \\ \hline b & 0 & I \end{array} \right], \tag{5.11}$$

$$\left[\begin{array}{c|c} \hat{P}^* & \\ \hline & I \end{array} \right] [\mathbf{P}(\Sigma\Sigma^*)] \left[\begin{array}{c|c} \hat{P}^{(-1)} & \\ \hline & I \end{array} \right] = \left[\begin{array}{cc|c} \hat{a}_{11} & \hat{a}_{12} & \hat{c} \\ & \hat{a}_{22} & 0 \\ \hline \hat{b}_1 & \hat{b}_2 & I \end{array} \right] \tag{5.12}$$

for some properly partitioned block diagonals $a, b, c, \hat{a}, \hat{b}, \hat{c}$ where a_{22} and \hat{a}_{22} are locally of maximal rank. Let uniformly positive definite $\Phi \in \mathcal{D}$ and $\hat{\Psi}_o \in \mathcal{D}$ be respectively the reachability and the observability Gramians corresponding to the causal AB- and AC-pairs; (a_{11}, b) and (\hat{a}_{11}, \hat{c}) from realizations (5.11)-(5.12);

$$a_{11}^* \Phi a_{11} + b^* b = \Phi^{(-1)}, \tag{5.13}$$

$$\hat{a}_{11} \hat{\Psi}_o^{(-1)} \hat{a}_{11}^* + \hat{c} \hat{c}^* = \hat{\Psi}_o. \tag{5.14}$$

Then,

(1) $c_1 = 0$ and $\hat{b}_1 = 0$.

(2) It holds that

$$\hat{P}^* Q = \begin{bmatrix} & \pi_1^* \\ \pi_2^* & \end{bmatrix} \tag{5.15}$$

for some unitaries $\pi_i \in \mathcal{D}$, $i = 1, 2$.

(3) Q_{11}, Q_{22} (and $\hat{P}_{12}, \hat{P}_{21}$) are locally square invertible matrices and it holds that

$$A_c^*(Q_{11}\Phi Q_{11}^*)A_c + B_c^*B_c = (Q_{11}\Phi Q_{11}^*)^{(-1)} \quad (= A_c^{(-1)}) \tag{5.16}$$

$$A_a^*(Q_{22}\pi_1\hat{\Psi}_o\pi_1^*Q_{22}^*)^{(-1)}A_a + B_a^*B_a = Q_{22}\pi_1\hat{\Psi}_o\pi_1^*Q_{22}^* \quad (= A_a) \tag{5.17}$$

where $A_c \in \mathcal{D}$ and $A_a \in \mathcal{D}$ are uniformly positive definite solutions for (5.3) and (5.4) respectively.

(4) $Y_o := Q_{12}Q_{22}^{-1}$ is a solution for the following set equations

$$Y_o A_a^* - A_c Y_o^{(-1)} = C_c C_a^*, \tag{5.18}$$

$$A_c A_{o,c}^{(-1)} B_c^* + C_c D^* = Y_o B_a^*, \tag{5.19}$$

$$B_c Y_o^{(-1)} + B_a A_{o,a} A_a^* + D C_a^* = 0. \tag{5.20}$$

PROOF:

(1) We prove that $c_1 = 0$ and $\hat{b}_1 = 0$.

After applying the state-space transformation Q and after extracting the maximal unreachable subspace from $[\mathbf{P}(\Sigma\Sigma^*)]$, the remaining realization; $\mathbf{R} := \{a_{11}, b, c_1, I\}$, has the uniformly reachable AB-pair; (a_{11}, b) , with the s-dim sequence $\#(a_{11}) \geq \#(A_c)$. However, since $[\mathbf{P}(\Sigma\Sigma^*)]$ is completely nonminimal, the realization \mathbf{R} is an entirely unobservable realization. Equivalently, the observability Gramian corresponding to the AC-pair (a_{11}, c_1) is a zero block diagonal. The latter is possible if and only if $c_1 = 0$. The proof that $\hat{b}_1 = 0$ goes similarly.

(2) Define $K = Q^*\hat{P}$. Then $K \in \mathcal{D}^{2 \times 2}$ relates the realizations given by (5.11) and (5.12) through

$$\left[\begin{array}{cc|c} K_{11} & K_{12} & \\ \hline K_{21} & K_{22} & \\ \hline & & I \end{array} \right] \left[\begin{array}{cc|c} \hat{a}_{11} & \hat{a}_{12} & \hat{c} \\ \hline & \hat{a}_{22} & 0 \\ \hline 0 & \hat{b}_2 & I \end{array} \right] = \left[\begin{array}{cc|c} a_{11} & & 0 \\ \hline a_{21} & a_{22} & c_2 \\ \hline b & 0 & I \end{array} \right] \left[\begin{array}{cc|c} K_{11}^{(-1)} & K_{12}^{(-1)} & \\ \hline K_{21}^{(-1)} & K_{22}^{(-1)} & \\ \hline & & I \end{array} \right].$$

From the previous identity, we find that

$$K_{11}[\hat{a}_{11} \ \hat{c}] = [a_{11}K_{11}^{(-1)} \ 0].$$

Let us postmultiply the previous equation with $\left[\begin{array}{c|c} \hat{\Psi}_o^{(-1)} & \\ \hline & I \end{array} \right]$ where $\hat{\Psi}_o$ is defined by (5.14), and then with its transpose. When the equation (5.14) is substituted in the result we obtain the trivial Lyapunov-Stein equation

$$K_{11}\hat{\Psi}_oK_{11}^* = a_{11}(K_{11}\hat{\Psi}_oK_{11}^*)^{(-1)}a_{11}^*$$

which, because $\ell_{a_{11}} < 1$, has zero as its only bounded fixed point. Hence, $K_{11}\hat{\Psi}_oK_{11}^* = 0$. Because of the uniform positive definiteness of $\hat{\Psi}_o$, we conclude that $K_{11} = 0$. Since K is the product of two unitary matrices, it is unitary on its own. Because $K_{11} = 0$, the latter fact implies that there are some unitaries $\pi_i \in \mathcal{D}$, $i = 1, 2$ for which

$$(\hat{P}^*Q) \ K^* = \begin{bmatrix} & \pi_1^* \\ \pi_2^* & \end{bmatrix}.$$

This completes the proof for the first assertion.

(3) Instead of the equation (5.11), we consider the equivalent equation

$$\left[\begin{array}{c|c} Q^* & \\ \hline & I \end{array} \right] [\mathbf{P}(\Sigma\Sigma^*)] = \left[\begin{array}{cc|c} a_{11} & & c_1 \\ \hline a_{21} & a_{22} & c_2 \\ \hline b & 0 & I \end{array} \right] \left[\begin{array}{c|c} Q^{*(-1)} & \\ \hline & I \end{array} \right] \tag{5.21}$$

from which we find that

$$Q_{11}^*A_c = a_{11}Q_{11}^{*(-1)}. \tag{5.22}$$

Let us postmultiply the equation (5.22) with $Q_{11}^{(-1)}$ and employ the identity $Q_{11}^* Q_{11} = I - Q_{21}^* Q_{21}$ into the result. We obtain

$$Q_{11}^* A_c Q_{11}^{(-1)} = a_{11} - a_{11} Q_{21}^* Q_{21}^{(-1)}.$$

Hence,

$$\#(Q_{11}^* A_c Q_{11}^{(-1)}) = \#(a_{11}) \tag{5.23}$$

Let us now premultiply the equation (5.22) with Q_{11} and employ the identity $Q_{11} Q_{11}^* = I - Q_{12} Q_{12}^*$. This yields

$$A_c - Q_{12} Q_{12}^* A_c = Q_{11} a_{11} Q_{11}^{*(-1)}$$

from which we conclude that

$$\#(A_c) = \#(Q_{11} a_{11} Q_{11}^{*(-1)}). \tag{5.24}$$

Equations (5.23) and (5.24) together provide the proof that Q_{11} is a locally square block diagonal. Since Q is locally square, we have that Q_{22} is locally square as well. *Viz.* the equation (5.15), we conclude that $\hat{P}_{12}(= Q_{11} \pi_2)$ and $\hat{P}_{21}(= Q_{22} \pi_1)$ are locally square too.

Now, we prove that $Q_{11} \in \mathcal{D}$ is locally invertible. From (5.21) we find that

$$\begin{bmatrix} Q_{11}^* A_c \\ B_c \end{bmatrix} = \begin{bmatrix} a_{11} \\ b \end{bmatrix} Q_{11}^{*(-1)}.$$

Let us premultiply the previous equation first with $\begin{bmatrix} \Phi \\ I \end{bmatrix}$ where Φ is defined by (5.13), and then with its transpose. When the identity (5.13) is plugged into the result, we obtain the equation (5.16). The proof that Q_{22} ($= \hat{P}_{21} \pi_1^*$) is invertible as well follows immediately by applying the argument of strict contractivity for Q_{12} and Q_{21} .

To prove the identity (5.17) let us start from the identity

$$[\mathbf{P}(\Sigma \Sigma^*)] \begin{bmatrix} \hat{P}^{(-1)} \\ I \end{bmatrix} = \begin{bmatrix} \hat{P} \\ I \end{bmatrix} \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \hat{c} \\ \hat{a}_{22} & 0 & \\ 0 & \hat{b}_2 & I \end{bmatrix} \tag{5.25}$$

from which we find that

$$[A_a^* \hat{P}_{21}^{(-1)} B_a^*] = \hat{P}_{21} [\hat{a}_{11} \hat{c}]. \tag{5.26}$$

Let us postmultiply the equation (5.26) first with $\begin{bmatrix} \hat{\Psi}_o^{(-1)} \\ I \end{bmatrix}$ where $\hat{\Psi}_o \in \mathcal{D}$ is the uniformly positive definite Hermitian from the statement and then with its transpose. When the equation (5.14) is plugged into the result together with the fact that $\hat{P}_{21}(= Q_{22} \pi_1)$, the equation (5.17) follows straightforwardly.

(4) To prove the last statement, we first premultiply the equation (5.21) with $\begin{bmatrix} Q \\ I \end{bmatrix}$ and then extract (1 - 2), (2 - 2) and (3 - 2)-block identity from the result. This produces

$$A_c Q_{12}^{(-1)} + C_c C_a^* Q_{22}^{(-1)} = Q_{12} a_{22}, \tag{5.27}$$

$$A_a^* Q_{22}^{(-1)} = Q_{22} a_{22} \quad \text{and} \tag{5.28}$$

$$Q_{11}^* (A_c A_{o,c}^{(-1)} B_c^* + C_c D^*) + Q_{21}^* B_a^* = c_1 \tag{5.29}$$

respectively where $c_1 = 0$ viz. the first statement. After premultiplying the equation (5.29) with Q_{11}^{-*} and applying the identity $Q_{11}^{-*} Q_{21}^* = -Q_{12} Q_{22}^{-1}$ ($= -Y_o$), we obtain the equation (5.19). Since Q_{22} is invertible, we solve the equation (5.27) in a_{22} and substitute the result into the equation (5.27). After a little algebra, we produce the equation (5.18) as we needed to show. The equation (5.20) follows immediately after extraction of the equation corresponding to the zero block in B-matrix from the realization on the right side of the equation (5.11), postmultiplying it with $Q_{22}^{-(-1)}$ and putting $Y_o = Q_{12} Q_{22}^{-1}$ into the result.

This completes the proof of the theorem. ■

Corollary 5.7. *Let the conditions from Theorem 5.6 hold for an isometric $\Sigma \in \mathcal{X}$. Then, the set equations (5.18) and (5.19) is equivalent to the set equations (5.18) and (5.20).*

PROOF: The proof follows immediately after conceiving that the (single) unitary state-space transformation Q as introduced in Theorem 5.6 must bring the realization $[\mathbf{P}(\Sigma\Sigma^*)]$ in both Kalman canonical forms simultaneously as the equation (5.11) and the fact $c_1 = 0$ announce. ■

In the following lemma we evaluate the reachability and the observability Gramians for the realization $[\mathbf{P}(\Sigma\Sigma^*)]$ from Proposition 5.4.

Lemma 5.8. *Let $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be an isometric operator given by a minimal, u.e.s additive realization $[\Sigma]$. Let $Y_o \in \mathcal{D}$ be as defined in Theorem 5.6. Then,*

$$\begin{bmatrix} Y_o \\ I \end{bmatrix} A_a [Y_o^* \ I] \quad \text{and} \tag{5.30}$$

$$\begin{bmatrix} I \\ -Y_o^* \end{bmatrix} A_c [I \ -Y_o]. \tag{5.31}$$

are respectively the observability and the reachability Gramians for the realization $[\mathbf{P}(\Sigma\Sigma^*)]$ from Proposition 5.4.

PROOF: Let a properly partitioned $\Pi_o = \begin{bmatrix} \Pi_{o,11} & \Pi_{o,12} \\ \Pi_{o,12}^* & \Pi_{o,22} \end{bmatrix} \in \mathcal{D}^{2 \times 2}$ be the observability Gramian for $[\mathbf{P}(\Sigma\Sigma^*)]$. Then, Π_o is the bounded solution for

$$\begin{bmatrix} A_c & C_c C_a^* \\ & A_a^* \end{bmatrix} \Pi_o^{(-1)} (\cdot)^* + \begin{bmatrix} A_c A_{o,c}^{(-1)} B_c^* + C_c D^* \\ B_a^* \end{bmatrix} (\cdot)^* = \Pi_o.$$

Viz. the equation (5.19) from the last additional statement from Theorem 5.6 we have that

$$\begin{bmatrix} A_c A_{o,c}^{(-1)} B_c^* + C_c D^* \\ B_a^* \end{bmatrix} (\cdot)^* = \begin{bmatrix} Y_o \\ I \end{bmatrix} (\Lambda_a - A_a^* A_a^{(-1)} A_a) [Y_o^* \ I]$$

where we also employed the identity $B_a^* B_a = \Lambda_a - A_a^* A_a^{(-1)} A_a$. By considering each block from Π_o separately, one sets up three constitutive identities from the Lyapunov-Stein equation given above:

$$\begin{aligned} A_a^* \Pi_{o,22}^{(-1)} A_a + B_a^* B_a &= \Pi_{o,22}, \\ A_a^* \Pi_{o,12}^{(-1)} A_c^* + A_a^* A_a^{(-1)} C_a C_c^* + (\Lambda_a - A_a^* A_a^{(-1)} A_a) Y_o^* &= \Pi_{o,12}^*, \\ A_c^* \Pi_{o,11}^{(-1)} A_c^* + C_c C_a^* \Pi_{o,12}^{(-1)} A_c^* + A_c \Pi_{o,12}^{(-1)} C_a C_c^* + C_c C_a^* \Pi_{o,22}^{(-1)} C_a C_c^* \\ &\quad + Y_o (\Lambda_a - A_a^* A_a^{(-1)} A_a) Y_o^* = \Pi_{o,11}. \end{aligned}$$

As $\ell_{A_a} < 1$, the bounded solution for the first equation is $\Pi_{o,22} = \Lambda_a$. When this value is substituted into the second one and the Sylvester equation (5.18) is employed to substitute $C_a C_c^*$ we obtain the trivial Lyapunov-Stein equation

$$A_a^* (\Pi_{o,12}^* - \Lambda_a Y_o^*)^{(-1)} A_c^* = \Pi_{o,12}^* - \Lambda_a Y_o^*$$

with zero as its only bounded solution. Hence, $\Pi_{o,12} = Y_o \Lambda_a$.

Finally, we substitute the values for $\Pi_{o,i2}$, $i = 1, 2$ in the third equation from the set above and employ the equation (5.18) again. After a little algebra, we end up with the trivial Lyapunov-Stein equation

$$A_c (\Pi_{o,11} - Y_o \Lambda_a Y_o^*)^{(-1)} A_c^* = \Pi_{o,11} - Y_o \Lambda_a Y_o^*.$$

Because $\ell_{A_c} < 1$, we conclude that $\Pi_{o,11} = Y_o \Lambda_a Y_o^*$ and complete the proof that the expression (5.30) represents the observability Gramian for $[\mathbf{P}(\Sigma\Sigma^*)]$.

The proof that (5.31) is the reachability Gramian for $[\mathbf{P}(\Sigma\Sigma^*)]$ uses the equations (5.18) and (5.20). It follows the same line of arguments as the previous one and will be omitted. ■

The following lemma states that if a solution Y_o for (5.18)-(5.19) is known then, it is possible to construct a boundedly invertible state-space transformation, which brings the AC-pair from the realization $\mathbf{P}(\Sigma\Sigma^*)$ given in Proposition 5.4 into a Kalman canonical form.

Lemma 5.9. *Let conditions from Theorem 5.6 hold for an isometric $\Sigma \in \mathcal{X}$. Let $Y_o \in \mathcal{D}$ be a solution for (5.18)-(5.19). Let q_{22} be any locally invertible block diagonal of an appropriate s -dim sequence. Define $q_{12} = Y_o q_{22}$. Let $R, Q_{12}, Q_{22} \in \mathcal{D}$ be such that*

$$\begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix} = \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} R$$

is a QR-decomposition for (locally left invertible) $\begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix}$. Let also $Q_{11}, Q_{21} \in \mathcal{D}$ be

such that $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ is a unitary matrix.

Then, Q brings the partially unobservable pair from the realization $\mathbf{P}(\Sigma\Sigma^)$ as introduced in Proposition 5.4 into a Kalman canonical form.*

PROOF: Since q_{22} and R are invertible, we will have that q_{11}, Q_{11}, Q_{22} are invertible too. Also, we have that $Q_{i2} = q_{i2}R^{-1}$, $i = 1, 2$ and because Q is unitary, it holds that

$$Q_{11}^{-*} Q_{21}^* = -Q_{12} Q_{22}^* = -q_{12} q_{22}^{-1} = -Y_o. \quad (5.32)$$

Let us apply the unitary state-space transformation $Q \in \mathcal{D}^{2 \times 2}$ on the AC-pair from the realization $\mathbf{P}(\Sigma\Sigma^*)$. We have that

$$Q^* \begin{bmatrix} A_c & C_c C_a^* \\ & A_a^* \end{bmatrix} \begin{bmatrix} A_c A_{o,c}^{(-1)} B_c^* + C_c D^* \\ B_a^* \end{bmatrix} \begin{bmatrix} Q^{(-1)} \\ I \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \end{bmatrix}. \quad (5.33)$$

By extraction of (1-2) and (1-3)-block from the previous identity, we obtain that

$$Q_{11}^* A_c Q_{12}^{(-1)} + Q_{11}^* C_c C_a^* Q_{22}^{(-1)} + Q_{21}^* A_a^* Q_{22}^{(-1)} = a_{12}, \quad (5.34)$$

$$Q_{11}^* (A_c A_{o,c}^{(-1)} B_c^* + C_c D^*) + Q_{21}^* B_a^* = c_1. \quad (5.35)$$

The second equation from the previous set is equivalent to

$$A_c A_{o,c}^{(-1)} B_c^* + C_c D^* - Y_o B_a^* = Q_{11}^{-*} c_1$$

and since its left side is zero by assumption, we conclude that $c_1 = 0$. Because $Q_{ii}, i = 1, 2$ are invertible, we pre- and postmultiply the first equation from the previous set with Q_{11}^{-*} and $Q_{22}^{-(-1)}$ and employ the identity (5.32). We obtain

$$A_c (q_{12} q_{22}^{-1})^{(-1)} + C_c C_a^* - (q_{12} q_{22}^{-1}) A_a^* = Q_{11}^{-*} a_{12} Q_{22}^{-(-1)}.$$

By the assumption, the left side from the previous equation is equal zero. Hence, $a_{12} = 0$ and we complete the proof that the right side of (5.33) is a Kalman canonical form. This completes the proof. \blacksquare

The dual result to that stated in Theorem 5.6 considers coisometric operators of a mixed causality and is stated in the next theorem.

Theorem 5.10. *Let $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be a coisometric operator given by a minimal, u.e.s realization $[\Sigma]$ in which (A_c, C_c) and (A_a, C_a) are respectively a causal uniformly observable and anti-causal uniformly observable pairs. Let $\hat{Q} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix} \in \mathcal{D}^{2 \times 2}$ and $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{D}^{2 \times 2}$ be properly partitioned unitary state-space transformations which bring the partially reachable, partially observable and entirely nonminimal realization $[\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)]$ from Proposition 5.5, into Kalman canonical forms*

$$\left[\frac{\hat{Q}^{(-1)}}{I} \right] [\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)] \left[\frac{\hat{Q}^*}{I} \right] = \begin{bmatrix} \hat{\alpha}_{11} & \hat{\alpha}_{22} & \hat{\gamma}_1 \\ \hat{\beta} & 0 & I \end{bmatrix}, \tag{5.36}$$

$$\left[\frac{P^{(-1)}}{I} \right] [\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)] \left[\frac{P^*}{I} \right] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \gamma \\ \alpha_{22} & 0 & \\ \beta_1 & \beta_2 & I \end{bmatrix} \tag{5.37}$$

for some properly partitioned $\alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \mathcal{D}$ where α_{22} and $\hat{\alpha}_{22}$ are locally of maximal rank. Let uniformly positive definite $\hat{\Phi} \in \mathcal{D}$ and $\Psi_o \in \mathcal{D}$ be respectively the reachability and the observability Gramians corresponding to the anti-causal AB- and AC-pairs: $(\hat{\alpha}_{11}, \hat{\beta})$ and (α_{11}, γ) from the realizations (5.36)-(5.37);

$$\hat{\alpha}_{11}^* \hat{\Phi}^{(-1)} \hat{\alpha}_{11} + \hat{\beta}^* \hat{\beta} = \hat{\Phi}, \tag{5.38}$$

$$\alpha_{11} \Psi_o \alpha_{11}^* + \gamma \gamma^* = \Psi_o^{(-1)}. \tag{5.39}$$

Then,

(1) $\hat{\gamma}_1 = 0$ and $\beta_1 = 0$.

(2) It holds that

$$P \hat{Q}^* = \begin{bmatrix} \tau_2^* \\ \tau_1^* \end{bmatrix} \tag{5.40}$$

for some unitary block diagonals $\tau_i \in \mathcal{D}$, $i = 1, 2$,

(3) P_{11}, P_{22} (and $\hat{Q}_{12}, \hat{Q}_{21}$) are locally square invertible matrices and it holds that

$$A_c (P_{22}^* \tau_2^* \hat{\Phi} \tau_2 P_{22})^{(-1)} A_c^* + C_c C_c^* = P_{22}^* \tau_2^* \hat{\Phi} \tau_2 P_{22} \quad (= \Lambda_{o,c}) \tag{5.41}$$

$$A_a (P_{11}^* \Psi_o P_{11}) A_a^* + C_a C_a^* = (P_{11}^* \Psi_o P_{11})^{(-1)} \quad (= \Lambda_{o,a}^{(-1)}) \tag{5.42}$$

where $\Lambda_{o,c} \in \mathcal{D}$ and $\Lambda_{o,a} \in \mathcal{D}$ are uniformly positive definite solutions for (5.5) and (5.6) respectively.

(4) $Y_c := P_{12}^* P_{11}^{-1}$ is a solution for the following set equations

$$Y_c^{(-1)} A_a - A_c^* Y_c = B_c^* B_a, \tag{5.43}$$

$$A_c^* \Lambda_c C_c + B_c^* D = Y_c^{(-1)} C_a, \tag{5.44}$$

$$C_a^* \Lambda_a^{(-1)} A_a + D^* B_a + C_c^* Y_c = 0. \tag{5.45}$$

Dually to Lemma 5.8 we state the following result without the proof.

Lemma 5.11. *Let $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be a coisometric operator given by a minimal, u.e.s. additive realization $[\Sigma]$. Let $Y_c \in \mathcal{D}$ be as defined in Theorem 5.10.*

Then,

$$\begin{bmatrix} I \\ Y_c \end{bmatrix} A_{o,a} [I \ Y_c^*] \quad \text{and} \quad (5.46)$$

$$\begin{bmatrix} -Y_c^* \\ I \end{bmatrix} A_{o,c} [-Y_c \ I] \quad (5.47)$$

are respectively the observability and the reachability Gramians for the realization $[\mathbf{P}(\Sigma^ \Sigma)]$ from Proposition 5.5.*

The following proposition states that it is always possible to factorize a mixed causality isometric operator into the product of a causal-isometric and an anti-inner terms in a minimal way. Moreover, it states necessary and sufficient conditions for a mixed causality operator to be isometric.

Proposition 5.12. *Let an operator $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s. additive realization $[\Sigma] = \{A_c, B_c, C_c, A_a, B_a, C_a, D\}$. Let $[\Sigma_a] = \{A_a, b_a, C_a, d_a\}$ be a minimal, u.e.s. realization for an anti-inner operator Σ_a .*

Then, Σ is isometric if and only if there exists a properly partitioned unitary $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathcal{D}^{2 \times 2}$ for which $Y_o = Q_{12} Q_{22}^{-1}$ satisfies equations (5.18) and (5.20) from Theorem 5.6 and the minimal, u.e.s. realization

$$[\Sigma_c] = \begin{bmatrix} A_c & C_c d_a^* - Y_o b_a^* \\ B_c & B_a A_{o,a} b_a^* + D d_a^* \end{bmatrix} \quad (5.48)$$

is a realization for the isometric operator Σ_c .

In addition, the factorization $\Sigma = \Sigma_c \Sigma_a$ is a minimal causal-isometric/anti-inner factorization and

(1) Y_o is the bounded solution for the Lyapunov-Stein equation

$$A_c^* (A_c Y_o) A_c^* - B_c^* B_a A_{o,a} A_c^* - (A_c^* A_c C_c + B_c^* D) C_a^* = (A_c Y_o)^{(-1)}, \quad (5.49)$$

(2) the observability Gramian for $[\Sigma_c]$ equals

$$A_{o,c} + Y_o A_{o,a}^{-1} Y_o^*. \quad (5.50)$$

PROOF: (\Rightarrow) Let Σ be isometric. Since $[\Sigma]$ is assumed minimal and u.e.s, the observability pair (A_a, C_a) has the invertible Gramian $\Lambda_{o,a}$, and the anti-inner Σ_a with u.e.s, the anti-inner Σ given by u.e.s, unitary realization $[\Sigma_a]$ from the statement is such that $\Sigma_c := \Sigma \Sigma_a^*$ is a causal isometric operator. By evaluating the expression $\Sigma \Sigma_a^*$ in which we apply the mixed partial fraction lemma to decompose the mixed causality terms, one obtains a u.e.s realization for Σ_c in the form

$$[\Sigma_c]_{non-min.} = \left[\begin{array}{c|c} A_c & C_c C_a^* \\ \hline & A_a^* \\ \hline B_c B_a \Lambda_{o,a} A_a^* + D C_a^* & B_a \Lambda_{o,a} b_a^* + D d_a^* \end{array} \right].$$

But the AB-pair from the previous realization is the same as the AB-pair from the realization $[\mathbf{P}(\Sigma \Sigma^*)]$ given in Proposition 5.4. On the other side, Theorem 5.6 states that the unitary state-space transformation Q brings the previously mentioned AB-pair in the Kalman canonical form. After successively, applying Q on the realization $[\Sigma_c]_{non-min.}$, extracting all unreachable modes, and applying the invertibility argument on Q_{11} , we obtain the u.e.s realization $[\Sigma_c]$ for Σ_c given in the statement. By Theorem 5.6, $Y_o = Q_{12} Q_{22}^{-1}$ satisfies the Sylvester equation 5.18. That $[\Sigma_c]$ is a minimal realization can easily be seen by applying the contradiction argument. Indeed, if $[\Sigma_c]$ would not be a minimal realization then $\#(\Sigma_c) < \#(A_c)$ implying that $\#(\Sigma_c \Sigma_a) < \#(\Sigma)$ which, in turn, is in contradiction with $\Sigma = \Sigma_c \Sigma_a$.

(\Leftarrow) Let the unitary Q , as defined in the statement, be such that $Y_o = Q_{12} Q_{22}^{-1}$ is a solution for the Sylvester equation (5.18) from Theorem 5.6 and let Y_o satisfy the equation (5.20). Define $L \in \mathcal{D}$ to be the bounded solution for the Lyapunov-Stein equation

$$A_c L^{(-1)} A_a + \{C_c d_a^* - Y_o b_a^*\} b_a = L \tag{5.51}$$

and note that $\{C_c d_a^* - Y_o b_a^*\} b_a = (Y_o A_a^* - C_c C_a^*) \Lambda_{o,a}^{(-1)} A_a - Y_o \Lambda_{o,a}^{-1}$. Viz. the Sylvester equation (5.18), the Lyapunov-Stein equation (5.51) reduces to

$$A_c (L + Y_o \Lambda_{o,a}^{-1})^{(-1)} A_a = L + Y_o \Lambda_{o,a}^{-1}$$

whose only bounded solution implies that $Y_o = -L \Lambda_{o,a}$. Then, by applying the mixed partial fraction lemma and the coisometricity argument for Σ_a , we find that

$$\begin{aligned} \Sigma_c \Sigma_a &= D + B_a Z^* (I - A_a Z^*)^{-1} C_a + B_c Z (I - A_c Z)^{-1} C_c \\ &\quad - \{B_c Y_o^{(-1)} + B_a \Lambda_{o,a} A_a^* + D C_a^*\} \Lambda_{o,a}^{(-1)} C_a \\ &\quad - \{B_c Y_o^{(-1)} + B_a \Lambda_{o,a} A_a^* + D C_a^*\} \Lambda_{o,a}^{(-1)} A_a Z^* (I - A_a Z^*)^{-1} C_a \\ &\quad + B_c Z (I - A_c Z)^{-1} \{Y_o A_a^* - A_c Y_o^{(-1)} - C_c C_a^*\} \Lambda_{o,a}^{(-1)} C_a. \end{aligned}$$

Because of the Sylvester equation (5.18) and the equation (5.20), we conclude that $\Sigma_c \Sigma_a = \Sigma = D + B_a Z^* (I - A_a Z^*)^{-1} C_a + B_c Z (I - A_c Z)^{-1} C_c$ what we needed to prove. The proof for the first part of the additional statement follows immediately if we write equations (5.18) and (5.20) together as

$$\begin{bmatrix} A_c \\ B_c \end{bmatrix} Y_o^{(-1)} = \begin{bmatrix} Y_o \\ -B_a A_{o,a} \end{bmatrix} A_a^* - \begin{bmatrix} C_c \\ D \end{bmatrix} C_a^*.$$

After premultiplying the latter equation with $[A_c^* A_c \ B_c^*]$ followed by employing the identity $A_c^* A_c A_c + B_c^* B_c = A_c^{(-1)}$, we obtain the equation (5.49).

Let us prove that the expression (5.50) is the observability Gramian for $[\Sigma_c]$. Let $X \in \mathcal{D}$ be the unique bounded solution for

$$A_c X^{(-1)} A_c^* + (C_c d_a^* - Y_o b_a^*)(d_a C_c^* - b_a Y_o^*) = X.$$

By applying the coisometricity argument for $[\Sigma_a]$ on the second factor on the left side of the previous equation, followed by employing the Sylvester equation (5.18) and rearranging the terms, one obtains

$$A_c (X - Y_o A_{o,a}^{-1} Y_o^*)^{-1} A_c^* + C_c C_c^* = X - Y_o A_{o,a}^{-1} Y_o^*.$$

But since $A_c A_{o,c}^{(-1)} A_c^* + C_c C_c^* = A_{o,c}$, we conclude that $X = A_{o,c} + Y_o A_{o,a}^{-1} Y_o^*$. This completes the proof. ■

Corollary 5.13. *Let the conditions from Proposition 5.12 hold for $\Sigma \in \mathcal{X}$ given by a minimal, u.e.s additive realization $[\Sigma]$ for which*

$$DD^* + B_c A_{o,c}^{(-1)} B_c^* + B_a A_{o,a} B_a^* = I.$$

Then, Σ is isometric if and only if the set equations (5.18) and (5.19) shares a solution with the set equations (5.18) and (5.20).

PROOF: The sufficiency has already been proved by Corollary 5.7.

We prove the necessity by applying a contradiction argument. Assume Σ is isometric and Y_o is a solution for the pair of equations (5.18) and (5.19) which is not a solution for the pair (5.18) and (5.20). Then, by Lemma 5.9, it is possible to construct unitary state-space transformation $Q \in \mathcal{D}^{2 \times 2}$ which brings the AC-pair from the realization $[\mathbf{P}(\Sigma \Sigma^*)]$ into a Kalman canonical form. A candidate for Q can be obtained by considering the QR-decomposition

$$\begin{bmatrix} Y_o \\ I \end{bmatrix} = \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} Q_{22}^{-1}$$

for an appropriately partitioned coisometric $\begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix}$. Then, any unitary extension of the latter coisometric factor

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

by some $Q_{11}, Q_{21} \in \mathcal{D}$, produces the unitary state-space transformation needed. When applied on $[\mathbf{P}(\Sigma \Sigma^*)]$, this yields a partially unobservable realization in Kalman canonical form. Extraction of the maximal unobservable part followed by application of the

back transformation; $Q_{22}(\cdot)Q_{22}^{(-1)}$, yields the uniformly observable alternative realization to $[\mathbf{P}(\Sigma\Sigma^*)]$;

$$[\mathbf{P}(\Sigma\Sigma^*)]_{alt.} = \left[\begin{array}{c|c} A_a^* & B_a^* \\ \hline B_c Y_o^{(-1)} + B_a \Lambda_{o,a} A_a^* + DC_a^* & I \end{array} \right].$$

But it holds by assumption that Y_o is not a solution for (5.18) and (5.20). Hence, $B_c Y_o^{(-1)} + B_a \Lambda_{o,a} A_a^* + DC_a^* \neq 0$. Because $l_{A_a} < 1$, this implies that the AB-pair

$$(A_a^*, B_c Y_o^{(-1)} + B_a \Lambda_{o,a} A_a^* + DC_a^*)$$

from the latter realization has, at least, one-dimensional uniformly reachable subspace. This implies that $[\mathbf{P}(\Sigma\Sigma^*)]_{alt.}$ is not entirely nonminimal. Hence, $\mathbf{P}(\Sigma\Sigma^*) \neq 0$ which is in the contradiction with the assumption that Σ is isometric. This completes the proof. ■

We state the dual formulation of Proposition 5.12.

Proposition 5.14. *Let an operator $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s, additive realization $[\Sigma] = \{A_c, B_c, C_c, A_a, B_a, C_a, D\}$. Let $[\Sigma'_a] = \{A_a, B_a, c'_a, d'_a\}$ be a minimal, u.e.s realization for an anti-inner operator Σ'_a .*

Then, Σ is coisometric if and only if there exists a properly partitioned unitary $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ for which $Y_c = P_{12}^ P_{11}^{-*}$ satisfies the equations (5.43)-(5.45) from Theorem 5.10 and*

$$C_c^* Y_c + C_a^* \Lambda_a^{(-1)} A_a + D^* B_a = 0 \tag{5.52}$$

and the minimal, u.e.s realization

$$[\Sigma'_c] = \left[\begin{array}{c|c} A_c & C_c \\ \hline d'^*_a B_c - c'^*_a Y_c^{*(-1)} & c'^*_a \Lambda_a^{(-1)} C_a + d'^*_a D \end{array} \right]$$

is a realization for the causal coisometric operator Σ'_c .

In addition, the factorization $\Sigma = \Sigma'_a \Sigma'_c$ is a minimal anti-inner/causal-coisometric factorization and

(1) Y_c is the bounded solution for the Lyapunov-Stein equation

$$A_c(\Lambda_{o,c} Y_c)^{(-1)} A_a - C_c C_a^* \Lambda_a^{(-1)} A_a - (A_c \Lambda_{o,c}^{(-1)} B_c^* + C_c D^*) B_a = \Lambda_{o,c} Y_c, \tag{5.53}$$

(2) the reachability Gramian for $[\Sigma'_c]$ equals

$$A_c + Y_c \Lambda_a^{-1} Y_c^*. \tag{5.54}$$

Unlike the causal-isometric/anti-inner factorization for an isometric $\Sigma \in \mathcal{X}$ which, viz. the additional statements from Proposition 5.12 always exists, an inner/anti-causal-isometric factorization for Σ need not exist as the following proposition states.

Proposition 5.15. *Let an isometric operator $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s, additive realization $[\Sigma] = \{A_c, B_c, C_c, A_a, B_a, C_a, D\}$. Let $[\hat{\Sigma}_c] = \{A_c, B_c, \hat{c}_c, \hat{d}_c\}$ be a minimal, u.e.s realization for an inner operator $\hat{\Sigma}_c$. Then, Σ admits a minimal inner/anti-causal-isometric factorization*

$$\Sigma = \hat{\Sigma}_c \hat{\Sigma}_a \quad (5.55)$$

for an anti-causal isometric $\hat{\Sigma}_a$ if and only if there exists a $Y_c \in \mathcal{D}$ which satisfies equations (5.43)-(5.44).

In addition, if such a $Y_c \in \mathcal{D}$ exists then

(1)

$$[\hat{\Sigma}_a] = \left[\begin{array}{c|c} A_a & C_a \\ \hline \hat{d}_c^* B_a + \hat{c}_c^* Y_c & \hat{c}_c^* A_c C_c + \hat{d}_c^* D \end{array} \right]$$

is a minimal, u.e.s realization for $\hat{\Sigma}_a$,

(2) the reachability Gramian for $[\hat{\Sigma}_a]$ is equal to

$$A_a + Y_c^* A_c^{-1} Y_c, \quad (5.56)$$

(3) the realization

$$[\Sigma^* \Sigma] = \left[\begin{array}{c|c} A_a^* & A_a^* A_a^{(-1)} C_a + B_a^* D + Y_c^* C_c \\ \hline A_a & C_a \end{array} \right] \\ \left[\begin{array}{c|c} C_a^* C_a^* A_a^{(-1)} A_a + D^* B_a + C_c^* Y_c & D^* D + C_c^* A_c C_c + C_a^* A_a^{(-1)} C_a \end{array} \right] \quad (5.57)$$

is a possibly nonminimal, u.e.s additive realization for $\Sigma^* \Sigma$,

(4) the reachability Gramian assigned to the anti-causal part from $[\Sigma^* \Sigma]$ is equal to

$$A_a + Y_c^* A_{o,c} Y_c + A_a Y_o^* Y_c + Y_c^* Y_o A_a - A_a A_{o,a} A_a. \quad (5.58)$$

PROOF: (\Rightarrow) To find a realization for the isometric $\hat{\Sigma}_c$, we apply the external factorization argument and the mixed partial fraction lemma 2.17 on the product $\hat{\Sigma}_c^* \Sigma (=:\hat{\Sigma}_a)$. We obtain the u.e.s realization

$$[\hat{\Sigma}_a]_{alt.} = \left[\begin{array}{c|c} A_a & C_a \\ \hline B_c^* B_a A_c^* & A_c^* A_c C_c + B_c^* D \\ \hat{d}_c^* B_a & \hat{c}_c^* A_c C_c + \hat{d}_c^* D \end{array} \right]$$

for the isometric $\hat{\Sigma}_a \in \mathcal{L}$. Note that $[\hat{\Sigma}_a]_{alt.}$ is a uniformly reachable realization. Assume Y_c satisfying (5.43)-(5.44) exists. Then, the dual formulation of Lemma 5.9 states that if for some properly partitioned isometric $[P_{11} \ P_{12}]$ the representation

$$\begin{bmatrix} I \\ Y_c \end{bmatrix} = \begin{bmatrix} P_{11}^* \\ P_{12}^* \end{bmatrix} P_{11}^{-*}$$

is a QR factorization for the left invertible $\begin{bmatrix} I \\ Y_c \end{bmatrix}$ then, for $P_{21}, P_{22} \in \mathcal{D}$ for which $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ is unitary, the state-space transformation P brings the anti-causal AC-pair pair from $[\hat{\Sigma}_a]_{alt.}$ into the Kalman canonical form

$$\left[\begin{array}{c|c} P^{(-1)} & \\ \hline & I \end{array} \right] [\hat{\Sigma}_a]_{alt.} \left[\begin{array}{c|c} P^* & \\ \hline & I \end{array} \right] = \left[\begin{array}{cc|c} \alpha_{11} & \alpha_{12} & \gamma \\ & \alpha_{22} & 0 \\ \hline \hat{d}_c^* B_a P_{11}^* + \hat{c}_c^* P_{12}^* & \beta_2^* & \hat{c}_c^* A_c C_c + \hat{d}_c^* D \end{array} \right]. \quad (5.59)$$

Note that $P_{ii}, i = 1, 2$ are locally square invertible. After extraction of the maximal unobservable part from the realization on the right side of (5.59), and applying the back transformation P_{11}^* on the obtained minimal realization, we obtain the realization $[\hat{\Sigma}_a]$ from the additional statement. $[\hat{\Sigma}_a]$ is minimal since $\#(\Sigma) = \#(A_c) + \#(A_a)$ as we needed to show.

(\Leftarrow) Assume $\hat{\Sigma}_c \hat{\Sigma}_a = \Sigma$ is a minimal factorization for Σ . Let us evaluate the expression $\hat{\Sigma}_c \hat{\Sigma}_a$ starting from minimal realizations $[\hat{\Sigma}_a] = \{A_a, \hat{b}_a, C_a, \hat{d}_a\}$ and $[\hat{\Sigma}_c] = \{A_c, \hat{b}_c, C_c, \hat{d}_c\}$. We obtain

$$\begin{aligned} \hat{\Sigma}_c \hat{\Sigma}_a &= \hat{d}_c \hat{d}_a + B_c K^{(-1)} C_a \\ &\quad + (\hat{d}_c \hat{b}_a + B_c K^{(-1)} A_a) Z^* (I - A_a Z^*)^{-1} C_a \\ &\quad + B_c Z (I - A_c Z)^{-1} (\hat{c}_c \hat{d}_a + A_c K^{(-1)} C_a) \end{aligned}$$

where $K \in \mathcal{D}$ is the bounded solution for $A_c K^{(-1)} A_a + \hat{c}_c \hat{b}_a = K$. Since $\hat{\Sigma}_c \hat{\Sigma}_a = \Sigma$, it holds that

$$\begin{bmatrix} A_c \\ B_c \end{bmatrix} K^{(-1)} \begin{bmatrix} A_a & C_a \end{bmatrix} = \begin{bmatrix} K & C_c \\ B_a & D \end{bmatrix} - \begin{bmatrix} \hat{c}_c \\ \hat{d}_c \end{bmatrix} \begin{bmatrix} \hat{b}_a & \hat{d}_a \end{bmatrix}.$$

After premultiplication of the previous identity with $[A_c^* A_c \ B_c^*]$ and employing the fact that $[\hat{\Sigma}_c]$ is A_c^{-1} -unitary realization, we obtain

$$(A_c K)^{(-1)} \begin{bmatrix} A_a & C_a \end{bmatrix} = \begin{bmatrix} A_c^* & B_c^* \end{bmatrix} \begin{bmatrix} A_c K & A_c C_c \\ B_a & D \end{bmatrix}.$$

By defining $Y_c := A_c K$ we prove that the previous equation consists from equations (5.43) and (5.44) as we needed to show in order to complete the proof for the main and the first additional statement.

To prove the second additional statement, define $\Omega \in \mathcal{D}$ to be the bounded solution for

$$A_a^* \Omega^{(-1)} A_a + (\cdot)^* (\hat{d}_c^* B_a + \hat{c}_c^* Y_c) = \Omega.$$

Using first the fact that $[\hat{\Sigma}_c]$ is A_c^{-1} -unitary, it is possible to express the 'hatted' terms from the previous equation through A_c , B_c and Λ_c . However, new terms $B_a^* B_a$ and $B_a^* B_c$ appears which, in turn, can be substituted by equations (5.4) and (5.43). After these substitutions, the obtained equation takes the form of a trivial Lyapunov-Stein equation;

$$A_a^* \{ \Omega - Y_c^* A_c^{-1} Y_c - \Lambda_a \}^{(-1)} A_a = \Omega - Y_c^* A_c^{-1} Y_c - \Lambda_a$$

with zero as its only bounded solution. Hence, the expression (5.56) represents the sought Gramian.

To prove the third additional statement we evaluate a realization for the product $\Sigma^* \Sigma$. Since the latter operator is a Hermitian, it suffices to evaluate a realization for $\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)$. After calculations are completed, we obtain a realization which coincides with $[\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)]$ from Proposition 5.5. But the AC-pair from $[\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)]$ is the same as the AC-pair from $[\hat{\Sigma}_c]$. Since the minimal factorization exists by assumption, there is a state-space transformation which brings the AC-pair in a Kalman canonical form.

Application of the state-space transformation $P^* \begin{bmatrix} P_{11}^{-*} & \\ & I \end{bmatrix}$ with P introduced before, on $[\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)]$ shall produce a nonminimal realization in a Kalman canonical form which, after extraction of the maximal nonobservable part, yields

$$[\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)]_{alt.} = \left[\begin{array}{c|c} \frac{A_a}{C_a^* A_a^{(-1)} A_a + D^* B_a + C_c^* Y_c} & \frac{C_a}{D^* D + C_c^* \Lambda_c C_c + C_a^* A_a^{(-1)} C_a} \end{array} \right].$$

The realization $[\mathbf{P}_{\mathcal{L}}(\Sigma^* \Sigma)]_{alt.}$ is not necessarily minimal and easily extends to the realization (5.57) as we needed to show.

To prove the fourth additional statement, define Π to be the bounded solution for

$$A_a^* \Pi^{(-1)} A_a + (\cdot)^* (C_a^* A_a^{(-1)} A_a + D^* B_a + C_c^* Y_c) = \Pi.$$

Then, we evaluate the left hand-side of the previous equation where we encounter the following terms: $C_a C_a^*$, $C_a D^*$, $C_a C_c^*$, DD^* and their transposes. These terms allow for substitution by identities (5.6), (5.20), (5.18) and (5.8). When the substitution is done, new terms arise; $B_c^* B_a$, $B_a^* B_a$ which, in turn, can be replaced by (5.43) and (5.4). After all substitutions are completed, a lot of cancellations occurs ending up with the trivial Lyapunov-Stein equation

$$\begin{aligned} A_a^* \{ \Pi - \Lambda_a - Y_c^* A_{o,c} Y_c + \Lambda_a A_{o,a} \Lambda_a - \Lambda_a Y_o^* Y_c - Y_c^* Y_o \Lambda_a \}^{(-1)} A_a \\ = \Pi - \Lambda_a - Y_c^* A_{o,c} Y_c + \Lambda_a A_{o,a} \Lambda_a - \Lambda_a Y_o^* Y_c - Y_c^* Y_o \Lambda_a. \end{aligned}$$

Since, the only bounded solution for the previous equation is zero, we complete the proof that (5.57) is the reachability Gramian. ■

The propositions 5.12 and 5.14 state a simple test under which a mixed causality operator is respectively isometric or coisometric. Moreover, in the case the operator is unitary, it is possible to find the relationship between solutions of Sylvester equations; Y_o and Y_c , defined by equations (5.18) and (5.43). We state this in the following lemma.

Lemma 5.16. *Let a unitary $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s, additive realization $[\Sigma]$.*

Then, for $Y_o, Y_c \in \mathcal{D}$ defined respectively by equations (5.18) and (5.43) it holds that

$$A_c Y_o = -Y_c A_{o,a}, \tag{5.60}$$

$$A_{o,c} Y_c = -Y_o A_a, \tag{5.61}$$

$$A_c^{-1} = A_{o,c} + Y_o A_{o,a}^{-1} Y_o^*, \tag{5.62}$$

$$A_{o,c}^{-1} = A_c + Y_c A_a^{-1} Y_c^*. \tag{5.63}$$

PROOF: For a unitary $\Sigma \in \mathcal{X}$ given by a minimal, u.e.s, additive realization $[\Sigma]$, the equation (5.49) holds. Let us eliminate the term $B_c^* B_c$ from the latter equation by means of the equation (5.43). After rearranging the terms, we obtain

$$(A_c Y_o + Y_c A_{o,a})^{(-1)} = A_c^* (A_c Y_o + Y_c A_{o,a}) A_a^* + \underline{(Y_c^{(-1)} C_a + A_c^* A_c C_c + B_c^* D) C_a^*}. \tag{5.64}$$

Since Σ is coisometric, Theorem 5.10 holds. Hence, the equation (5.45) implies that the underlined term from (5.64) is zero. As the consequence, the Lyapunov equation (5.64) reduces to the trivial one with zero as its only bounded solution. Therefore, $A_c Y_o = -Y_c A_{o,a}$ as the equation (5.60) states.

The proof for the equation (5.61) follows similar vein as the previous and uses equations (5.53) and (5.18).

The proof for (5.62) follows from the second additional statement of Proposition 5.12 for the case $[\Sigma_c]$ is A_c^{-1} -unitary realization. Then, its reachability and the observability Gramians are mutual inverses which, in turn immediately provides the truth that the equation (5.62) holds. The proof for (5.63) is similar to the previous one and uses $A_{o,c}$ -unitarity argument for $[\Sigma_c']$ and the second additional statement from Proposition 5.14. This completes the proof. ■

5.3 Isometric realizations for isometric operators from \mathcal{X}

Using the result stated in Proposition 5.12, it is possible to find an alternative minimal, u.e.s realization to the additive realization $[\Sigma]$ for an isometric $\Sigma \in \mathcal{X}$. As the following theorem states, this realization has its transition operator in block upper form and therefore is suitable for the minimal factorization purposes.

Theorem 5.17. *Let an isometric $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s, additive realization $[\Sigma]$. Let $Y_o \in \mathcal{D}$ be as defined in Theorem 5.6. Then, the u.e.s realization*

$$[\Sigma]_{alt.} = \left[\begin{array}{c|c} A_c & A_{ca} \\ \hline A_a & C_a \\ \hline B_c & x_{21} \end{array} \middle| \begin{array}{c} x_{12} \\ x_{22} \end{array} \right] \quad (5.65)$$

where

$$x = \begin{bmatrix} A_{ca} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -Y_o A_{o,a}^{-1} & C_c \\ B_a & D \end{bmatrix} + \begin{bmatrix} A_c \\ B_c \end{bmatrix} (Y_o A_{o,a}^{-1})^{(-1)} [A_a \ C_a]$$

is an alternative minimal realization for Σ .

In addition, it holds that

$$[\Sigma]_{alt.} \left[\begin{array}{c|c} A_c^{(-1)} & \\ \hline & A_{o,a} \\ \hline & I \end{array} \right] [\Sigma]_{alt.}^* = \left[\begin{array}{c|c} A_c^{-1} & \\ \hline & A_{o,a}^{(-1)} \\ \hline & I \end{array} \right]. \quad (5.66)$$

PROOF: By Proposition 5.12, we have that the isometric Σ admits a minimal causal-isometric/anti-inner factorization in the form $\Sigma = \Sigma_c \Sigma_a$ where the causal isometric Σ_c and the anti-inner Σ_a are given by minimal, u.e.s realizations (5.48) and $[\Sigma_a] = \{A_a, b_a, C_a, d_a\}$ respectively. Hence, Σ can be assembled through the product $\Sigma_c \Sigma_a$ in which the mixed causality term is not further decomposed by the mixed partial fraction lemma 2.17. As the result, we obtain a minimal, u.e.s realization for Σ in the form

$$[\Sigma]_{alt.} = \left[\begin{array}{c|c} A_c & (C_c d_a^* - Y_o b_a^*) b_a \\ \hline A_a & C_a \\ \hline B_c & (B_a A_{o,a} b_a^* + D d_a^*) b_a \end{array} \middle| \begin{array}{c} (C_c d_a^* - Y_o b_a^*) b_a \\ C_a \\ (B_a A_{o,a} b_a^* + D d_a^*) d_a \end{array} \right].$$

By means of the equation (5.20) from Theorem 5.17, the Sylvester equation (5.18) and the coisometricity property for Σ_a , it is possible to show that x given in the statement is such that

$$x = \begin{bmatrix} C_c d_a^* - Y_o b_a^* \\ B_a A_{o,a} b_a^* + D d_a^* \end{bmatrix} [b_a \ d_a]$$

which completes the proof of the statement.

The proof for the additional statement follows immediately after rewriting $[\Sigma]_{alt.}$ as

$$[\Sigma]_{alt.} = \left[\begin{array}{c|c} A_c & C_c d_a^* - Y_o b_a^* \\ \hline I & \\ \hline B_c & B_a A_{o,a} b_a^* + D d_a^* \end{array} \right] \left[\begin{array}{c|c} I & \\ \hline A_a & C_a \\ \hline b_a & d_a \end{array} \right]$$

and after recalling that realizations $[\Sigma_c]$ and $[\Sigma_a]$, contained in the matrices on the right side of the previous equation, are respectively $A_c^{(-1)}$ -isometric and $A_{o,a}$ -unitary. This makes the proof complete. \blacksquare

For isometric $\Sigma \in \mathcal{X}$ with the additional structure, namely ones which admit the inner/anti-causal-isometric minimal factorization, there is also an alternative realization which satisfies the equation (5.66) from Theorem 5.17. We state this fact in the following theorem. Its proof is a carbon-copy of the proof of Theorem 5.17 and will be omitted.

Theorem 5.18. *Let an isometric $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s, additive realization $[\Sigma]$. Let $Y_o \in \mathcal{D}$ be as defined in Theorem 5.6 and let $Y_c \in \mathcal{D}$ satisfy either (5.43) and (5.44) or (5.43) and (5.45). Then, the u.e.s realization*

$$[\Sigma]_{att.} = \left[\begin{array}{c|c} A_c \hat{A}_{ca} & \hat{x}_{12} \\ \hline A_a & C_a \\ \hline B_c \hat{x}_{21} & \hat{x}_{22} \end{array} \right] \tag{5.67}$$

where

$$\hat{x} = \begin{bmatrix} \hat{A}_{ca} & \hat{x}_{12} \\ \hat{x}_{21} & \hat{x}_{22} \end{bmatrix} = \begin{bmatrix} \Lambda_c^{-1} Y_c & C_c \\ B_a & D \end{bmatrix} - \begin{bmatrix} A_c \\ B_c \end{bmatrix} (\Lambda_c^{-1} Y_c)^{(-1)} [A_a \ C_a]$$

is an alternative minimal realization for Σ which satisfies the equation (5.66) from Theorem 5.17.

Realizations given by (5.65) and (5.67) for Σ from the theorems 5.17 and 5.18 do not necessarily coincide. However, if Σ is unitary then, viz. the equation (5.60) from Lemma 5.16, we have that $x = \hat{x}$ as the consequence of the fact that the isometric factors in the minimal factorizations $\Sigma = \Sigma_c \Sigma_a$ and $\Sigma = \hat{\Sigma}_c \hat{\Sigma}_a$ are inner and anti-inner respectively and, for some unitary block diagonal δ , are related via $\Sigma_c = \hat{\Sigma}_c \delta$ and $\Sigma_a = \delta^* \hat{\Sigma}_a$. The dual statement to Theorem 5.17 is given by the following theorem.

Theorem 5.19. *Let a coisometric $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s, additive realization $[\Sigma]$. Let $Y_c \in \mathcal{D}$ be as defined in Theorem 5.10. Then, the u.e.s realization*

$$[\Sigma]_{att.} = \left[\begin{array}{c|c} A_c & C_c \\ \hline A_{ac} \ A_a & y_{12} \\ \hline y_{21} \ B_a & y_{22} \end{array} \right]$$

where

$$y = \begin{bmatrix} A_{ac} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} -(\Lambda_a^{-1} Y_c^*)^{(-1)} & C_a \\ B_c & D \end{bmatrix} + \begin{bmatrix} A_a \\ B_a \end{bmatrix} (\Lambda_a^{-1} Y_c^*) [A_c \ C_c]$$

is an alternative minimal realization for Σ .

In addition, it holds that

$$[\Sigma]_{alt.}^* \left[\begin{array}{c|c} A_{o,c}^{-1} & \\ \hline & A_a^{(-1)} \\ \hline & & I \end{array} \right] [\Sigma]_{alt.} = \left[\begin{array}{c|c} A_{o,c}^{(-1)} & \\ \hline & A_a \\ \hline & & I \end{array} \right].$$

Theorems 5.17 and 5.19 introduce a more general class of minimal realizations than additive ones. For instance, the realization $[\Sigma]_{alt.}$ mentioned in Theorem 5.17 involves the causal/anti-causal product term

$$B_c Z (I - A_c Z)^{-1} A_{ca} Z^* (I - A_a Z^*)^{-1} C_a$$

which, by the mixed partial fraction lemma 2.17, can be further decomposed into the sum of a diagonal term and terms of single but opposite causality. As the consequence, an additive minimal realization for Σ is obtained. However, in the case of a general minimal realization which, beside a causal/anti-causal product, involves an anti-causal/causal product term, it is still an open question how to find an alternative minimal realization in the additive form for a mixed causality operator. If the mixed causality operator is isometric then the solution to the problem can be easily found. Consider again the realization $[\Sigma]_{alt.}$ from Theorem 5.17. Let A_{ac} be an arbitrary matrix of an appropriate dimension. Due to $\ell_{A_a} < 1$ and $\ell_{A_c} < 1$, the Lyapunov-Stein equation

$$A_a L A_c + A_{ac} = L^{(-1)}$$

has the unique bounded solution $L \in \mathcal{D}$. Then, by the mixed partial fraction lemma 2.17, it can be easily verified that the realization

$$[\Sigma]' = \left[\begin{array}{cc|c} A_c & A_{ca} & x_{12} \\ A_{ac} & A_a & C_a - A_a L x_{12} \\ \hline B_c - x_{21} L A_c & x_{21} & x_{22} - x_{21} L x_{12} \end{array} \right]$$

parameterized by A_{ac} is a minimal, u.e.s realization for the isometric Σ and, beside for $A_{ac} = 0$, it is not necessarily unitary. Note that $[\Sigma]'$ shares the top row block and the central column block with $[\Sigma]_{alt.}$ given in Theorem 5.17 for each A_{ac} and, in the special case when $A_{ac} = 0$, we even have that $[\Sigma]'$ and $[\Sigma]_{alt.}$ coincide.

5.4 One-pass unitary embedding algorithm for isometries from \mathcal{X}

To this end, we consider a unitary embedding procedure for mixed causality isometric and coisometric operators and set up the algorithm for finding a unitary, u.e.s realization for the extended (unitary) operator.

By Proposition 5.12, an isometric operator $\Sigma_{iso.}$ given by a minimal, additive u.e.s

realization $[\Sigma_{iso.}] = \{A_c, B_c, C_c, A_a, B_a, C_a, D\}$ can be always written in a minimal way as the product

$$\Sigma_{iso.} = \Sigma_{c,iso.} \Sigma_a$$

where Σ_a is anti-inner and $\Sigma_{c,iso.}$ is causal isometric. Minimal, u.e.s realizations for these factors; $[\Sigma_a]$ and $[\Sigma_{c,iso.}]$ have also been evaluated in the latter proposition. A common way to turn $\Sigma_{iso.}$ into an inner operator is to extend a basis for its input state-space $\mathcal{H}(\Sigma_{iso.})$ to the entire space \mathcal{U}_2 by a (causal) basis for the complementary subspace; $\mathcal{H}(\Sigma_{iso.}^\perp)$. The later subspace implies the existence of the another (causal)

isometry $\Sigma_{iso.}^\perp$ for which $\begin{bmatrix} \Sigma_{iso.} \\ \Sigma_{iso.}^\perp \end{bmatrix}$ is inner.

Then the operator $\Sigma_{iso.}^\perp := \Sigma_{c,iso.}^\perp \Sigma_a$ is isometric, of mixed causality and such that

$$\Sigma := \begin{bmatrix} \Sigma_{iso.} \\ \Sigma_{iso.}^\perp \end{bmatrix} = \begin{bmatrix} \Sigma_{c,iso.} \\ \Sigma_{c,iso.}^\perp \end{bmatrix} \Sigma_a$$

is unitary as needed. However, in order to find a minimal realization for Σ , the product on the right side of the previous identity must be decomposed by means of the mixed partial fraction lemma first. This step introduces solving an additional Lyapunov-Stein equation locally and increases the computational effort. Fortunately, by Theorem 5.17 all unnecessary work can be avoided by setting up a unitary embedding procedure on an isometric, u.e.s realization for $\Sigma_{iso.}$. The benefit is that this procedure can be formulated as an one-pass algorithm.

PROOF:(*Emb. algorithm*) The proof for the previous algorithm relies on time-forward equations defining $A_c, A_{o,a}$ and the equation (5.49) defining $K = A_c Y_o$ from Proposition 5.12. Once snapshots of these variables are known at time instants k , their values can be evaluated at the instant $k + 1$. The Cholesky factors: $\{R_{c,l}, R_{o,l}\}$, $l = k, k + 1$, allow for the evaluation as well and as state-space transformations and are used to pass to the alternative minimal realization $[\Sigma_{iso.}]'_k$ to $[\Sigma_{iso.}]_k$. By construction, $[\Sigma_{iso.}]'_k$ fulfills the equation (5.66) from Theorem 5.17 for the case $A_{c,k}$ and $A_{o,a,k}$ are set to the identity and hence it is an isometric realization. Since the isometric matrix $[\Sigma_{iso.}]'_k$ allows for a unitary extension by properly chosen $b_{1,k}, b_{2,k}$ and d_k , the resulting extended realization $[\Sigma]_k$ is a unitary realization for the unitary operator corresponding to the unitary extension of the original operator $\Sigma_{iso.}$. This completes the proof. ■

In the case $\Sigma \in \mathcal{X}$ is unitary, it admits both inner/anti-inner and anti-inner/inner minimal factorizations. By applying the propositions 5.12 and 5.14 on such an operator, we obtain that $\Sigma = \Sigma_c \Sigma_a = \Sigma'_a \Sigma'_c$ for some inner Σ'_c and anti-inner Σ'_a . We then evaluate two different minimal, u.e.s realizations for the inner $\Sigma'^*_a \Sigma_c (= \Sigma'_c \Sigma'^*_a)$ as

1. **input:** sequence of minimal, u.e.s additive realizations $[\Sigma_{iso.}]_k = \{A_{c,k}, B_{c,k}, C_{c,k}, A_{a,k}, B_{a,k}, C_{a,k}, D_k\}$ for isometric $\Sigma_{iso.} \in \mathcal{X}$ starting from some integer k_{init} on, initial K_{init} and initial invertible Hermitians: $\Lambda_{c,init}$ and $\Lambda_{o,a,init}$.
2. **output:** sequence of unitary, u.e.s realization $[\Sigma]_k$ for the unitarily embedded $\Sigma_{iso.}$
3. Set

$$\begin{aligned} \Lambda_{c,k_{init}} &:= \Lambda_{c,init}, \\ \Lambda_{o,a,k_{init}} &:= \Lambda_{o,a,init}, \\ K_{k_{init}} &:= K_{init}. \end{aligned}$$

For $k = k_{init}, k_{init} + 1, \dots$ do

4. Update

$$\begin{aligned} \Lambda_{c,k+1} &:= A_{c,k}^* \Lambda_{c,k} A_{c,k} + B_{c,k}^* B_{c,k} \\ \Lambda_{o,a,k+1} &:= A_{a,k} \Lambda_{o,a,k} A_{a,k}^* + C_{a,k} C_{a,k}^* \\ K_{k+1} &:= A_{c,k}^* K_k A_{a,k}^* - B_{c,k}^* B_{a,k} \Lambda_{o,a,k} A_{a,k}^* - (A_{c,k}^* \Lambda_{c,k} C_{c,k} + B_{c,k}^* D_k) C_{a,k}^*. \end{aligned}$$

5. Find invertible $R_{c,k}$ and $R_{o,a,k}$ for which

$$\begin{aligned} R_{c,k}^* R_{c,k} &:= \Lambda_{c,k}, \\ R_{o,a,k} R_{o,a,k}^* &:= \Lambda_{o,a,k} \end{aligned}$$

are Cholesky factorizations.

6. Define

$$\begin{aligned} \begin{bmatrix} A_{ca,k} & x_{12,k} \\ x_{21,k} & x_{22,k} \end{bmatrix} &:= \begin{bmatrix} -K_k & C_{c,k} \\ B_{a,k} & D_k \end{bmatrix} + \begin{bmatrix} A_{c,k} \\ B_{c,k} \end{bmatrix} K_{k+1} \begin{bmatrix} A_{a,k} & C_{a,k} \end{bmatrix}, \\ [\Sigma_{iso.}]'_k &:= \left[\begin{array}{c|c} R_{c,k} & \\ \hline R_{o,a,k+1}^{-1} & \\ \hline & I \end{array} \right] \begin{bmatrix} A_{c,k} & A_{ca,k} & x_{12,k} \\ A_{a,k} & C_{a,k} & \end{bmatrix} \left[\begin{array}{c|c} R_{c,k+1}^{-1} & \\ \hline R_{o,a,k} & \\ \hline & I \end{array} \right] \end{aligned}$$

7. Find $b_{1,k}, b_{2,k}, d_k$ such that

$$[\Sigma]_k := \begin{bmatrix} [\Sigma_{iso.}]'_k \\ b_{1,k} & b_{2,k} & d_k \end{bmatrix}$$

is a unitary realization.

8. End.

Fig. 5.1. Algorithm for embedding of a mixed causality isometry $\Sigma_{iso.}$ into a unitary operator $\Sigma = \begin{bmatrix} \Sigma_{iso.} \\ \Sigma_{iso.}^\perp \end{bmatrix}$.

$$[\Sigma'_a \Sigma_c] = \left[\begin{array}{c|c} A_c & C_c d_a^* - Y_o b_a^* \\ \hline B_a^* B_c & A_a^* B_a^* (B_a \Lambda_{o,a} b_a^* + D d_a^*) \\ \hline d_a^* B_c & c_a^* d_a^* (B_a \Lambda_{o,a} b_a^* + D d_a^*) \end{array} \right] \quad \text{and} \quad (5.68)$$

$$[\Sigma'_c \Sigma_a^*] = \left[\begin{array}{cc|c} A_c & C_c C_a^* & C_c d_a^* \\ \hline & A_a^* & b_a^* \\ \hline d_a^* B_c - c_a^* Y_c^{*(-1)} & (c_a^* A_a^{(-1)} C_a + d_a^* D) C_a^* & (c_a^* A_a^{(-1)} C_a + d_a^* D) d_a^* \end{array} \right] \quad (5.69)$$

respectively.

The realizations $[\Sigma'_a \Sigma_c]$ and $[\Sigma'_c \Sigma_a^*]$ are then related through a boundedly invertible state-space transformation. Evaluation of this transformation shall be investigated in the following section.

5.5 State-space properties of partitioned unitary operators

In this section, we evaluate Gramians and their relationships for isometric and coisometric operators when the latter are arbitrarily partitioned. The reason for the consideration of the partitioned operators is necessary in the case we want to relate them to their J-isometric (or J-coisometric) siblings. In this case a quantitative information about the Gramians can tell us when the closed formulae for evaluation of realization for the associated J-isometric operator are minimal. Let $\Sigma \in \mathcal{X}$ be either isometric or coisometric as before. Let $u = [a_1 \ b_2] \in \mathcal{M}_1 \oplus \mathcal{M}_2$, $y = [a_2 \ b_1] \in \mathcal{N}_1 \oplus \mathcal{N}_2$ where ' \oplus ' denotes the orthogonal sum of subspaces, be respectively input- and output sequences partitioned correspondingly to an arbitrary partitioning of the input- resp. output spaces of Σ . Along with this partitioning, Σ becomes the map

$$\Sigma = \left[\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array} \right] \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N}) \quad (5.70)$$

$$[a_2 \ b_1] = [a_1 \ b_2] \Sigma \quad (5.71)$$

and the realization $[\Sigma]$ introduced in the previous sections, takes the form

$$[\Sigma] = \left[\begin{array}{cc|cc} A_c & & C_{c,1} & C_{c,2} \\ & A_a & C_{a,1} & C_{a,2} \\ \hline B_{c,1} & B_{a,1} & D_{11} & D_{12} \\ B_{c,2} & B_{a,2} & D_{21} & D_{22} \end{array} \right] \quad (5.72)$$

$$[x_c^{(-1)} \ x_a | a_2 \ b_1] = [x_c \ x_a^{(-1)} | a_1 \ b_2] [\Sigma]. \quad (5.73)$$

Viz. the additional partitioning, we introduce Gramians $A_{c,i}, A_{a,i}, A_{o,c,i}, A_{o,a,i} \in \mathcal{D}$ to be bounded possibly singular solutions for the following Lyapunov equations

$$A_c^* A_{c,i} A_c + B_{c,i}^* B_{c,i} = A_{c,i}^{(-1)}, \quad (5.74)$$

$$A_a^* A_{a,i}^{(-1)} A_a + B_{a,i}^* B_{a,i} = A_{a,i}, \quad (5.75)$$

$$A_c A_{o,c,i}^{(-1)} A_c^* + C_{c,i} C_{c,i}^* = A_{o,c,i}, \quad (5.76)$$

$$A_a A_{o,a,i} A_a^* + C_{a,i} C_{a,i}^* = A_{o,a,i}^{(-1)}, \quad i = 1, 2. \quad (5.77)$$

It can be easily deduced from the previous Lyapunov equations that

$$\begin{aligned} A_c &= A_{c,1} + A_{c,2}, \quad A_{o,c} = A_{o,c,1} + A_{o,c,2}, \\ A_a &= A_{a,1} + A_{a,2}, \quad A_{o,a} = A_{o,a,1} + A_{o,a,2} \end{aligned}$$

where $A_c, A_a, A_{o,c}$ and $A_{o,a}$ are Gramians corresponding to the unpartitioned minimal, u.e.s realization $[\Sigma]$ and are boundedly invertible solutions of (5.3)-(5.6) respectively. Hence, for instance, $A_{c,1}$ and $A_{c,2}$ can not be singular on the common subspace of state-space vectors. Similar claims hold for the remaining pairs of the previously introduced Gramians.

The following proposition relates the unitary state-space transformation Q introduced in the statement of Theorem 5.6 to an arbitrary block partitioning of an isometric $\Sigma \in \mathcal{X}$ as given by (5.70)-(5.71).

Proposition 5.20. *Let an isometric $\Sigma \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s additive realization $[\Sigma]$. Consider A -, B - and C -matrices from the realization $[\mathbf{P}(\Sigma\Sigma^*)]$ given by (5.7) in Proposition 5.4 and let $Q \in \mathcal{D}$ be the properly partitioned and unitary state-space transformation defined by (5.11) from Theorem 5.6. Define*

$$K = \begin{bmatrix} I & -Q_{12}Q_{22}^{-1} \\ & -I \end{bmatrix} \in \mathcal{D}^{2 \times 2}.$$

Then, it holds that

$$K^{-1} \begin{bmatrix} A_c & C_{c,2}C_{a,2}^* \\ & A_a^* \end{bmatrix} K^{(-1)} = \begin{bmatrix} A_c & C_{c,1}C_{a,1}^* \\ & A_a^* \end{bmatrix}, \quad (5.78)$$

$$\begin{bmatrix} B_c & B_a A_{o,a,2} A_a^* + \begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix} C_{a,2}^* \end{bmatrix} K^{(-1)} = \begin{bmatrix} B_c & B_a A_{o,a,1} A_a^* + \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} C_{a,1}^* \end{bmatrix}, \quad (5.79)$$

$$-K^{-1} \begin{bmatrix} A_c A_{o,c,2}^{(-1)} B_c^* + C_{c,2} [D_{12}^* & D_{22}^*] \\ & B_a^* \end{bmatrix} = \begin{bmatrix} A_c A_{o,c,1}^{(-1)} B_c^* + C_{c,1} [D_{11}^* & D_{21}^*] \\ & B_a^* \end{bmatrix}. \quad (5.80)$$

PROOF: Consider the equation corresponding to the zero block from the state transition operator on the right side of the equation (5.11) from Theorem 5.6

$$Q_{11}^* A_c Q_{12}^{(-1)} + Q_{11}^* C_c C_a^* Q_{22}^{(-1)} + Q_{21}^* A_a^* Q_{22}^{(-1)} = 0.$$

Let us pre- and postmultiply it with Q_{11}^{-*} and $-Q_{22}^{(-1)}$ respectively and employ the identity $-Q_{12}Q_{22}^{-1} = Q_{11}^{-*}Q_{21}^*$ into the result. We obtain an equivalent equation to the previous one in the form

$$A_c (-Q_{12}Q_{22}^{-1})^{(-1)} - C_c C_a^* + Q_{12}Q_{22}^{-1} A_a^* = 0.$$

Let us substitute $C_c C_a^* = C_{c,1} C_{a,1}^* + C_{c,2} C_{a,2}^*$ into the previous equation. After rearranging the terms we obtain

$$C_{c,2} C_{a,2}^* - Q_{12}Q_{22}^{-1} A_a^* = A_c (-Q_{12}Q_{22}^{-1})^{(-1)} - C_{c,1} C_{a,1}^*.$$

Identities: $A_c = A_c$ and $-A_a^* = -A_a^*$ together with the previous equation yield

$$\begin{bmatrix} I - Q_{12}Q_{22}^{-1} \\ -I \end{bmatrix} \begin{bmatrix} A_c & C_{c,2}C_{a,2}^* \\ & A_a^* \end{bmatrix} = \begin{bmatrix} A_c & C_{c,1}C_{a,1}^* \\ & A_a^* \end{bmatrix} \begin{bmatrix} I - Q_{12}Q_{22}^{-1} \\ -I \end{bmatrix}^{(-1)}.$$

We note that $(K =) \begin{bmatrix} I - Q_{12}Q_{22}^{-1} \\ -I \end{bmatrix}$ is such that $K^{-1} = K$ and rewrite the previous equation as

$$K^{-1} \begin{bmatrix} A_c & C_{c,2}C_{a,2}^* \\ & A_a^* \end{bmatrix} K^{(-1)} = \begin{bmatrix} A_c & C_{c,1}C_{a,1}^* \\ & A_a^* \end{bmatrix}$$

which is, in fact, the equation (5.78) from the statement.

Consider the zero block in B-matrix from the right side of the equation (5.11) from Theorem 5.6 corresponding to the maximal unreachable subspace of $[\mathbf{P}(\Sigma\Sigma^*)]$:

$$B_c Q_{12}^{(-1)} + (B_a \Lambda_{o,a} A_a^* + DC_a^*) Q_{22}^{(-1)} = 0.$$

Because $Q_{22} \in \mathcal{D}$ is locally square invertible, the previous equation is equivalent to $B_c(Q_{12}Q_{22}^{-1})^{(-1)} + B_a \Lambda_{o,a} A_a^* + DC_a^* = 0$. When the additional partitioning is taken into account together with the identity $\Lambda_{o,a} = \Lambda_{o,a,1} + \Lambda_{o,a,2}$ where the Gramians $\Lambda_{o,a}$, $\Lambda_{o,a,1}$ and $\Lambda_{o,a,2}$ are bounded solutions for (5.6), (5.77) for $i = 1, 2$ respectively, the previous equation reads

$$B_c(Q_{12}Q_{22}^{-1})^{(-1)} + B_a \Lambda_{o,a,1} A_a^* + B_a \Lambda_{o,a,2} A_a^* + D \begin{bmatrix} C_{a,1}^* \\ 0 \end{bmatrix} + D \begin{bmatrix} 0 \\ C_{a,2}^* \end{bmatrix} = 0.$$

Let us rearrange the previous equation as

$$-B_c(Q_{12}Q_{22}^{-1})^{(-1)} - B_a \Lambda_{o,a,2} A_a^* - D \begin{bmatrix} 0 \\ C_{a,2}^* \end{bmatrix} = B_a \Lambda_{o,a,1} A_a^* + D \begin{bmatrix} C_{a,1}^* \\ 0 \end{bmatrix}.$$

By extending the column space of the previous equation with B_c , we obtain

$$\begin{aligned} & \begin{bmatrix} B_c & B_a \Lambda_{o,a,2} A_a^* + \begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix} C_{a,2}^* \end{bmatrix} \begin{bmatrix} I - Q_{12}Q_{22}^{-1} \\ -I \end{bmatrix}^{(-1)} \\ & = \begin{bmatrix} B_c & B_a \Lambda_{o,a,1} A_a^* + \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} C_{a,1}^* \end{bmatrix} \end{aligned}$$

which is, in fact, the second equation from the statement.

To prove the remaining equation, let us consider the equation (5.12) from Theorem 5.6 and write the equation corresponding to the zero block in the C-matrix of the 'hatted' realization. We have that

$$\hat{P}_{12}^* (A_c \Lambda_{o,c}^{(-1)} B_c^* + C_c D^*) + \hat{P}_{22}^* B_a^* = 0.$$

Since \hat{P}_{12} ($= Q_{11}\pi_2$) is invertible, employing the identity $\hat{P}_{11}\hat{P}_{21}^{-1} = -\hat{P}_{12}^{-*}\hat{P}_{22}^*$ and $\Lambda_{o,c} = \Lambda_{o,c,1} + \Lambda_{o,c,2}$ into the previous equation, we get

$$A_c \Lambda_{o,c,1}^{(-1)} B_c^* + A_c \Lambda_{o,c,2}^{(-1)} B_c^* + C_c D^* - (\hat{P}_{11} \hat{P}_{21}^{-1}) B_a^* = 0.$$

After rearranging the terms from the previous equation, we obtain that

$$-A_c \Lambda_{o,c,2}^{(-1)} B_c^* - [0 \ C_{c,2}] D^* + (\hat{P}_{11} \hat{P}_{21}^{-1}) B_a^* = A_c \Lambda_{o,c,1}^{(-1)} B_c^* + [C_{c,1} \ 0] D^*.$$

By equipping the previous equation with the identity $B_a^* = B_a^*$, we obtain

$$\begin{bmatrix} -I & \hat{P}_{11} \hat{P}_{21}^{-1} \\ & I \end{bmatrix} \begin{bmatrix} A_c \Lambda_{o,c,2} B_c^* + C_{c,2} [D_{12}^* \ D_{22}^*] \\ B_a^* \end{bmatrix} = \begin{bmatrix} A_c \Lambda_{o,c,1} B_c^* + C_{c,1} [D_{11}^* \ D_{21}^*] \\ B_a^* \end{bmatrix}$$

which, viz. $\hat{P}_{11} = Q_{12} \pi_1$ and $\hat{P}_{21} = Q_{22} \pi_1$ implies that $-K = \begin{bmatrix} -I & \hat{P}_{11} \hat{P}_{21}^{-1} \\ & I \end{bmatrix}$ and provides the truth for the equation (5.80). This completes the proof. ■

We also state the following simple consequence of Proposition 5.20.

Corollary 5.21. *Let the condition from Proposition 5.20 hold. Then, it holds that*

$$K^{-1} \begin{bmatrix} A_c & C_c C_a^* \\ & A_a^* \end{bmatrix} K^{(-1)} = \begin{bmatrix} A_c & \\ & A_a^* \end{bmatrix}. \quad (5.81)$$

PROOF: To prove the corollary, recall that

$$\begin{bmatrix} A_c & C_c C_a^* \\ & A_a^* \end{bmatrix} = \begin{bmatrix} A_c & C_{c,2} C_{a,2}^* \\ & A_a^* \end{bmatrix} + \begin{bmatrix} 0 & C_{c,1} C_{a,1}^* \\ 0 & 0 \end{bmatrix}.$$

Applying the state transformation K on each matrix from the previous equation, followed by employing the equation (5.78) from Proposition 5.20 and the fact that

$$K^{-1} \begin{bmatrix} 0 & C_{c,1} C_{a,1}^* \\ 0 & 0 \end{bmatrix} K^{(-1)} = \begin{bmatrix} 0 & -C_{c,1} C_{a,1}^* \\ 0 & 0 \end{bmatrix}$$

provides the truth for the additional statement. This completes the proof. ■

The state-space transformation K as introduced in Proposition 5.20 contains a solution for the Sylvester equation (5.18) $P_o = Q_{12} Q_{22}^{-1}$. Hence, it is worth investigating how to evaluate this quantity. Before we start to evaluate it, let us state the dual to Proposition 5.20.

Proposition 5.22. *Let a coisometric $\Sigma \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s additive realization $[\Sigma]$. Consider the partially nonreachable pair from the realization $[\mathbf{P}'(\Sigma^* \Sigma)]$ given by the equation (5.9) from Proposition 5.5 and let $P \in \mathcal{D}$ be the properly partitioned and unitary state-space transformation defined by (5.36) from*

Theorem 5.10. Define $N := \begin{bmatrix} -I & \\ -P_{12}^* P_{11}^{-*} & I \end{bmatrix}$.

Then, it holds that

$$N^{(-1)} \begin{bmatrix} A_a & \\ B_{c,2}^* B_{a,2} & A_c^* \end{bmatrix} N^{-1} = \begin{bmatrix} A_a & \\ B_{c,1}^* B_{a,1} & A_c^* \end{bmatrix}, \quad (5.82)$$

$$\begin{bmatrix} C_a^* A_{a,2}^{(-1)} A_a + \begin{bmatrix} D_{21}^* \\ D_{22}^* \end{bmatrix} B_{a,2} C_c^* & \\ & \end{bmatrix} N^{-1} = \begin{bmatrix} C_a^* A_{a,1}^{(-1)} A_a + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} B_{a,1} C_c^* & \\ & \end{bmatrix}, \quad (5.83)$$

$$-N^{(-1)} \begin{bmatrix} C_a & \\ A_c^* A_{c,2} C_c + B_{c,2}^* [D_{21} \ D_{22}] & \end{bmatrix} = \begin{bmatrix} C_a & \\ A_c^* A_{c,1} C_c + B_{c,1}^* [D_{11} \ D_{12}] & \end{bmatrix}. \quad (5.84)$$

The proof for the following corollary of Proposition 5.22 follows the dual vein to that of Corollary 5.21 and will be omitted.

Corollary 5.23. Let the conditions from Proposition 5.22 hold.

Then, it holds that

$$N^{(-1)} \begin{bmatrix} A_a & \\ B_c^* B_a & A_c^* \end{bmatrix} N^{-1} = \begin{bmatrix} A_a & \\ & A_c^* \end{bmatrix}. \quad (5.85)$$

Corollaries 5.23 and 5.23 together produce the following result.

Lemma 5.24. Let K and N be the state-space transformations from the propositions 5.20 and 5.22 respectively. Denote $Y_o = Q_{12} Q_{22}^{-1}$ and $Y_c = P_{12}^* P_{11}^{-*}$.

Then,

$$\begin{bmatrix} A_c & \\ B_a^* B_c & A_a^* \end{bmatrix} = \begin{bmatrix} I & -Y_o \\ -Y_c^* & I + Y_c^* Y_o \end{bmatrix} \begin{bmatrix} A_c & C_c C_a^* \\ & A_a^* \end{bmatrix} \begin{bmatrix} I + Y_o Y_c^* & Y_o \\ Y_c^* & I \end{bmatrix}^{(-1)} \quad (5.86)$$

whereby

$$\begin{bmatrix} I + Y_o Y_c^* & Y_o \\ Y_c^* & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -Y_o \\ -Y_c^* & I + Y_c^* Y_o \end{bmatrix}. \quad (5.87)$$

PROOF: The proof follows immediately after successively: bringing together equations (5.81) and (5.85) from the corollaries 5.21 and 5.23 respectively by applying the additional state-space transformation $\begin{bmatrix} & I \\ I & \end{bmatrix}$ on the anti-causal A-matrix given by (5.85) and transposing the result to obtain the causal A-matrix. This completes the proof. ■

In the following proposition we assume $\Sigma \in \mathcal{X}$ is unitary, and evaluate the reachability and the observability Gramians corresponding to the following causal triples

$$(A_{\sigma_i}, B_{\sigma_i}, \hat{C}_{\sigma_i}) := \left(\begin{bmatrix} A_c & C_{c,i} C_{a,i}^* \\ & A_a^* \end{bmatrix}, \begin{bmatrix} B_c & B_a \Lambda_{o,a,i} A_a^* + \begin{bmatrix} D_{1i} \\ D_{2i} \end{bmatrix} C_{a,i}^* \end{bmatrix}, \right. \quad (5.88)$$

$$\left. \begin{bmatrix} A_c \Lambda_{o,c,i} B_c^* + C_{c,1} [D_{1i}^* & D_{2i}^*] \\ & B_a^* \end{bmatrix} \right), \quad i = 1, 2 \quad (5.89)$$

as introduced in Proposition 5.20.

Proposition 5.25. *Let a unitary $\Sigma \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s realization $[\Sigma]$. Let properly partitioned Hermitians; $(\Pi)_i = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}_i \in \mathcal{D}^{2 \times 2}$, $i =$*

1, 2 and $(\hat{\Pi}_o)_i = \begin{bmatrix} \hat{\Pi}_{o,11} & \hat{\Pi}_{o,21}^ \\ \hat{\Pi}_{o,21} & \hat{\Pi}_{o,22} \end{bmatrix}_i \in \mathcal{D}^{2 \times 2}$, $i = 1, 2$ be bounded solutions for respectively*

$$A_{\sigma_i}^* (\Pi)_i A_{\sigma_i} + B_{\sigma_i}^* B_{\sigma_i} = (\Pi)_i^{(-1)}, \quad i = 1, 2 \quad (5.90)$$

and

$$A_{\sigma_i} (\hat{\Pi}_o)_i^{(-1)} A_{\sigma_i}^* + \hat{C}_{\sigma_i} \hat{C}_{\sigma_i}^* = (\hat{\Pi}_o)_i, \quad i = 1, 2 \quad (5.91)$$

being Lyapunov-Stein equations corresponding to the causal AB- and AC-pairs defined by (5.88)-(5.89).

Then,

$$(\Pi)_i = \begin{bmatrix} A_c & P_{12}^* P_{11}^{-*} \Lambda_{o,a,i} \\ \Lambda_{o,a,i} P_{11}^{-1} P_{12} & \Lambda_{o,a,i} - \Lambda_{o,a,i} \Lambda_a \Lambda_{o,a,i} \end{bmatrix}, \quad (5.92)$$

$$(\hat{\Pi}_o)_i = \begin{bmatrix} \Lambda_{o,c,i} - \Lambda_{o,c,i} \Lambda_c \Lambda_{o,c,i} & \Lambda_{o,c,i} P_{12}^* P_{22}^{-*} \\ P_{11}^{-1} P_{22} \Lambda_{o,c,i} & \Lambda_a \end{bmatrix}, \quad i = 1, 2. \quad (5.93)$$

PROOF: To prove the proposition, we shall prove each constitutive equation from equations (5.90) and (5.91) separately. From simplicity reasons it suffices to consider the case $i = 1$ only. The proof for the case $i = 2$ follows the same tract entirely.

To avoid work with unnecessary indices, we simply set $\Pi = (\Pi)_1$. Let us start with the equation (5.90). Its (1-1) block identity is $A_c^* \Pi_{11} A_c + B_c^* B_c = \Pi_{11}^{(-1)}$, which because $\ell_{A_c} < 1$, produces

$$\Pi_{11} = A_c. \quad (5.94)$$

The (1-2)-block identity from (5.90) is

$$A_c^* \Pi_{11} C_{c,1} C_{a,1}^* + A_c^* \Pi_{12} A_a^* + \underline{B_c^* B_a} \Lambda_{o,a,1} A_a^* + \underline{B_c^*} \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} C_{a,1}^* = \Pi_{12}^{(-1)}. \quad (5.95)$$

From the zero block in C-matrix on the right hand side of (5.37), we evaluate

$$B_c^* \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} = -(P_{22}^{-1} P_{12})^{(-1)} C_{a,1} - A_c^* \Lambda_c C_{c,1}. \quad (5.96)$$

Similarly, from the zero block in the A-matrix from the right hand side of (5.37) we evaluate

$$B_c^* B_a = (P_{12}^* P_{11}^{-*})^{(-1)} A_a - A_c^* (P_{12}^* P_{11}^{-*}). \quad (5.97)$$

After replacing the underlined terms in (5.95) with the previous two equations, we obtain a (trivial) Lyapunov-Stein equation

$$A_a^* \{ \Pi_{12} - (P_{12}^* P_{11}^{-*}) \Lambda_{o,a,1} \} A_a^* = \{ \Pi_{12} - (P_{12}^* P_{11}^{-*}) \Lambda_{o,a,1} \}^{(-1)}$$

whose only bounded fixed point is the zero block diagonal because $\ell_{A_c}, \ell_{A_a} < 1$. Hence,

$$\Pi_{12} = (P_{12}^* P_{11}^{-*}) \Lambda_{o,a,1}. \quad (5.98)$$

The (2 - 2)-block identity from (5.90) is

$$\begin{aligned} \Pi_{22}^{(-1)} &= C_{a,1} \underline{C_{c,1}^* \Lambda_c C_{c,1}} C_{a,1}^* + A_a \Pi_{12}^* C_{c,1} C_{a,1}^* + C_{a,1} C_{c,1}^* \Pi_{12} A_a^* + A_a \Pi_{22} A_a^* \\ &\quad + A_a \Lambda_{o,a,1} B_a^* B_a \Lambda_{o,a,1} A_a^* + A_a \Lambda_{o,a,1} B_a^* \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} C_{a,1}^* \\ &\quad + C_{a,1} \underline{[D_{11}^* \ D_{21}^*]} B_a \Lambda_{o,a,1} A_a^* + C_{a,1} \underline{[D_{11}^* \ D_{21}^*]} \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} C_{a,1}^*. \end{aligned}$$

From $\mathbf{P}_o(\Sigma^* \Sigma) = I$, we find that

$$[D_{11}^* \ D_{21}^*] \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} + C_{c,1}^* \Lambda_c C_{c,1} = I - C_{a,1} \Lambda_a^{(-1)} C_{a,1}. \quad (5.99)$$

By considering the subequation from the equation (5.37) corresponding to $\beta_1 (= 0)$, we evaluate

$$[D_{11}^* \ D_{21}^*] B_a = -C_{a,1}^* \Lambda_a^{(-1)} A_a - C_{c,1}^* (P_{12}^* P_{11}^{-*}). \quad (5.100)$$

After plugging the previous two equations in the above equation for $\Pi_{22}^{(-1)}$, we obtain

$$\begin{aligned} \Pi_{22}^{(-1)} &= C_{a,1} C_{a,1}^* - C_{a,1} C_{a,1}^* \Lambda_a C_{a,1} C_{a,1}^* + C_{a,1} C_{c,1}^* \{ \Pi_{12} - (P_{12}^* P_{11}^{-*}) \Lambda_{o,a,1} \} A_a^* \\ &\quad - C_{a,1} C_{a,1}^* \Lambda_a^{(-1)} A_a \Lambda_{o,a,1} A_a^* - A_a \Lambda_{o,a,1} A_a^* \Lambda_a^{(-1)} C_{a,1} C_{a,1}^* \\ &\quad + A_a \{ \underline{\Pi_{12}^*} - \Lambda_{o,a,1} (P_{11}^{-1} P_{12}) \} C_{c,1} C_{a,1}^* + A_a (\Pi_{22} + \Lambda_{o,a,1} \underline{B_a^* B_a} \Lambda_{o,a,1}) A_a^* \end{aligned}$$

where, viz. (5.98), the single underlined terms equal zero. Let us consider the following terms from the previous equation

$$\begin{aligned} X &:= C_{a,1} C_{a,1}^* - C_{a,1} C_{a,1}^* \Lambda_a C_{a,1} C_{a,1}^* \\ &\quad - C_{a,1} C_{a,1}^* \Lambda_a^{(-1)} A_a \Lambda_{o,a,1} A_a^* - A_a \Lambda_{o,a,1} A_a^* \Lambda_a^{(-1)} C_{a,1} C_{a,1}^*. \end{aligned}$$

After substitution of the identity $C_{a,1}C_{a,1}^* = \Lambda_{o,a,1}^{(-1)} - A_a \Lambda_{o,a,1} A_a^*$ in all terms from the previous equation, and after cancellations have been done, we eliminate the occurrence of $C_{a,1}C_{a,1}^*$ in X and obtain

$$X = \Lambda_{o,a,1}^{(-1)} - A_a \Lambda_{o,a,1} A_a^* + A_a \Lambda_{o,a,1} \underline{A_a^* \Lambda_a^{(-1)} A_a} \Lambda_{o,a,1} A_a^* - (\Lambda_{o,a,1} A_a \Lambda_{o,a,1})^{(-1)}.$$

After substituting X back into the equation for $\Pi_{22}^{(-1)}$, replacing the doubly underlined terms with the identity $A_a^* \Lambda_a^{(-1)} A_a + B_a^* B_a = \Lambda_a$ and rearranging the terms from the result, we obtain

$$A_a \{ \Pi_{22} + \Lambda_{o,a,1} A_a \Lambda_{o,a,1} - \Lambda_{o,a,1} \} A_a^* = \{ \Pi_{22} + \Lambda_{o,a,1} A_a \Lambda_{o,a,1} - \Lambda_{o,a,1} \}^{(-1)}$$

with the zero block diagonal as its only bounded fixed point solution because $\ell_{A_a} < 1$. Hence,

$$\Pi_{22} = A_{o,a,1} - A_{o,a,1} A_a \Lambda_{o,a,1}. \tag{5.101}$$

The equations (5.94), (5.98) and (5.101) together provide the proof that the equation (5.92) defines the reachability Gramian for the causal AB-pair (A_σ, B_σ) .

We proceed with the proof concerning the observability Gramian $\hat{\Pi}_{o,1}$ and, similar as above, we avoid the work with unnecessary indices by setting $\hat{\Pi}_o = (\hat{\Pi}_o)_1$.

The (2-2) constitutive equation from (5.91); $A_a^* \hat{\Pi}_{o,22}^{(-1)} A_a + B^* B_a = \hat{\Pi}_{o,22}^{(-1)}$ and the u.e.s argument; $\ell_{A_a} < 1$, lead to

$$\hat{\Pi}_{o,22} = A_a. \tag{5.102}$$

The (1-2)-term from (5.91) is

$$A_c \hat{\Pi}_{o,21}^{*(-1)} A_a + C_{c,1} C_{a,1}^* \hat{\Pi}_{o,22}^{(-1)} A_a + A_c \Lambda_{o,c,1}^{(-1)} \underline{B_c^* B_a} + C_{c,1} [\underline{D_{11}^*} \quad \underline{D_{21}^*}] B_a = \hat{\Pi}_{21}^*.$$

After replacing the underlined terms from the previous equation with equations (5.97) and (5.100) respectively, and after cancellations have been done, the remaining terms can be rearranged and written as

$$A_c \{ \hat{\Pi}_{o,21}^* - \Lambda_{o,c,1} (P_{12}^* P_{22}^{*-}) \}^{(-1)} A_a = \hat{\Pi}_{o,21}^* - \Lambda_{o,c,1} (P_{12}^* P_{22}^{*-}).$$

Viz. $\ell_{A_c} < 1$, the only bounded fixed point for the previous (trivial) Lyapunov-Stein equation implies that

$$\hat{\Pi}_{o,21}^* = \Lambda_{o,c,1} (P_{12}^* P_{22}^{*-}). \tag{5.103}$$

The (1-1)-block identity from (5.91) is

$$\begin{aligned} \hat{\Pi}_{o,11} &= A_c \hat{\Pi}_{o,11}^{*(-1)} A_c^* + C_{c,1} C_{a,1}^* \hat{\Pi}_{o,21}^{(-1)} A_c^* + A_c \hat{\Pi}_{o,21}^{*(-1)} C_{a,1} C_{c,1}^* \\ &\quad + C_{c,1} \underline{C_{a,1}^* \Lambda_a^{(-1)} C_{a,1}} C_{c,1}^* + A_c \Lambda_{o,c,1}^{(-1)} B_c^* B_c \Lambda_{o,c,1} A_c^* + A_c \Lambda_{o,c,1}^{(-1)} B_c^* \underline{\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}} C_{c,1}^* \\ &\quad + C_{c,1} [\underline{D_{11}^*} \quad \underline{D_{21}^*}] B_c \Lambda_{o,c,1} A_c^* + C_{c,1} \underline{\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}} C_{c,1}^*. \end{aligned}$$

After replacing the underlined terms from the previous identity with equations (5.99) and (5.96), we obtain

$$\begin{aligned} \hat{\Pi}_{o,11} = & A_c \hat{\Pi}_{o,11}^{(-1)} A_c^* + \underline{C_{c,1} C_{a,1}^* \hat{\Pi}_{o,21}^{(-1)} A_c^*} + \underline{\underline{A_c \hat{\Pi}_{21}^{*(-1)} C_{a,1} C_{c,1}^*}} \\ & + C_{c,1} (I - C_{c,1}^* A_c C_{c,1}) C_{c,1}^* + A_c^* A_{o,c,1}^{(-1)} B_c^* B_c A_{o,c,1}^{(-1)} A_c^* \\ & - \underline{\underline{A_c A_{o,c,1}^{(-1)} (P_{22}^{-1} P_{12})^{(-1)} C_{a,1} C_{c,1}^*}} - A_c A_{o,c,1}^{(-1)} A_c^* A_c C_{c,1} C_{c,1}^* \\ & - \underline{\underline{C_{c,1} C_{a,1}^* (P_{12}^* P_{22}^{-*})^{(-1)} A_{o,c,1}^{(-1)} A_c^*}} - C_{c,1} C_{c,1}^* A_c A_c A_{o,c,1}^{(-1)} A_c^* \end{aligned}$$

where, viz. (5.103) the single underlined and doubly underlined terms separately cancel. After the cancellations are done, the remaining terms do not contain factors $C_{a,1} C_{c,1}^*$. After substitution of the identity $C_{c,1} C_{c,1}^* = A_{o,c} - A_c A_{o,c}^{(-1)} A_c^*$ in all other terms containing the factor $C_{c,1} C_{c,1}^*$ a lot of cancellations occurs. After rearranging the remaining terms, we finally end up with the (trivial) Lyapunov-Stein equation

$$A_c \{ \hat{\Pi}_{o,11} - A_{o,c,1} + A_{o,c,1} A_c A_{o,c,1} \}^{(-1)} A_c^* = \hat{\Pi}_{o,11} - A_{o,c,1} + A_{o,c,1} A_c A_{o,c,1}.$$

Since $\ell_{A_c} < 1$, the previous equation has the zero as its only bounded solution. Hence,

$$\hat{\Pi}_{o,11} = A_{o,c,1} - A_{o,c,1} A_c A_{o,c,1}. \tag{5.104}$$

Equations (5.102), (5.103) and (5.104) together constitute the observability Gramian $\hat{\Pi}_o$ as we needed to show. This completes the proof. ■

In the following proposition we evaluate the reachability and the observability Gramians $(\Omega)_i \in \mathcal{D} \times \mathcal{D}$ and $(\hat{\Omega}_o)_i \in \mathcal{D} \times \mathcal{D}$ for $i = 1, 2$ of the following two triples of A-, B-, and C-matrices introduced in Proposition 5.22

$$(A_{\zeta_i}, B_{\zeta_i}, \hat{C}_{\zeta_i}) := \left(\begin{bmatrix} A_a \\ B_{c,i}^* B_{a,i} A_c^* \end{bmatrix}, \begin{bmatrix} C_a^* A_{a,i}^{(-1)} A_a + \begin{bmatrix} D_{i1}^* \\ D_{i2}^* \end{bmatrix} B_{a,i} C_c^* \end{bmatrix}, \right) \tag{5.105}$$

$$\begin{bmatrix} C_a \\ A_c^* A_{c,i} C_c + B_{c,i}^* [D_{i1} \ D_{i2}] \end{bmatrix}, \quad i = 1, 2. \tag{5.106}$$

Proposition 5.26. *Let a unitary $\Sigma \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s realization $[\Sigma]$. Let properly partitioned Hermitians; $(\Omega)_i = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^* & \Omega_{22} \end{bmatrix}_i \in \mathcal{D}^{2 \times 2}$, $i = 1, 2$ and $(\hat{\Omega}_o)_i = \begin{bmatrix} \hat{\Omega}_{o,11} & \hat{\Omega}_{o,21}^* \\ \hat{\Omega}_{o,21} & \hat{\Omega}_{o,22} \end{bmatrix}_i \in \mathcal{D}^{2 \times 2}$, $i = 1, 2$ be the unique bounded solutions for respectively*

$$A_{\zeta_i}^* (\Omega)_i^{(-1)} A_{\zeta_i} + B_{\zeta_i}^* B_{\zeta_i} = (\Omega)_i, \quad i = 1, 2 \tag{5.107}$$

and

$$A_{\zeta_i}(\hat{\Omega}_o)_i A_{\zeta_i}^* + \hat{C}_{\zeta_i} \hat{C}_{\zeta_i}^* = (\hat{\Omega}_o)_i^{(-1)}, \quad i = 1, 2 \quad (5.108)$$

being Lyapunov-Stein equations corresponding to the anti-causal AB- and AC-pairs defined by (5.105)-(5.106).

Then,

$$(\Omega)_i = \begin{bmatrix} \Lambda_{a,i} - \Lambda_{a,i} \Lambda_{o,a} \Lambda_{a,i} & \Lambda_{a,i} Q_{22}^{-*} Q_{12}^* \\ Q_{12} Q_{22}^{-1} \Lambda_{a,i} & \Lambda_{o,c} \end{bmatrix}, \quad (5.109)$$

$$(\hat{\Omega}_o)_i = \begin{bmatrix} \Lambda_{o,a} & Q_{22}^{-*} Q_{12}^* \Lambda_{a,i} \\ \Lambda_{a,i} Q_{12} Q_{22}^{-1} & \Lambda_{a,i} - \Lambda_{o,a,i} \Lambda_{o,c} \Lambda_{o,i} \end{bmatrix}, \quad i = 1, 2. \quad (5.110)$$

PROOF: The proof will be omitted since it follows similar tract to that for Proposition 5.25. It uses the equations corresponding to the block entries from the canonical unreachable realization given at the right hand side of the equation (5.11) from Theorem 5.6. ■

The following lemma states the necessary and sufficient conditions for the uniform reachability for the causal AB-pairs $(A_{\sigma,i}, B_{\sigma,i})$, $i = 1, 2$.

Lemma 5.27. *Reachability Gramians $(\Pi)_i$, $i = 1, 2$ corresponding to the causal AB-pairs $(A_{\sigma,i}, B_{\sigma,i})$, $i = 1, 2$ and defined in Proposition 5.25 are uniformly positive definite if and only if the observability Gramian $\Lambda_{o,a,1}$ is uniformly positive definite and*

$$\Lambda_{o,a,1}^{-1} > \Lambda_a + P_{11}^{-1} P_{12} \Lambda_c^{-1} P_{12}^* P_{11}^{-*}. \quad (5.111)$$

Or equivalently, if and only if the observability Gramian $\Lambda_{o,a,2}$ is uniformly positive definite and

$$\Lambda_{o,a,2}^{-1} > \Lambda_a + P_{11}^{-1} P_{12} \Lambda_c^{-1} P_{12}^* P_{11}^{-*}. \quad (5.112)$$

PROOF: (\Leftrightarrow) (Case $i = 1$) Let $(\Pi)_1$ be uniformly positive definite. Since Λ_c is invertible, we apply the matrix inversion lemma 2.17 and find that the $(\Pi)_1$ is invertible if and only if the Schur complement for Λ_c :

$$\Lambda_{o,a,1} - \Lambda_{o,a,1} \Lambda_a \Lambda_{o,a,1} - \Lambda_{o,a,1} P_{11}^{-1} P_{12} \Lambda_c^{-1} P_{12}^* P_{11}^{-*} \Lambda_{o,a,1}$$

is uniformly positive definite. Or equivalently if and only if

$$\Lambda_{o,a,1} > \Lambda_{o,a,1} \{ \Lambda_a + P_{11}^{-1} P_{12} \Lambda_c^{-1} P_{12}^* P_{11}^{-*} \} \Lambda_{o,a,1} \quad (> 0). \quad (5.113)$$

Since the right hand side from the previous inequality is uniformly positive definite, we achieve the necessary condition for the uniform positive definiteness of $(\Pi)_1$;

$$\Lambda_{o,a,1} > 0 \quad (\Leftrightarrow \Lambda_{o,a,1} \text{ is invertible}).$$

But the previous is equivalent to saying that there exists a locally square invertible Cholesky factor $\lambda_{o,a,1} \in \mathcal{D}$ for $\Lambda_{o,a,1}$; $\Lambda_{o,a,1} = \lambda_{o,a,1}^* \lambda_{o,a,1}$. When this decomposition for $\Lambda_{o,a,1}$ is substituted into (5.113), we obtain that

$$\Lambda_{o,a,1}^{-1} > A_a + P_{11}^{-1} P_{12} \Lambda_c^{-1} P_{12}^* P_{11}^{-*}$$

which, in turn, is the (sufficiency) condition (5.111) stated in the lemma.

The proof for the case $i = 2$, follows the same tract entirely.

The proof that the statements are equivalent follows immediately after recalling that, by Proposition 5.20, reachability Gramians $(\Pi)_1$ and $(\Pi)_2$ are related through the invertible state-space transformation $K = \begin{bmatrix} I & -Q_{12}Q_{22}^{-1} \\ & -I \end{bmatrix}$ as

$$(\Pi)_2 = K^*(\Pi)_1 K.$$

This completes the proof. ■

The following lemma proves similarly as Lemma 5.27. Hence, its proof will be omitted.

Lemma 5.28. *Reachability Gramians $(\Omega)_i$, $i = 1, 2$ corresponding to the anti-causal AB-pairs $(A_{\zeta_i}, B_{\zeta_i})$, $i = 1, 2$ and defined in Proposition 5.26 are uniformly positive definite if and only if the reachability Gramian $\Lambda_{a,1}$ is uniformly positive definite and*

$$\Lambda_{a,1}^{-1} > \Lambda_{o,a} + Q_{22}^{-*} Q_{12}^* \Lambda_{o,c}^{-1} Q_{12} Q_{22}^{-1}. \quad (5.114)$$

Or equivalently, if and only if the reachability Gramian $\Lambda_{a,2}$ is uniformly positive definite and

$$\Lambda_{a,2}^{-1} > \Lambda_{o,a} + Q_{22}^{-*} Q_{12}^* \Lambda_{o,c}^{-1} Q_{12} Q_{22}^{-1}. \quad (5.115)$$

5.6 Conjugation of unitary operators

As we have seen in the first section, the theory developed therein does not allow in general to check the unitarity property by means of a single Sylvester equation. Given a Sylvester equation, more is needed to say how a unitary operator is related to the terms the equation is defined in. However, if we pose additional conditions of regularity on the terms then, due to the additional geometric structure, the problem becomes more suitable to solve.

In this section we make connections between the theory developed in the previous section and the so called conjugation technique. The conjugation technique is widely

used in the signal processing, the filter design and the robust control [Kim97] and relies, in the LTI-context, on shifting the poles of a given inner matrix valued function with entries from the ring of proper stable real rational functions [Alb03] through the unit circle by means of a bilinear map to achieve the anti-innerness while preserving the same complexity. Actually, the conjugation is nothing else than the minimal degree external factorization applied on block columns (or block rows) of a partitioned matrix valued inner function. To allow this, we need to pose additional assumptions namely, that minimal realizations for the block columns (resp. block rows) of such a function are both uniformly observable (resp. uniformly reachable). Viz. these technical conditions, the conjugation extends to isometric and coisometric operators of mixed causality as well and, as we shall see in the case of unitary operators, it allows to test a mixed causality operator on the unitarity property by testing certain expressions on uniform positive definiteness.

Let us start from the unitary operator $\Sigma \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ given by the minimal realization (5.72) and having uniformly observable AC-pairs $(A_a, C_{a,i})$, $i = 1, 2$. Viz. the block column partitioning in $\Sigma \in \mathcal{X}$, we write

$$\Sigma = [\Sigma_1 \ \Sigma_2] \in \mathcal{X}^{1 \times 2}(\mathcal{M}, \mathcal{N})$$

where Σ_i , $i = 1, 2$ denote the left resp. the right column block from Σ . Then, there are $C_{W_i}, D_{W_i} \in \mathcal{D}$, $i = 1, 2$ for which

$$[W_i] = \left[\begin{array}{c|c} A_a^* & C_{W_i} \\ \hline C_{a,i}^* & D_{W_i} \end{array} \right], \quad i = 1, 2 \quad (5.116)$$

are u.e.s and respectively $\Lambda_{o,a,1}$ - and $\Lambda_{o,a,2}$ -unitary realizations for inner W_i , $i = 1, 2$ such that

$$\Sigma_i = \sigma_i W_i^*, \quad i = 1, 2 \quad (5.117)$$

for some causal, coisometric σ_i , $i = 1, 2$. Equivalently said σ_i , $i = 1, 2$ are minimal degree left external factors for Σ_i , $i = 1, 2$. Their respective u.e.s realizations can be obtained in forms

$$[\sigma_i] = \left[\begin{array}{cc|cc} A_c & C_{c,i} C_{a,i}^* & & C_{c,i} D_{W_i} \\ & A_a^* & & C_{W_i} \\ \hline B_c & B_a \Lambda_{o,a,i} A_a^* + \begin{bmatrix} D_{1i} \\ D_{2i} \end{bmatrix} & C_{a,i}^* & B_a \Lambda_{o,a,i} C_{W_i} + \begin{bmatrix} D_{1i} \\ D_{2i} \end{bmatrix} D_{W_i} \end{array} \right], \quad i = 1, 2. \quad (5.118)$$

In obtaining them, we have applied the mixed partial fraction lemma 2.17 to decompose terms of the mixed causality arising from products $\sigma_i = \Sigma_i W_i$, $i = 1, 2$. To keep notations simple, we replace the terms from previous realizations with shorter ones by defining

$$[\sigma_i] = \left[\begin{array}{c|c} A_{\sigma_i} & C_{\sigma_i} \\ \hline B_{\sigma_i} & D_{\sigma_i} \end{array} \right], \quad i = 1, 2 \quad (5.119)$$

where the AB-pairs $(A_{\sigma_i}, B_{\sigma_i})$, $i = 1, 2$ are the same as those from A-B-C-triples defined in (5.88)-(5.89).

Concurrently to the previously introduced conjugation, we introduce the conjugation at the input side of the anti-causal operator part of Σ . To allow this, we assume that the Gramians $A_{a,i}$, $i = 1, 2$ corresponding to respectively $(A_a, B_{a,i})$, $i = 1, 2$ are both uniformly reachable. Then, for some block diagonals $B_{V_i}, D_{V_i} \in \mathcal{D}$, $i = 1, 2$, realizations

$$[V_i] = \left[\begin{array}{c|c} A_a^* & B_{a,i}^* \\ \hline B_{V_i} & D_{V_i} \end{array} \right], \quad i = 1, 2$$

are u.e.s, minimal realizations for inner V_i , $i = 1, 2$ respectively. Viz. the block row partitioning in $\Sigma \in \mathcal{X}$, we write

$$\Sigma = \left[\begin{array}{c} \Sigma' \\ \Sigma'' \end{array} \right] \in \mathcal{X}^{2 \times 1}(\mathcal{M}, \mathcal{N})$$

where Σ' and Σ'' denote the top and the bottom block row from Σ . Then, there are causal isometric operators ζ_i , $i = 1, 2$ such that Σ admits an external factorization in the form

$$\Sigma = \left[\begin{array}{c} \Sigma' \\ \Sigma'' \end{array} \right] = \left[\begin{array}{c|c} V_1^* & \\ \hline & V_2^* \end{array} \right] \left[\begin{array}{c} \zeta_1 \\ \zeta_2 \end{array} \right].$$

U.e.s realizations $[\zeta_i]$, $i = 1, 2$ for ζ_i , $i = 1, 2$ are then possible to be obtained as

$$[\zeta_i] = \left[\begin{array}{c|c} A_a^* & B_{a,i}^* B_{c,i} \\ \hline & A_c \end{array} \middle| \begin{array}{c} A_a^* A_{a,i}^{(-1)} C_a + B_{a,i}^* [D_{i1} \ D_{i2}] \\ C_c \end{array} \right], \quad i = 1, 2. \quad (5.120)$$

We adopt the following short notation

$$[\zeta_i] = \left[\begin{array}{c|c} A_{\zeta_i}^* & B_{\zeta_i}^* \\ \hline C_{\zeta_i}^* & D_{\zeta_i}^* \end{array} \right], \quad i = 1, 2 \quad (5.121)$$

where we recognize $(A_{\zeta_i}, B_{\zeta_i})$, $i = 1, 2$ being AB-pairs from A-B-C-triples defined in (5.105)-(5.106).

The following result is a straightforward consequence of the propositions 5.20 and 5.22.

Corollary 5.29. *For u.e.s realizations $[\sigma_i]$, $i = 1, 2$ given by (5.118), it holds that $[\sigma_1]$ is uniformly reachable if and only if $[\sigma_2]$ is uniformly reachable.*

For u.e.s realizations $[\zeta_i]$, $i = 1, 2$ given by (5.118), it holds that $[\zeta_1]$ is uniformly observable if and only if $[\zeta_2]$ is uniformly observable.

PROOF: By Proposition 5.20 we conclude that the reachability pairs of $[\sigma_1]$, $i = 1, 2$ are related through the boundedly invertible state-space transformation $K =$

$\begin{bmatrix} I - Q_{12}Q_{22}^{-1} \\ -I \end{bmatrix}$ whose action on the AB-pair from $[\sigma_2]$ is defined by equations (5.78)-(5.79). Hence, the uniform reachability of the AB-pair from $[\sigma_1]$ implies the minimality for the AB-pair from $[\sigma_2]$ and vice versa. This completes the proof for the first statement.

The proof for the second statement will be omitted since it follows the same tract as the previous one. It is based on Proposition 5.22 which considers pairs $(A_{\zeta_i}, B_{\zeta_i})$ given by (5.105)-(5.106). This completes the proof. ■

By Corollary 5.29 and the propositions 5.20 and 5.22, the realizations

$$[\sigma] := \left[\begin{array}{cc|cc} A_c & C_{c,1}C_{a,1}^* & C_{c,1}D_{W_1} & K^{-1} \begin{bmatrix} C_{c,2}D_{W_2} \\ C_{W_2} \end{bmatrix} \\ 0 & A_a^* & C_{W_1} & \\ \hline B_c & B_a A_{o,a,1} A_a^* + \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} & C_{a,1}^* & \begin{bmatrix} D_{\sigma_1} \\ D_{\sigma_2} \end{bmatrix} \end{array} \right], \quad (5.122)$$

$$[\zeta] := \left[\begin{array}{cc|c} A_a^* B_{a,1}^* B_{c,1} & A_a^* A_{a,1}^{(-1)} C_a + B_{a,1}^* [D_{11} \ D_{12}] & \\ 0 & A_c & C_c \\ \hline B_{V_1} & D_{V_1} B_{c,1} & D_{\zeta_1}^* \\ [B_{V_2} \ D_{V_2} B_{c,2}] & N^{*(-1)} & D_{\zeta_2}^* \end{array} \right] \quad (5.123)$$

are u.e.s realizations for inner operators $\sigma = [\sigma_1 \ \sigma_2]$ and $\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$ respectively. Although AC-pairs $(A_a, C_{a,i})$, $i = 1, 2$ are assumed uniformly observable, $[\sigma_i]$, $i = 1, 2$ need not be reachable or observable. Similar conclusion regarding AB-pairs $(A_a, B_{a,i})$, $i = 1, 2$ holds for realizations $[\zeta_i]$, $i = 1, 2$ too. In the following lemma we prove that $[\sigma]$ as defined by (5.122) is a minimal u.e.s realization.

Lemma 5.30. *The realization $[\sigma]$ for the inner σ as given by (5.122) is a minimal, u.e.s realization.*

PROOF: By Proposition 5.12, we have that Σ admits the minimal inner/anti-inner factorization $\Sigma = \Sigma_c \Sigma_a$ where, for appropriate $c_c, d_c, b_a, d_{a,1}, d_{a,2} \in \mathcal{D}$, realizations:

$$[\Sigma_c] = \left[\begin{array}{c|c} A_c & c_c \\ \hline B_c & d_c \end{array} \right], \quad [\Sigma_a] = \left[\begin{array}{c|cc} A_a & C_{a,1} & C_{a,2} \\ \hline b_a & d_{a,1} & d_{a,2} \end{array} \right]$$

are minimal, u.e.s realizations for respectively the inner Σ_c and the anti-inner Σ_a . Since both $(A_a, C_{a,i})$, $i = 1, 2$ are uniformly observable, there is some inner Σ_m of the degree A_a^* , such that

$$\Sigma_a = \Sigma_m \begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix}$$

for inner W_i , $i = 1, 2$ defined by (5.116).

Since Σ admits the representation $\Sigma = \sigma \begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix}$ too, we conclude that

$$\sigma = \Sigma_c \Sigma_m$$

and, as the product of two inner terms, σ has the degree $\#(\sigma) = \#(\Sigma_c) + \#(\Sigma_m) = \#(A_c) + \#(A_a^*)$. Because the realization $[\sigma]$ is of this degree, we conclude that $[\sigma]$ is a minimal, u.e.s realization. This completes the proof. ■

The dual result to that stated in Lemma 5.30, considers the minimality of the realization $[\zeta]$ given by (5.123) and is given by the following lemma.

Lemma 5.31. *The realization $[\zeta]$ for the inner ζ as given by (5.123) is a minimal, u.e.s realization.*

The following lemma relates the uniform observability for $[\sigma_i]$, $i = 1, 2$ to the uniform observability for $(A_a, C_{a,i})$, $i = 1, 2$.

Lemma 5.32. *Realization $[\sigma_1]$ given by (5.118) is uniformly observable if and only if the causal AC-pair $(A_c, C_{c,1})$ is uniformly observable.*

Realization $[\sigma_2]$ given by (5.118) is uniformly observable if and only if the causal AC-pair $(A_c, C_{c,2})$ is uniformly observable.

PROOF: Consider the realization $[\sigma_1]$ given by (5.118). It suffices to show that its AC-pair admits the following Lyapunov-Stein equation

$$\begin{aligned} & \begin{bmatrix} A_c & C_{c,1} C_{a,1}^* \\ & A_a^* \end{bmatrix} \begin{bmatrix} A_{o,c,1}^{(-1)} & \\ & A_{o,a,1}^{(-1)} \end{bmatrix} \begin{bmatrix} A_c^* & \\ C_{a,1} C_{c,1}^* & A_a \end{bmatrix} \\ & + \begin{bmatrix} C_{c,1} D_{W_1} \\ C_{W_1} \end{bmatrix} \begin{bmatrix} D_{W_1}^* C_{c,1}^* & C_{W_1}^* \end{bmatrix} = \begin{bmatrix} A_{o,c,1} & \\ & A_{o,a,1}^{-1} \end{bmatrix} \end{aligned}$$

where $A_{o,c,1}$ and $A_{o,a,1}$ are bounded solutions for respectively (5.76) and (5.77) for $i = 1$. Hence, a uniform positive definiteness of the (bounded) solution for the previous Lyapunov equation is the necessary and sufficient condition for the observability of the realization $[\sigma_1]$.

By writing out the constitutive block equations from the previous Lyapunov-Stein equation, we obtain

$$\begin{aligned} A_a^* A_{o,a,1}^{(-1)} A_a + C_{W_1} C_{W_1}^* &= A_{o,a,1}^{-1}, \\ C_{c,1} (C_{a,1}^* A_{o,a,1}^{(-1)} A_a + D_{W_1} C_{W_1}^*) &= 0, \\ C_{c,1} (C_{a,1}^* A_{o,a,1}^{(-1)} C_{a,1} + D_{W_1} D_{W_1}^*) C_{c,1}^* + A_c A_{o,c,1}^{(-1)} A_c &= A_{o,c,1}. \end{aligned}$$

Because of the fact that $[W_1] = \{A_a^*, C_{a,1}^*, C_{W_1}, D_{W_1}\}$ is $A_{o,a,1}$ -unitary, u.e.s realization, the first two equations from the previous set become identities and $A_{o,a,1} > 0$. Since the third equation is equivalent to

$$A_c A_{o,c,1}^{(-1)} A_c^* + C_{c,1} C_{c,1}^* = A_{o,c,1}$$

we conclude that $\begin{bmatrix} A_{o,c,1} & \\ & A_{o,a,1}^{-1} \end{bmatrix} > 0$ if and only if the AC-pair $(A_c, C_{c,1})$ is uniformly observable. The proof for the second statement goes similarly. ■

In the following lemma we evaluate the observability Gramian for the realization $[\sigma]$ given by (5.122).

Proposition 5.33. *Consider AC-pairs from the u.e.s realizations $[\sigma_i]$, $i = 1, 2$ given by (5.118) for coisometric causal minimal degree left external factors σ_i , $i = 1, 2$ corresponding to Σ_i , $i = 1, 2$. Let $K = \begin{bmatrix} I & -Q_{12}Q_{22}^{-1} \\ & -I \end{bmatrix}$ be the state-space transformation as introduced in Proposition 5.20 and let $\Pi_o \in \mathcal{D}$ be the observability Gramian corresponding to the causal AC-pair*

$$\left(\begin{bmatrix} A_c & C_{c,1} C_{a,1}^* \\ & A_a^* \end{bmatrix}, \begin{bmatrix} C_{c,1} D_{W_1} & K^{-1} \begin{bmatrix} C_{c,2} D_{W_2} \\ C_{W_2} \end{bmatrix} \\ C_{W_1} & \end{bmatrix} \right)$$

from u.e.s realization $[\sigma]$ given by (5.122).

Then,

$$(1) \quad \Pi_o = \begin{bmatrix} A_{o,c,1} & \\ & A_{o,a,1}^{-1} \end{bmatrix} + K^{-1} \begin{bmatrix} A_{o,c,2} & \\ & A_{o,a,2}^{-1} \end{bmatrix} K^{-*} \quad (5.124)$$

(2) Π_o is uniformly positive definite, i.e $\Pi_o > 0$.

PROOF:

(1) Similarly as it has been done in the proof of Lemma 5.32, we first convince ourselves that

$$\Pi_{o,i} := \begin{bmatrix} A_{o,c,i} & \\ & A_{o,a,i}^{-1} \end{bmatrix}, \quad i = 1, 2$$

are the observability Gramians for

$$\left(\begin{bmatrix} A_c & C_{c,i} C_{a,i}^* \\ & A_a^* \end{bmatrix}, \begin{bmatrix} C_{c,i} D_{W_i} \\ C_{W_i} \end{bmatrix} \right), \quad i = 1, 2$$

and are not necessarily uniformly positive definite.

Then, application of the state-space transformation $K \in \mathcal{D}$ whose action on the latter AC-pair is defined by $(K^{-1}(\cdot)K^{(-1)}, K^{-1}(\cdot))$, yields

$$\Pi_o = \Pi_{o,1} + K^{-1} \Pi_{o,2} K^{-*}$$

(2) By Lemma 5.30, we have that $[\sigma]$ is a minimal, u.e.s realization. Hence, Π_o is a uniformly positive definite Hermitian as we needed to show. We also provide an alternative proof for this fact.

We evaluate Π_o and obtain that

$$\Pi_o = \begin{bmatrix} \Lambda_{o,c} + Q_{12}Q_{22}^{-1}\Lambda_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^* & Q_{12}Q_{22}^{-1}\Lambda_{o,a,2}^{-1} \\ \Lambda_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^* & \Lambda_{o,a,1}^{-1} + \Lambda_{o,a,2}^{-1} \end{bmatrix}.$$

Then, Π_o is uniformly positive definite if and only if the Schur complement of $\Pi_{o,11}$ with respect to Π_o ;

$$X := \Lambda_{o,c} + Q_{12}Q_{22}^{-1}\Lambda_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^* - Q_{12}Q_{22}^{-1}\Lambda_{o,a,2}^{-1}(\Lambda_{o,a,1}^{-1} + \Lambda_{o,a,2}^{-1})^{-1}\Lambda_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^*$$

is uniformly positive definite (i.e. $X > 0$).

By substitution of the identity

$$\begin{aligned} (\Lambda_{o,a,1}^{-1} + \Lambda_{o,a,2}^{-1})^{-1} &= \left(\Lambda_{o,a,2}^{-1}(\Lambda_{o,a,1} + \Lambda_{o,a,2})\Lambda_{o,a,1}^{-1} \right)^{-1} \\ &= \Lambda_{o,a,1}\Lambda_{o,a}^{-1}\Lambda_{o,a,2} = (\Lambda_{o,a} - \Lambda_{o,a,2})\Lambda_{o,a}^{-1}\Lambda_{o,a,2} \\ &= \Lambda_{o,a,2} - \Lambda_{o,a,2}\Lambda_{o,a}^{-1}\Lambda_{o,a,2} \end{aligned}$$

into the condition $X > 0$, we obtain

$$\Lambda_{o,c} + Q_{12}Q_{22}^{-1}\Lambda_{o,a}^{-1}Q_{22}^{-*}Q_{12}^* > 0$$

which is the equivalent condition to $X > 0$ and is satisfied since $\Lambda_{o,c} > 0$. Hence, $\Pi_o > 0$. This completes the proof. ■

The following proposition states relationships between unitary state-space transformations P and Q introduced in the theorems 5.6 and 5.10 and Gramians: Λ_c , Λ_a , $\Lambda_{o,c}$ and $\Lambda_{o,a}$.

Proposition 5.34. *For the minimal, u.e.s realization $[\sigma]$ for the inner σ given by (5.122) it holds that*

$$\Lambda_a\Lambda_{o,a} = I + P_{11}^{-1}P_{12}Q_{12}Q_{22}^{-1}, \tag{5.125}$$

$$0 = P_{11}^{-1}P_{12}\Lambda_{o,c} + \Lambda_aQ_{22}^{-*}Q_{12}^*, \tag{5.126}$$

$$0 = \Lambda_cQ_{12}Q_{22}^{-1} + P_{12}^*P_{11}^{-*}\Lambda_{o,a}, \tag{5.127}$$

$$\Lambda_c\Lambda_{o,c} = I + P_{12}^*P_{11}^{-*}Q_{22}^{-*}Q_{12}^* \tag{5.128}$$

where $P_{11}, P_{12}, Q_{12}, Q_{22} \in \mathcal{D}$ constitute unitary state-space transformations P and Q introduced in the theorems 5.6 and 5.10 respectively.

PROOF: The u.e.s realization $[\sigma]$ given by (5.122) is a minimal realization for the inner operator σ . Hence, its reachability and observability Gramians Π and Π_o as defined by (5.92) and (5.124) are uniformly positive definite and such that $\Pi^{-1} = \Pi_o$.

Π_o as defined in Proposition 5.33 is equal to

$$\Pi_o = \begin{bmatrix} A_{o,c} + Q_{12}Q_{22}^{-1}A_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^* & Q_{12}Q_{22}^{-1}A_{o,a,2}^{-1} \\ A_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^* & A_{o,a,1}^{-1} + A_{o,a,2}^{-1} \end{bmatrix}.$$

We complete the proof by showing that each equation from (5.125)-(5.128) corresponds to an appropriate block identity from

$$\Pi \Pi_o = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (5.129)$$

Indeed, consider (2-2)-block identity from the previous equation:

$$I = A_{o,a,1}P_{11}^{-1}P_{12}Q_{12}Q_{22}^{-1}A_{o,a,2}^{-1} + (A_{o,a,1} - A_{o,a,1}A_aA_{o,a,1})(A_{o,a,1}^{-1} + A_{o,a,2}^{-1}).$$

The previous equation, after employing the fact that

$$A_{o,a,1}A_{o,a,2}^{-1} = (A_{o,a} - A_{o,a,2})A_{o,a,2}^{-1} = A_{o,a}A_{o,a,2}^{-1} - I \quad (5.130)$$

and after successively: canceling the identity matrix, pre- and postmultiplying with $A_{o,a,1}^{-1}$ and $A_{o,a,2}$ respectively, leads to the equation (5.125).

Consider now the (2-1) block identity from (5.129). It holds that

$$0 = A_{o,a,1}P_{11}^{-1}P_{12}(A_{o,c} + Q_{12}Q_{22}^{-1}A_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^*) + A_{o,a,1}(I - A_aA_{o,a,1})(A_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^*).$$

After premultiplying the previous equation with $A_{o,a,1}^{-1}$, employing the identity (5.130) and the already obtained equation (5.125), we obtain (5.126).

The (1-2)-block identity from (5.129) is

$$0 = A_cQ_{12}Q_{22}^{-1}A_{o,a,2}^{-1} + P_{12}^*P_{11}^{-*}A_{o,a,1}(A_{o,a,1}^{-1} + A_{o,a,2}^{-1}).$$

After postmultiplying the previous equation with $A_{o,a,2}$ and employing the fact that $A_{o,a} = A_{o,a,1} + A_{o,a,2}$ into the result we obtain (5.127).

Finally, the (1-1)-block identity from (5.129) is

$$I = A_c(A_{o,c} + Q_{12}Q_{22}^{-1}A_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^*) + P_{12}^*P_{11}^{-*}A_{o,a,1}A_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^*.$$

When the term $A_{o,a,1}A_{o,a,2}^{-1}$ from the latter equation is replaced by means of the identity (5.130), we obtain that

$$I = A_cA_{o,c} + (A_cQ_{12}Q_{22}^{-1} + P_{12}^*P_{11}^{-*}A_{o,a})A_{o,a,2}^{-1}Q_{22}^{-*}Q_{12}^* - P_{12}^*P_{11}^{-*}Q_{22}^{-*}Q_{12}^*$$

which, because of (5.127), reduces to (5.128) as we needed to show. This makes the proof complete. ■

We note that equations (5.126) and (5.127) from Proposition 5.34 are the same as (5.60)-(5.61) from Lemma 5.16. The remaining equations (5.125) and (5.128) reveal the additional geometric structure of partitioned unitary operators arising from the condition of regularity posed on their minimal realizations. As the consequence of the regularity assumptions, the entire Σ can be evaluated by means of data in which a Sylvester equation is defined as follows. First, assume the Sylvester equation (5.18) in Y_o is given. Then, minimal, u.e.s realizations $[W_i], i = 1, 2$ as given by (5.116) for inner $W_i, i = 1, 2$ can be constructed by appropriate embeddings of pairs $(A_a, C_{a,i}), i = 1, 2$ respectively. Note that then the AC-pair of $[\sigma]$ is defined completely in: the terms from the equation (5.18), its solution Y_o and the embedding terms $C_{W_i}, D_{W_i} \in \mathcal{D}$. If the solution Y_o is known, a Π_o -unitary embedding for the AC-pair of $[\sigma]$ produces a minimal, u.e.s Π_o -unitary realization for the inner σ . Finally, a unitary operator Σ corresponding to the Sylvester equation can be evaluated through its defining equation

$$\Sigma = [\sigma_1 \ \sigma_2] \begin{bmatrix} W_1^* & \\ & W_2^* \end{bmatrix}.$$

The statements made in the following two lemmas are direct consequences of minimality of realizations $[\sigma]$ and $[\zeta]$.

Lemma 5.35. *Let a unitary $\Sigma = [\Sigma_1 \ \Sigma_2] \in \mathcal{X}^{1 \times 2}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s additive realization $[\Sigma]$ in which $(A_a, C_{a,i}), i = 1, 2$ are uniformly observable pairs.*

Then,

$$A_{o,a,1}^{-1} > A_a + P_{11}^{-1} P_{12} A_c^{-1} P_{12}^* P_{11}^{-*} \tag{5.131}$$

where $P_{11}, P_{12} \in \mathcal{D}$ constitute the unitary matrix P defined in Theorem 5.10.

PROOF: The proof follows immediately after convincing ourselves that the realization $[\sigma]$ given by (5.122) and mentioned in the statement is a minimal realization if and only if it is uniformly reachable. By Lemma 5.27, the latter fact is true if and only if the condition (5.131) is satisfied for the block diagonals P_{11}, P_{12} introduced by Theorem 5.10. ■

Dually to the statement in Lemma 5.35, we formulate the following result.

Lemma 5.36. *Let a unitary $\Sigma = \begin{bmatrix} \Sigma' \\ \Sigma'' \end{bmatrix} \in \mathcal{X}^{2 \times 1}(\mathcal{M}, \mathcal{N})$ be given by a minimal, u.e.s additive realization $[\Sigma]$ in which $(A_a, B_{a,i}), i = 1, 2$ are uniformly reachable pairs.*

Then,

$$A_{a,1}^{-1} > A_{o,a} + Q_{22}^{-*} Q_{12}^* A_{o,c}^{-1} Q_{12} Q_{22}^{-1} \tag{5.132}$$

where $Q_{12}, Q_{22} \in \mathcal{D}$ constitute the unitary matrix Q defined in Theorem 5.6.

As we have already discovered in the first section, locally finite unitary operator admits both the minimal inner/anti-inner and the anti-inner/inner factorization. In the following proposition we evaluate reachability Gramians corresponding to the realizations (5.68) and (5.69) as well as the state-space transformation relating them.

Proposition 5.37. *Assume the conditions from Lemma 5.35 hold for a minimal, u.e.s, additive realization $[\Sigma]$ for a unitary $\Sigma = [\Sigma_1 \ \Sigma_2] \in \mathcal{X}^{1 \times 2}(\mathcal{M}, \mathcal{N})$. Then, the reachability Gramians corresponding to realizations $[\Sigma'_a \Sigma_c]$ and $[\Sigma'_c \Sigma_a^*]$ given by (5.68) and (5.69) are respectively*

$$\begin{bmatrix} \Lambda_c \\ \Lambda_a^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Lambda_{o,c}^{-1} \\ \Lambda_{o,a} \end{bmatrix}. \quad (5.133)$$

In addition, for the block diagonals Y_c and Y_o as introduced in Lemma 5.24, it holds that

$$\begin{bmatrix} I & -Y_o \\ -Y_c^* & I + Y_c^* Y_o \end{bmatrix} \begin{bmatrix} \Lambda_{o,c} \\ \Lambda_{o,a}^{-1} \end{bmatrix} \begin{bmatrix} I & -Y_c \\ -Y_o^* & I + Y_o^* Y_c \end{bmatrix} = \begin{bmatrix} \Lambda_c^{-1} \\ \Lambda_a \end{bmatrix}. \quad (5.134)$$

PROOF: By direct inspection, similarly as it has been done in the proof for Lemma 5.32, we convince ourselves that the Gramians mentioned in the statement are indeed the reachability Gramians for realizations $[\Sigma'_a \Sigma_c]$ and $[\Sigma'_c \Sigma_a^*]$ given by (5.68) and (5.69).

To prove the additional statement, we put the equation (5.134) into the equivalent form

$$\begin{bmatrix} I & -Y_o \\ -Y_c^* & I + Y_c^* Y_o \end{bmatrix} \begin{bmatrix} \Lambda_{o,c} \\ \Lambda_{o,a}^{-1} \end{bmatrix} = \begin{bmatrix} \Lambda_c^{-1} \\ \Lambda_a \end{bmatrix} \begin{bmatrix} I + Y_c Y_o^* & Y_c \\ Y_o^* & I \end{bmatrix}$$

where we have used the equation (5.87) from Lemma 5.24 for the inverse of the most right block diagonal matrix on the left side of (5.134). After invocation of equations (5.125)-(5.128) from Proposition 5.34 and recalling that $Y_o = Q_{12} Q_{22}^{-1}$ and $Y_c = P_{12}^* P_{11}^{-*}$, the proof follows straightforwardly. ■

Recall that in the previous section we have posed the goal to evaluate the state-space transformation between the realizations $[\Sigma'_a \Sigma_c]$ and $[\Sigma'_c \Sigma_a^*]$ defined by (5.68) and (5.69). By straightforward inspection it shows that this transformation is the same as that stated in Lemma 5.24.

5.7 Conclusions

We have considered the minimal factorization problem for semi-separable isometric and coisometric operators mapping between non-uniform Hilbert spaces and given

by minimal, u.e.s realizations. The propositions 5.12 and 5.14 state that the minimal factorization problem for such an operator (Σ) always has a solution. They also state how operators given by minimal, u.e.s realizations can be tested on the isometricity and the coisometricity. As we have shown, the characterization of operators involves Sylvester equations (in Y_o, Y_c) where, in the particular case of a unitary operator, it is possible to find relationships between solutions of Sylvester equations and Gramians corresponding to the minimal realization (see the lemmas 5.8 and 5.11). Solving the Sylvester equations was however not needed for purposes of minimal factorization.

The theorems 5.17 and 5.19 and in particular their additional statements reveal algebraic properties of realizations and explain how the minimal factorization problem is related to factorizations of the realizations themselves. As the consequence, they bring the study of single causality and mixed causality isometric and coisometric operators on equal footing. We have also considered embeddings of isometric and coisometric operators of mixed causality into a unitary operator and formulated an one-pass algorithm for doing this completion.

From the section 5.5 on, we introduced the additional partitioning in a unitary operator and in a minimal additive realization for it. We then considered the state-space properties of the operator's parts induced by the partitioning and characterized these parts by means of (uniform) positivity of the bounded solutions of some Lyapunov-Stein equations. On one side, this made possible (see Lemma 5.24) to state the relationship between the state transition operators from the realizations $[\mathbf{P}(\Sigma\Sigma^*)]$ and $[\mathbf{P}_{\mathcal{L}}(\Sigma^*\Sigma)]$. On the other side, finding the transformation (as the Cholesky factor for the bounded Hermitian solution of some Lyapunov-Stein equation defined in terms from either $[\mathbf{P}(\Sigma\Sigma^*)]$ or $[\mathbf{P}_{\mathcal{L}}(\Sigma^*\Sigma)]$) made possible to find the link between Y_o, Y_c and the bounded solution for the Lyapunov-Stein equation.



6

MINIMAL FACTORIZATIONS OF OPERATORS FROM \mathcal{X}

Factorization theory is concerned about representing a function or an operator as the product of, at least, two factors. Factorizations of polynomial and rational matrix functions appears in many branches of analysis and applications. In network theory [Bel68],[DewBelNewc71],[DD81], minimal factorizations are of special interest in order to design a network as a cascade connection of sections with simple structure. As the notion of 'simplicity' is intrinsically related to the McMillan degree of the network in question, emphasizing numerical computability, it became of interest to relate the factorization to state-space realizations.

6.1 Introduction

In this chapter we study necessary and sufficient conditions for an invertible operator of mixed causality to have a *minimal factorization*. The approach we follow throughout is geometric and relies upon investigating the properties of appropriate Hankel maps associated to a minimal realization for the operator in question. Throughout this chapter, $[T] = \{A_c, B_c, C_c, A_a, B_a, C_a, D\}$ denotes a minimal additive, u.e.s realization for a left invertible bounded operator $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$. Let $t \in \mathcal{X}(\mathcal{N}, \mathcal{M})$ denote a left inverse for T , i.e. $tT = I_{\mathcal{N}}$ and let $[t] = \{a_c, b_c, c_c, a_a, b_a, c_a, d\}$ be a minimal, additive u.e.s realization for t .

In the time-invariant realization theory [DD81],[BartGohKaa79] it is known that a minimal factorization for a causal T requires partitioning of poles and zeros of T into two sets to form the factors. Of course, this partitioning is not unique and a possible candidate for the factor, if exists, can be generated by finding the state-space transformation $Q \in \mathcal{D}$ which brings the state transition operators from $[T]$ into an appropriate block triangular form. This transformation is not arbitrary however and is related to another appropriate transformation $q \in \mathcal{D}$ which block-triangularizes the state transition operators from $[t]$ and therefore introduces a minimal factorization for t i.e. $t = t''t'$ where for some intermediate space \mathcal{M}' it holds that $t' \in \mathcal{X}(\mathcal{N}, \mathcal{M}')$ and $t'' \in \mathcal{X}(\mathcal{M}', \mathcal{M})$. Concurrently T splits in a minimal way as $T = T'T''$ for some $T' \in \mathcal{X}(\mathcal{M}, \mathcal{M}')$ and $T'' \in \mathcal{X}(\mathcal{M}', \mathcal{N})$. As a consequence of the factorizations in T and t , a well chosen Q and q shall imply that $t'T' = I_{\mathcal{M}'}$ and $t''T'' = I_{\mathcal{N}}$.

An important issue of the factorization theory is how to express the necessary and sufficient conditions for a minimal factorization of T through the algebraic properties of the transformations Q and q .

In the subsequent sections, in order of appearance, we recall some relevant facts from the realization theory of mixed causality operators. Then, we study the relationship between the cancellation in a product between operators and observability and reachability of realizations of the product itself. This study cumulates in finding algebraic conditions under which a product of two operators is an operator of lower complexity.

The problem of completion of a partially specified invertible operator when its partially specified inverse is given is also considered and solved. The problem of a minimal causal/anti-causal factorization J-unitary operator is worked out and necessary and sufficient conditions for the factorization are stated. Moreover, all possible minimal causal/anti-causal factorizations are produced. Finally, the problem of extraction of intrinsic inner factors from a J-unitary operator is considered and an algorithm is stated.

6.2 Product of operators from \mathcal{X}

Let us first introduce the class of factorizations we will deal with throughout.

Definition 6.1. *Let $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ be a bounded locally finite operator. If there exist an intermediate non-uniform space \mathcal{M}' and operators $T' \in \mathcal{X}(\mathcal{M}, \mathcal{M}')$ and $T'' \in \mathcal{X}(\mathcal{M}', \mathcal{N})$ such that*

$$T = T'T'' \quad \text{and} \quad s\text{-dim}(T) = s\text{-dim}(T') + s\text{-dim}(T'') \quad (6.1)$$

where $s\text{-dim}(\cdot)$ denote the dimension sequence for the operator \cdot , then the factorization (6.1) is called a minimal (or minimal degree) factorization.

Let us also state some elementary facts regarding the product of two operators from \mathcal{X} .

Lemma 6.2. *Let*

$$[T'] = \{A'_c, B'_c, C'_c, A'_a, B'_a, C'_a, D'\}$$

and

$$[T''] = \{A''_c, B''_c, C''_c, A''_a, B''_a, C''_a, D''\}$$

be minimal, additive u.e.s realizations for $T' \in \mathcal{X}(\mathcal{M}, \mathcal{M}')$ and $T'' \in \mathcal{X}(\mathcal{M}', \mathcal{N})$. Then a u.e.s, additive realization for $T'T'' \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ is given by

$$[T'T''] = \left[\begin{array}{cc|cc|c} A'_c & C'_c B''_c & & & C'_c D'' + A'_c L_{ca}^{(-1)} C''_a \\ & A''_c & & & C''_c \\ \hline & A'_a & C'_a B''_a & & C'_a D'' + A'_a K_{ac} C''_c \\ & & A''_a & & C''_a \\ \hline B'_c & D' B''_c & B'_a & D' B''_a & D' D'' + B'_a K_{ac} C''_c \\ & + B'_a K_{ac} A''_c & & + B'_c L_{ca}^{(-1)} A''_a & + B'_c L_{ca}^{(-1)} C''_a \end{array} \right] \quad (6.2)$$

where $L_{ca}, K_{ac} \in \mathcal{D}$ are unique bounded solutions for

$$A'_c L_{ca}^{(-1)} A''_a + C'_c B''_a = L_{ca}, \quad (6.3)$$

$$A'_a K_{ac} A''_c + C'_a B''_c = K_{ac}^{(-1)}. \quad (6.4)$$

PROOF: The proof follows after evaluation of the product $T'T''$ analytically. The product introduces: one diagonal term, two product terms of mixed causality, one causal/causal product term and one anti-causal/anti-causal product term. After the mixed causality terms are decomposed by virtue of the mixed partial fraction lemma 2.17 and the Lyapunov-Stein equations (6.3) and (6.4), the obtained terms allow for putting into an additive realization in the form (6.2). ■

It is also possible to obtain (6.2) by working on realizations $[T']$ and $[T'']$ straightforwardly. We first adjust these realizations by inserting extra block rows and columns and obtain that

$$\left[\begin{array}{c|c|c} A'_c & & C'_c \\ \hline I & & \\ \hline & A'_a & C'_a \\ & & I \\ \hline B'_c & B'_a & D' \end{array} \right] \left[\begin{array}{c|c|c} I & & C''_c \\ \hline & A''_c & \\ \hline & & I \\ & & A''_a \\ & & C''_a \\ \hline B''_c & B''_a & D'' \end{array} \right] = \left[\begin{array}{cc|cc|c} A'_c & C'_c B''_c & & C'_c B''_a & C'_c D'' \\ & A''_c & & C''_c & C''_c \\ \hline & C'_a B''_c & A'_a & C'_a B''_a & C'_a D'' \\ & & A''_a & C''_a & C''_a \\ \hline B'_c & D' B''_c & B'_a & D' B''_a & D' D'' \end{array} \right]$$

To show that the realization on the right side of the previous equation is an alternative realization to the realization (6.2) from Lemma 6.2, it suffices to decompose the terms of mixed causality governed by $C'_c B''_a$ and $C'_a B''_c$ by means of the mixed partial fraction lemma 2.17 and the Lyapunov-Stein equations (6.3) and (6.4). These terms are then distributed into the strictly causal, the strictly anti-causal and the constant term yielding the realization (6.2).

Realization (6.2) from Lemma 6.2 is not necessarily minimal. It is minimal if and only if its causal and the anti-causal parts are both minimal on their own (see Chapter 2).

Definition 6.3. $T'' \in \mathcal{X}(\mathcal{M}', \mathcal{N})$ is said to cancel the dynamics of $T' \in \mathcal{X}(\mathcal{M}, \mathcal{M}')$ on the right side if $\mathcal{H}_{o,c}(T'T'') \subseteq \mathcal{H}_{o,c}(T'')$ and $\mathcal{H}_{o,a}(T'T'') \subseteq \mathcal{H}_{o,a}(T'')$.

The qualification 'right' mentioned in Definition 6.3 will be left out whenever it is clear from the context what is being meant.

In Definition 6.3 we do not assume (partial) invertibility for neither T' nor T'' in order to have a canceling dynamics. Note that even in cases when T' and T'' are such that $T'T'' = 0$, we still have a canceling dynamics although none of the involved operators need be partially invertible.

In the following lemma we state an algebraic test for such a cancellation.

Lemma 6.4. Let T' and T'' be as in Lemma 6.2. T'' cancels the dynamics of T' if and only if there are invertible $r_c, r_a \in \mathcal{D}^{2 \times 2}$ with $r_{c,ii}, r_{a,ii}$, $i = 1, 2$ invertible such that

$$\begin{aligned} r_c^{-1} \begin{bmatrix} A'_c & C'_c B''_c \\ & A''_c \end{bmatrix} r_c^{(-1)} &= \begin{bmatrix} \alpha_{c,11} & \\ \alpha_{c,21} & A''_c \end{bmatrix}, \\ r_c^{-1} \begin{bmatrix} C'_c D'' + A'_c L_{ca}^{(-1)} C''_a \\ C''_c \end{bmatrix} &= \begin{bmatrix} 0 \\ C''_c \end{bmatrix} \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} r_a^{-(-1)} \begin{bmatrix} A'_a & C'_a B''_a \\ & A''_a \end{bmatrix} r_a &= \begin{bmatrix} \alpha_{a,11} & \\ \alpha_{a,21} & A''_a \end{bmatrix}, \\ r_a^{-(-1)} \begin{bmatrix} C'_a D'' + A'_a K_{ac} C''_c \\ C''_a \end{bmatrix} &= \begin{bmatrix} 0 \\ C''_a \end{bmatrix}. \end{aligned} \quad (6.6)$$

In addition, r_c and r_a may be chosen equal

$$r_c = Q_c \begin{bmatrix} I & \\ & Q_{c,22}^{-1} \end{bmatrix}, \quad r_a = Q_a \begin{bmatrix} I & \\ & Q_{a,22}^{-1} \end{bmatrix}$$

for some properly partitioned unitary transformations $Q_c, Q_a \in \mathcal{D}^{2 \times 2}$ having invertible $Q_{c,ii}, Q_{a,ii}$, $i = 1, 2$.

PROOF: (\Rightarrow) If T'' cancels the dynamics of T' then $\mathcal{H}_{o,c}(T'T'') \subseteq \mathcal{H}_{o,c}(T'')$ and $\mathcal{H}_{o,a}(T'T'') \subseteq \mathcal{H}_{o,a}(T'')$. Let us consider the causal operator part from $T'T''$ given by realization (6.2). Because of the nonminimality in the causal AC-pair, by the Kalman canonical observer form theorem, we have that there exists a unitary state-space transformation $Q_c \in \mathcal{D}^{2 \times 2}$ such that

$$\left[\begin{array}{cc|c} A'_c & C'_c B''_c & C'_c D''_c \\ & & + A'_c L_{ca}^{(-1)} C''_a \\ \hline & A''_c & C''_c \\ B'_c & D' B''_c & * \\ + B'_a K_{ac} A''_c & & \end{array} \right] \left[\begin{array}{c|c} Q_c^{(-1)} & \\ \hline & I \end{array} \right] = \left[\begin{array}{c|c} Q_c & \\ \hline & I \end{array} \right] \left[\begin{array}{cc|c} \alpha_{c,11} & & \underline{0} \\ \alpha_{c,21} & \alpha_{c,22} & \gamma_c \\ \hline \beta_{c,1} & \beta_{c,2} & * \end{array} \right]$$

for some properly partitioned block diagonals $\alpha_c, \beta_c, \gamma_c$ and the underlined zero block of maximal row rank.

From the previous identity we find

$$\left[A''_c Q_{c,22}^{(-1)} \quad C''_c \right] = Q_{c,22} \left[\alpha_{c,22} \quad \gamma_c \right].$$

As (A''_c, C''_c) uniformly observable, we conclude that

$$\begin{aligned} A''_c (Q_{c,22} \Omega_c Q_{c,22}^*)^{(-1)} A_c^* + C''_c C_c^{**} &= Q_{c,22} (\alpha_{c,22} \Omega_c^{(-1)} \alpha_{c,22}^* + \gamma_c \gamma_c^*) Q_{c,22}^* \\ &= Q_{c,22} \Omega_c Q_{c,22}^* \quad (> 0) \end{aligned}$$

where Ω_c is the (nonsingular) observability Gramian for the causal AC-pair $(\alpha_{c,22}, \gamma_c)$. On the other side, the condition $\mathcal{H}_{o,c}(T' T'') \subseteq \mathcal{H}_{o,c}(T'')$ implies that $s\text{-dim}(\alpha_{22}) \leq s\text{-dim}(A''_c)$. Hence, $Q_{c,22}$ from the invertible Gramian $Q_{c,22} \Omega_c Q_{c,22}^*$ must be locally square invertible. As $Q_{c,22}$ is invertible and Q_c is unitary, the term $Q_{c,11}$ is locally square invertible as well¹ as we needed to show. By straightforward inspection one verifies that

$$r_c := Q_c \begin{bmatrix} I \\ Q_{c,22}^{-1} \end{bmatrix}$$

is the state-space transformation fulfilling (6.5).

The proof for (6.6) considers the anti-causal part from (6.2) and goes similarly. This completes the proof. \blacksquare

The following proposition is an immediate consequence of Lemma 6.4.

Proposition 6.5. *Let $T', T'' \in \mathcal{X}$ be as in Lemma 6.2. Then $T' T'' \in \mathcal{D}$ if and only if there is a properly partitioned unitary state-space transformation $Q_c, Q_a \in \mathcal{D}^{2 \times 2}$ with invertible $Q_{c,ii}, Q_{a,ii}, i = 1, 2$ such that*

$$\left[\begin{array}{c|c} R_c^{-1} & \\ \hline & I \end{array} \right] \left[\begin{array}{cc|c} A'_c & C'_c B''_c & C'_c D''_c \\ & & + A'_c L_{ca}^{(-1)} C''_a \\ \hline & A''_c & C''_c \\ B'_c & D' B''_c & * \\ + B'_a K_{ac} A''_c & & \end{array} \right] \left[\begin{array}{c|c} R_c^{(-1)} & \\ \hline & I \end{array} \right] = \left[\begin{array}{cc|c} A'_c & & 0 \\ x_{c,21} & A''_c & C''_c \\ \hline B'_c & 0 & * \end{array} \right], \quad (6.7)$$

¹Since Q_c is unitary with locally invertible $Q_{c,22}$, substitute $Q_{c,12} = -Q_{c,11} Q_{c,21}^* Q_{c,22}^{-*}$ into identity $Q_{c,11} Q_{c,11}^* + Q_{c,12} Q_{c,12}^* = I$. After rearranging the terms one obtains $Q_{c,11} (I + Q_{c,21}^* Q_{c,22}^{-*} Q_{c,22}^{-1} Q_{c,21}) Q_{c,11}^* = I$. As $Q_{c,11}$ is locally square too and the expression between brackets is invertible, the latter identity provides the proof that $Q_{c,11}$ is invertible.

$$\left[\begin{array}{c|c} R_a^{-(-1)} & I \end{array} \right] \left[\begin{array}{cc|c} A'_a & C'_a B''_a & C'_a D'' \\ \hline & A''_a & +A'_a K_{ac} C''_c \\ B'_a & D' B''_a & C''_a \\ \hline & +B'_c L_{ca}^{(-1)} A''_a & * \end{array} \right] \left[\begin{array}{c|c} R_a & I \end{array} \right] = \left[\begin{array}{cc|c} A'_a & & 0 \\ x_{a,21} & A''_a & C''_a \\ \hline B'_a & 0 & * \end{array} \right] \quad (6.8)$$

where

$$\begin{aligned} R_c &:= Q_c \begin{bmatrix} Q_{c,11}^* & \\ & Q_{c,22}^{-1} \end{bmatrix}, \quad R_a := Q_a \begin{bmatrix} Q_{a,11}^* & \\ & Q_{a,22}^{-1} \end{bmatrix}, \\ x_{c,21} &:= A''_c (Q_{c,21} Q_{c,11}^*)^{(-1)} - (Q_{c,21} Q_{c,11}^*) A'_c, \\ x_{a,21} &:= A''_a (Q_{a,21} Q_{a,11}^*)^{(-1)} - (Q_{a,21} Q_{a,11}^*) A'_a. \end{aligned}$$

and the block diagonals L_{ca} and K_{ac} are given by (6.3)-(6.4).

PROOF: In Lemma 6.4 we have evaluated the transformation r_c which fulfills (6.5)-(6.6). This respectively explains the block triangular forms in the AC-pairs from (6.7) and (6.8). The transformation r_c can further be composed with $\begin{bmatrix} Q_{c,11}^* & \\ & I \end{bmatrix}$ to define R_c from the statement. The unitary transformation Q_c , when applied on the causal realization part from (6.2), leads to

$$\begin{bmatrix} Q_{c,11}^* A'_c \\ B'_c \end{bmatrix} = \begin{bmatrix} \alpha_{c,11} \\ \beta_{c,1} \end{bmatrix} Q_{c,11}^{*(-1)}$$

for some uniformly reachable $(\alpha_{c,11}, \beta_{c,1})$. After having it composed and applied in the way the left side of the equation (6.7) specifies, we obtain the realization from its right side where $x_{c,21}$ can be found by extracting the (2,1) block from the identity

$$\begin{bmatrix} A'_c & C'_c B''_c \\ & A''_c \end{bmatrix} R_c^{(-1)} = R_c \begin{bmatrix} A''_c \\ x_{c,21} & A'_c \end{bmatrix}.$$

This completes the proof that $\mathcal{H}_{o,c}(T'T'') \subset \mathcal{H}_{o,c}(T'')$. The proof that $\mathcal{H}_{o,a}(T'T'') \subset \mathcal{H}_{o,a}(T'')$ goes in a similar vein and uses the transformation r_a from Lemma 6.4 to define new transformation R_a satisfying (6.8). The proof for the opposite direction is straightforward. Since the right sides of (6.7)-(6.8) represent completely nonminimal realizations for $\mathbf{P}(T'T'')$ and $\mathbf{P}'(T'T'')$, it follows that $T'T'' \in \mathcal{D}$. This completes the proof. \blacksquare

Determination of the state-space transformations R_c and R_a mentioned in Proposition 6.5 simplifies after noting that the realizations from the right sides of (6.7) and (6.8) are such that

$$\left[\begin{array}{c|c} N_c & \\ \hline & I \end{array} \right] \left[\begin{array}{c|c} A'_c & 0 \\ x_{c,21} & A''_c \\ \hline B'_c & C''_c \\ & * \end{array} \right] \left[\begin{array}{c|c} N_c^{-(-1)} & \\ \hline & I \end{array} \right] = \left[\begin{array}{c|c} A'_c & 0 \\ \hline B'_c & C''_c \\ & * \end{array} \right], \tag{6.9}$$

$$\left[\begin{array}{c|c} N_a^{-(-1)} & \\ \hline & I \end{array} \right] \left[\begin{array}{c|c} A'_a & 0 \\ x_{a,21} & A''_a \\ \hline B'_a & C''_a \\ & * \end{array} \right] \left[\begin{array}{c|c} N_a & \\ \hline & I \end{array} \right] = \left[\begin{array}{c|c} A'_a & 0 \\ \hline B'_a & C''_a \\ & * \end{array} \right] \tag{6.10}$$

where

$$N_c := \begin{bmatrix} I & \\ Q_{c,21} Q_{c,11}^* & I \end{bmatrix} \quad \text{and} \quad N_a := \begin{bmatrix} I & \\ -Q_{a,21} Q_{a,11}^* & I \end{bmatrix}. \tag{6.11}$$

Reformulation of Proposition 6.5 is now easy since the presence of $x_{c,21}$ and $x_{a,21}$ can be avoided by an appropriate choice of the state-space transformation. To see this define

$$F_c := R_c N_c \quad \text{and} \quad F_a := R_a N_a.$$

Then, the cancellation conditions require that F_c and F_a fulfill the following Lyapunov-Stein equations

$$\left[\begin{array}{c|c} A'_c & C'_c B''_c \\ \hline B''_c & A''_c \end{array} \right] (F_c \begin{bmatrix} 0 & \\ & A''_{o,c} \end{bmatrix})^{(-1)} \begin{bmatrix} A'_c & \\ & A''_* \end{bmatrix} + \left[\begin{array}{c|c} C'_c D'' + A'_c L_{ca}^{(-1)} C''_c \\ \hline & 0 \end{array} \right] \begin{bmatrix} 0 & C''_* \end{bmatrix} = F_c \begin{bmatrix} 0 & \\ & A''_{o,c} \end{bmatrix}, \tag{6.12}$$

$$\left[\begin{array}{c|c} A'_c & \\ \hline B''_* & C'_c \end{array} \right] (F_c \begin{bmatrix} A'_c & \\ & 0 \end{bmatrix}) \begin{bmatrix} A'_c & \\ & A''_c \end{bmatrix} + \left[\begin{array}{c|c} B''_* D'' + A'_c K_{ac}^* B'_a & \\ \hline & B_c \end{array} \right] \begin{bmatrix} B_c & 0 \end{bmatrix} = (F_c \begin{bmatrix} A'_c & \\ & 0 \end{bmatrix})^{(-1)}, \tag{6.13}$$

$$\left[\begin{array}{c|c} A'_a & C'_a B''_a \\ \hline B''_a & A''_a \end{array} \right] (F_a \begin{bmatrix} 0 & \\ & A''_{o,a} \end{bmatrix}) \begin{bmatrix} A'_a & \\ & A''_* \end{bmatrix} + \left[\begin{array}{c|c} C'_a D'' + A'_a K_{ac} C''_c \\ \hline & 0 \end{array} \right] \begin{bmatrix} 0 & C''_* \end{bmatrix} = (F_a \begin{bmatrix} 0 & \\ & A''_{o,a} \end{bmatrix})^{(-1)}, \tag{6.14}$$

$$\left[\begin{array}{c|c} A'_a & \\ \hline B''_* & C'_a \end{array} \right] (F_a \begin{bmatrix} A'_a & \\ & 0 \end{bmatrix})^{(-1)} \begin{bmatrix} A'_a & \\ & A''_a \end{bmatrix} + \left[\begin{array}{c|c} B''_* D'' + A'_a L_{ca}^{(-1)} B'_a & \\ \hline & B_a \end{array} \right] \begin{bmatrix} B_a & 0 \end{bmatrix} = F_a \begin{bmatrix} A'_a & \\ & 0 \end{bmatrix} \tag{6.15}$$

where $A''_{o,c}$, A'_c , $A''_{o,a}$ and A'_a are the boundedly invertible solutions for

$$A''_c (A''_{o,c})^{(-1)} A''_* + C''_c C''_* = A''_{o,c}, \tag{6.16}$$

$$A'_c A'_c + B'_c B''_* = A'_c^{(-1)}, \tag{6.17}$$

$$A''_a A''_{o,a} A''_* + C''_a C''_* = (A''_{o,a})^{(-1)} \quad \text{and} \tag{6.18}$$

$$A'_a A'_a + B'_a B'_a = A'_a. \tag{6.19}$$

The equations (6.12)-(6.15) explain the simultaneously bringing the realizations for the causal and anti-causal part of $T'T''$ into the Kalman forms in which $x_{c,21} = 0$ and $x_{a,21} = 0$.

The previous consideration provides the proof for the following result.

Theorem 6.6. *Let $T', T'' \in \mathcal{X}$ be as in Lemma 6.2. Then $T'T'' \in \mathcal{D}$ if and only if there are block diagonals $F_c, F_a \in \mathcal{D}^{2 \times 2}$ that fulfill the Lyapunov-Stein equations (6.12)-(6.15).*

Working with mixed causality operators, in particular with their products, involves inspection of Gramians corresponding to certain Lyapunov-Stein equations. The latter

have opposite directions of evaluations and, as such, can not be evaluated in an one-pass fashion. As we shall see in the subsequent section, the theory of maximal-phase operators as introduced in Chapter 4 shall be more suitable here. By the theory, inspection whether $T'T'' \in \mathcal{D}$ holds true can be replaced by the inspection on $G'G''^* \in \mathcal{D}$ where G' and G'' are maximal-phase operators constructed from T' and T'' in an appropriate way.

We now aim at finding necessary conditions for a minimal factorization for a left-invertible $T \in \mathcal{X}$. Let $Q_c, Q_a \in \mathcal{D}$ be properly partitioned unitary state-space transformations which, when applied on the realization $[T]$ as $Q_c^*(\cdot c')Q_c^{(-1)}, Q_c^{*(-1)}(\cdot a')Q_a$, bring A_c and A_a into upper triangular forms. We introduce an alternative realization for T as

$$[T]_Q := \left[\begin{array}{cc|cc} A'_c & A_{c,12} & & C_{c,1} \\ & A''_c & & C''_c \\ \hline & & A'_a & A_{a,12} \\ & & & A''_a \\ \hline B'_c & B_{c,2} & B'_a & B_{a,2} \\ \hline & & & D \end{array} \right]. \tag{6.20}$$

Similarly, we bring the state transition operators in $[t]$ into a block triangular form by means of other properly partitioned unitary diagonal operators q_c, q_a

$$[t]_q := \left[\begin{array}{cc|cc} a''_c & a_{c,12} & & c_{c,1} \\ & a'_c & & c'_c \\ \hline & & a''_a & a_{a,12} \\ & & & a'_a \\ \hline b''_c & b_{c,2} & b'_a & b_{a,2} \\ \hline & & & d \end{array} \right]. \tag{6.21}$$

Induced by the above partitioning, define bounded operators T' and t' by choosing some bounded $C'_c, C'_a, D', b'_c, b'_a$ and d' for which

$$[T'] = \{A'_c, B'_c, C'_c, A'_a, B'_a, C'_a, D'\} \quad \text{and} \quad [t'] = \{a'_c, b'_c, c'_c, a'_a, b'_a, c'_a, d'\}$$

are minimal additive realizations. Then, by slightly generalizing Proposition 6.5 and by verifying if conditions

$$\begin{aligned} \mathcal{H}_{o,c}(t'T) &\subseteq \mathcal{H}_{o,c}(T), & \mathcal{H}_{o,a}(t'T) &\subseteq \mathcal{H}_{o,a}(T), \\ \mathcal{H}_c(tT') &\subseteq \mathcal{H}_c(t), & \mathcal{H}_a(tT') &\subseteq \mathcal{H}_a(t) \end{aligned}$$

are fulfilled, we convince ourselves whether the transformations Q and q are well chosen. If this is the case then there is a constant $\Delta \in \mathcal{D}$ and operators T'' and t'' such that $(I =) tT = t''(t'T')T'' = t''\Delta T''$. As we choose $C'_c, C'_a, D', b'_c, b'_a$ and d' such that $\Delta = I$ (i.e $t'T' = I$) then, $T'' := t'T$ is such that $t''T'' = I$ by the construction.

Proposition 6.7. *Let $T \in \mathcal{X}$ be a boundedly left invertible operator given by minimal, u.e.s realization $[T]_Q$ from (6.20). Let t be a left inverse for T given by minimal u.e.s additive realization $[t]_q$ from (6.21). Define $\psi^{ca}, \psi^{ac} \in \mathcal{D}^{1 \times 2}$ to be the unique bounded solutions for*

$$\begin{aligned}
 a'_c [\psi_1^{ca} \psi_2^{sa}]^{(-1)} \begin{bmatrix} A'_a & A_{a,12} \\ & A''_a \end{bmatrix} + c'_c [B'_a \ B_{a,2}] &= [\psi_1^{ca} \psi_2^{ca}], \\
 a'_a [\psi_1^{ac} \psi_2^{ac}] \begin{bmatrix} A'_c & A_{c,12} \\ & A''_c \end{bmatrix} + c'_a [B'_c \ B_{c,2}] &= [\psi_1^{ac} \psi_2^{ac}]^{(-1)}.
 \end{aligned}$$

Define also Hermitians $H_{o,c}, H_{o,a} \in \mathcal{D}^{3 \times 3}$ to be the unique bounded solutions for

$$\begin{aligned}
 \begin{bmatrix} a'_c & c'_c B'_c & c'_c B_{c,2} \\ & A'_c & A_{c,12} \\ & & A''_c \end{bmatrix} H_{o,c}^{(-1)}(\cdot)^* + \begin{bmatrix} c'_c D + a'_c [\psi_1^{ca} \psi_2^{ca}]^{(-1)} \begin{bmatrix} C_{a,1} \\ C''_a \end{bmatrix} \\ & C_{c,1} \\ & C''_c \end{bmatrix} (\cdot)^* &= H_{o,c}, \\
 \begin{bmatrix} a'_a & c'_a B'_a & c'_a B_{a,2} \\ & A'_a & A_{a,12} \\ & & A''_a \end{bmatrix} H_{o,a}(\cdot)^* + \begin{bmatrix} c'_a D + a'_a [\psi_1^{ac} \psi_2^{ac}] \begin{bmatrix} C_{c,1} \\ C''_c \end{bmatrix} \\ & C_{a,1} \\ & C''_a \end{bmatrix} (\cdot)^* &= H_{o,a}^{(-1)}.
 \end{aligned}$$

Assume there are some block diagonals $C'_c, C'_a, D', b'_c, b'_a$ and d' for which additive, u.e.s realizations

$$[T'] = \{A'_c, B'_c, C'_c, A'_a, B'_a, C'_a, D'\} \quad \text{and} \quad [t'] = \{a'_c, b'_c, c'_c, a'_a, b'_a, c'_a, d'\}$$

are minimal realizations for bounded T' and t' for which

$$t' T' = I. \tag{6.22}$$

It holds that

$$\mathcal{H}_{o,c}(t' T) = \mathcal{D}_2(I - A''_c Z)^{-1} C''_c, \tag{6.23}$$

$$\mathcal{H}_{o,a}(t' T) = \mathcal{D}_2(I - A''_a Z^*)^{-1} C''_a \tag{6.24}$$

if and only if $H_{o,c}$ and $H_{o,a}$ have singular parts in their SVD-decompositions of dimensions $s\text{-dim}(a'_c)$ and $s\text{-dim}(a'_a)$ respectively.

PROOF: (\Rightarrow) Let $C'_c, C'_a, D', b'_c, b'_a$ and d' be block diagonals for which

$$[T'] = \{A'_c, B'_c, C'_c, A'_a, B'_a, C'_a, D'\} \quad \text{and} \quad [t'] = \{a'_c, b'_c, c'_c, a'_a, b'_a, c'_a, d'\}$$

are minimal additive realizations for bounded T' and t' fulfilling (6.22). By straightforward evaluation we obtain

$$[t' T'] = \left[\begin{array}{cc|cc} a'_c & c'_c B'_c & & \\ & A'_c & & \\ \hline & & a'_a & c'_a B'_a \\ & & & A'_a \\ \hline b'_c & d' B'_c & b'_a & d' B'_a \\ & + b'_a \psi_1^{ac} A'_c & & + b'_c \psi_1^{ca(-1)} A'_a \end{array} \left\| \begin{array}{c} c'_c D' + a'_c \psi_1^{ca(-1)} C'_a \\ C'_c \\ c'_a D' + a'_a \psi_1^{ac} C'_c \\ C'_a \\ d' D' + b'_a \psi_1^{ac} C'_c \\ + b'_c \psi_1^{ca(-1)} C'_c \end{array} \right. \right].$$

with ψ^{ca}, ψ^{ac} as defined in the statement. As $t'T' = I$, the realization $[t'T']$ is completely nonminimal. In that case, there are properly partitioned state-space transformations R_c and R_a that simultaneously bring AB- and AC-pairs from $[t'T']$ into their Kalman canonical forms² as stated in Proposition 6.5.

Let us now evaluate realizations for the causal and the strictly anti-causal operator parts of $t'T$. By virtue of Lemma 6.4 we have that

$$[\mathbf{P}(t'T)] = \left[\begin{array}{ccc|c} a'_c & c'_c B'_c & c'_c B_{c,2} & (c'_c D \\ & A'_c & A_{c,12} & +a'_c [\psi_1^{ca} \ \psi_2^{ca}]^{(-1)} \begin{bmatrix} C_{c,1} \\ C''_c \end{bmatrix} \\ & & A''_c & C_{c,1} \\ \hline b'_c & (d' B'_c & (d' B_{c,2} & d' D \\ & +b'_c \psi_1^{ac} A'_c) + b'_c [\psi_1^{ac} \ \psi_2^{ac}] \begin{bmatrix} A_{c,12} \\ A''_c \end{bmatrix} & & C''_c \end{array} \right]$$

and

$$[\mathbf{P}'(t'T)] = \left[\begin{array}{ccc|c} a'_a & c'_a B'_a & c'_a B_{a,2} & (c'_a D \\ & A'_a & A_{a,12} & +a'_a [\psi_1^{ac} \ \psi_2^{ac}] \begin{bmatrix} C_{c,1} \\ C''_c \end{bmatrix} \\ & & A''_a & C_{a,1} \\ \hline b'_a & (d' B'_a & (d' B_{a,2} & 0 \\ & +b'_c \psi_1^{ca(-1)} A'_a) + b'_c [\psi_1^{ca} \ \psi_2^{ca}]^{(-1)} \begin{bmatrix} A_{a,12} \\ A''_a \end{bmatrix} & & C''_a \end{array} \right]$$

The observability Gramians for $[\mathbf{P}(t'T)]$ and $[\mathbf{P}'(t'T)]$ are respectively $H_{o,c}$ and $H_{o,a}$ from the statement. As the AB-pairs from $[t'T]$ appear in $[\mathbf{P}(t'T)]$ and $[\mathbf{P}'(t'T)]$ too, removal of the unreachable modes corresponding to A'_c and A'_a by R_c and R_a implies removal of the same modes from both the AC-pairs of $[\mathbf{P}(t'T)]$ and $[\mathbf{P}'(t'T)]$. Since $[t']$ is assumed minimal we have that (a'_c, b'_c) and (a'_a, b'_a) are uniformly reachable. As a consequence, having (6.23)-(6.24) fulfilled is possible only if the states corresponding to a'_c and a'_a are completely unobservable. This, in turn is equivalent to saying that the Hermitians $H_{o,c}$ and $H_{o,a}$ have singular parts in their SVD-decompositions of dimensions $s\text{-dim}(a'_c)$ and $s\text{-dim}(a'_a)$ respectively.

(\Leftarrow) (By reducing to the contradiction) If the states from $[\mathbf{P}(t'T)]$ and $[\mathbf{P}'(t'T)]$ corresponding to a'_c and a'_a are partially observable then the pairs (a'_c, b'_c) and (a'_a, b'_a) from $[t'T']$ must be partially unreachable in order to have $t'T$ that fulfills (6.23)-(6.24). Because this is in the contradiction with the assumption that $[t']$ is a minimal realization, the proof completes. ■

Proposition 6.7 requires that the bounded operators t' and T' are chosen such that $t'T' = I$. If the assumption (6.22) were $t'T' \in \mathcal{D}$ instead, it is quite clear that (6.23)-(6.24) need no longer be satisfied simply because $t'T$ may imply larger dimensions

²Moreover, by Theorem 6.6, these transformations can be chosen such that the quantities $x_{c,21}$ and $x_{a,21}$ from the right sides of (6.7) and (6.8) become zero.

for the unreachable subspaces in the realizations $[\mathbf{P}(t'T)]$ and $[\mathbf{P}'(t'T)]$ introduced in the proof of the proposition then it was the case for $t'T' = I$. Hence, if we are only concerned about the complexity reduction in the product $t'T$ and not about the minimality of the factorization, then t' need not be assumed right invertible.

Note the elementary fact that states that the minimality for some AC-pair will not change after postmultiplying its C-term with a (right) invertible block diagonal. The reason for introducing such a block diagonal lies in the freedom that we have when determining T' and t' for which $T't' = I$. If δ is such an diagonal of appropriate s-dim sequence, we have that $T'\delta$ and $\delta^\dagger t'$ fulfill $T'\delta\delta^\dagger t' = I$ too where δ^\dagger is an arbitrary right inverse for δ .

Lemma 6.8. *Let (A, C) be a causal observable pair. Then for any $\delta \in \mathcal{D}$ having proper s-dim sequence and full row rank, the pair $(A, C\delta)$ is observable too.*

PROOF: Since (A, C) is assumed observable, the sequence of the snapshots of the observability operator

$$\mathcal{O}_k := [C_k \ A_k C_{k+1} \ A_k A_{k+1} C_{k+2} \ \cdots]$$

has full row rank for each integer k . Then, for any $\delta \in \mathcal{D}$ having proper s-dim sequence and full row rank, all its snapshots $\{\delta_k\}$ have full row rank. This implies that the sequence of operators

$$\mathcal{O}_{\delta,k} := [C_k \ A_k C_{k+1} \ A_k A_{k+1} C_{k+2} \ \cdots] \begin{bmatrix} \delta_k & & & \\ & \delta_{k+1} & & \\ & & \delta_{k+2} & \\ & & & \ddots \end{bmatrix}$$

is such that $\ker(\cdot\mathcal{O}_{\delta,k}) = \{0\}$. Equivalently said $\mathcal{O}_{\delta,k}$ has full row rank for each k . Obviously $\mathcal{O}_{\delta,k}$ is the k -th snapshot of the observability operator corresponding to $(A, C\delta)$. By the definition of the observability, the full row rank of $\mathcal{O}_{\delta,k}, \forall k$ means that $(A, C\delta)$ is observable. This completes the proof. ■

6.3 Completion of a partially specified inverse

In this section we complete a u.e.s additive realization of a partially specified invertible operator T_f when its inverse exists and is also partially known.

Given minimal u.e.s causal and anti-causal AB-pairs $(A_c, B_c), (A_a, B_a)$ and minimal u.e.s causal and anti-causal AC-pairs (a_c, c_c) and (a_a, c_a) , determine block diagonals $C_{cf}, C_{af}, D_f, b_{cf}, b_{af}$ and d_f for which

$$[T_f] = \left[\begin{array}{c|c} A_c & C_{cf} (=?) \\ A_a & C_{af} (=?) \\ \hline B_c & B_a \\ \hline & D_f (=?) \end{array} \right] \quad \text{and} \quad [t_f] = \left[\begin{array}{c|c} a_c & c_c \\ & a_a \\ \hline b_{cf} & b_{af} \\ \hline (=?) & (=?) \end{array} \middle| \begin{array}{c} c_c \\ c_a \\ (=?) \end{array} \right]$$

are u.e.s, additive realizations for T_f and t_f such that $t_f T_f = I$. It is obvious to show that this problem always have a trivial solution independently of the given terms. To show this recall that although the presumed given data are minimal the realizations $[T_f]$ and $[t_f]$ are not assumed minimal. This means that, for instance, the choice $C_{cf} = 0$, $C_{af} = 0$, $b_{cf} = 0$ and $b_{af} = 0$ recasts the determination into finding D_f and d_f for which $D_f d_f = I$. In this trivial case the operators T_f and t_f reduce to the constants. It holds then that $T_f = D_f$ and $t_f = d_f$ independently of the known terms from $[T_f]$ and $[t_f]$.

In finding the unknowns C_{cf} , C_{af} , D_f , b_{cf} , b_{af} and d_f that give rise to nontrivial T_f and t_f we perform minimality test for (A_c, C_{cf}) , (A_a, C_{af}) , (a_c, b_{cf}) and (a_a, b_{af}) . Extraction of eventual nonminimality from the latter pairs reduces the realizations $[T_f]$ and $[t_f]$ so that the solution to the problem is achieved by T_f and t_f of smaller degree than presumed.

We refer to this problem as the completion problem for a partially specified inverse. We define it as follows.

Definition 6.9 (Completion problem for a partially specified inverse). *Given AB-pairs: causal (A_c, B_c) , anti-causal (A_a, B_a) and AC-pairs: causal (a_c, c_c) and anti-causal (a_a, c_a) all minimal and u.e.s, find all mixed causality T'_f and t'_f which are the inverses of each other and are such that*

$$\begin{aligned} \mathcal{H}_c(T'_f) &\subseteq \mathcal{D}_2(B_c Z (I - A_c Z)^{-1})^*, & \mathcal{H}_{o,c}(t'_f) &\subseteq \mathcal{D}_2(I - a_c Z)^{-1} c_c, \\ \mathcal{H}_a(T'_f) &\subseteq \mathcal{D}_2(B_a Z^* (I - A_a Z^*)^{-1})^*, & \mathcal{H}_{o,a}(t'_f) &\subseteq \mathcal{D}_2(I - a_a Z^*)^{-1} c_a. \end{aligned} \quad (6.25)$$

In addition, the problem is said to have a maximal degree solution if T'_f or t'_f can be chosen such that

$$s\text{-dim}(T'_f) = s\text{-dim}(A_c) + s\text{-dim}(A_a) \quad \text{or} \quad s\text{-dim}(t'_f) = s\text{-dim}(a_c) + s\text{-dim}(a_a)$$

As we work with invertible operators of mixed causality, it is more convenient for what follows in the sequel to adopt different notation for such an operator and its inverse. In this way, if such an operator is denoted by a capital symbol G , we denote its inverse with g^* . The reason is that this notation is more insightful and fits better in the context.

In the sequel, in order of appearance we shall do the following:

- (1) We first associate maximal-phase operators G and g to T_f and t_f . In this step the outputs are partially nonminimal but invertible realizations $[G]$ and $[g^*]$ for the maximal-phase operator G and its inverse $g^* := G^{-1}$.

- (2) As the given data from the realizations $[T_f]$ and $[t_f]$ may differ in their degrees, this property translates to $[G]$ and $[g^*]$ defined in the previous step. In this step we show how to adjust realizations $[G]$ and $[g^*]$ such that a degree balance is obtained. We also show that the number of unknown variables in parametrization can be reduced. The output in this step are again partially nonminimal and invertible realizations for new maximal-phase operators having zero block diagonals as their constant terms.
- (3) We show how to express the unknown terms from $[T_f]$ and $[t_f]$ in the terms generated in the previous step.

In the following section we consider the first and the second step from the list.

6.3.1 Construction of realizations for maximal-phase operators G and g

Instead of solving the problem specified in the begining of this section, we pass to the equivalent but simpler one defined as follows. First, let U and u be the minimal degree inner operators for which UT_f is a causal and $t_f u^*$ is an anti-causal operator. Such inner factors are expressed in the known terms by identifying them through $\mathcal{H}_o(U) = \mathcal{D}_2(I - A_a^* Z)^{-1} B_a^*$ and $\mathcal{H}(u) = \mathcal{D}_2\{c_c Z(I - a_c Z)^{-1}\}^*$. As a consequence of the identity $T_f t_f = I$, we have that

$$UT_f t_f u^* = U u^* \in \mathcal{X} \tag{6.26}$$

where $U u^*$ may introduce cancellations. After the cancellations are computed, we factorize the unitary $U u^*$ in a minimal way as $U u^* = W_m^* w_m$ for some inner W_m and w_m . As shown in Chapter 5, this is always possible. Then, for any inner W_f of an appropriate s-dim sequence we have that $W_m^* w_m = W_m^* W_f^* W_f w_m$.

The reason for introducing these inner factors is twofold:

- (1) They make possible to control some variables in the cancellations in such a way that we may achieve their annihilation or even invertibility.
- (2) The condition that $W_m^* w_m$ is a minimal anti-inner/inner factorization makes possible to balance the degrees of the not necessarily minimal realizations for the maximal-phase factors that we introduce in the sequel.

In this step we associate maximal-phase operators G and g where $g^* := G^{-1}$ to T_f and $t_f := T_f^{-1}$. Define $W = W_f W_m$ and $w = W_f w_m$. Then,

$$(WUT_f)(t_f u^* w^*) = I. \tag{6.27}$$

Introduce $G := WUT_f \in \mathcal{U}$. Then, $g^* = t_f u^* w^* \in \mathcal{L}$. The following fact is then obvious.

Lemma 6.10. *For G and T_f as defined above it holds that $\|G\| = \|T_f\|$.*

PROOF: The proof is a straightforward consequence of the Banach-space property

$$\begin{aligned} \|G\| &\leq \|WU\| \|T_f\| = \|T_f\| \quad \text{and} \\ \|T_f\| &\leq \|U^*W^*\| \|G\| = \|G\|. \end{aligned}$$

■

Let $[G] = \{A, B, C, \Delta\}$ be a u.e.s realization for G obtained by direct evaluation of the product WUT_f with the inner W and U as before. We find a u.e.s realization for G in the form

$$[G] = \left[\begin{array}{cc|c} A_W & C_W B_U & C_W(D_U D_f + B_U \Lambda_a^{(-1)} C_{af}) \\ A_a^* & B_a^* B_c & B_a^* D_f + A_a^* \Lambda_a^{(-1)} C_{af} \\ & A_c & C_{cf} \\ \hline B_W & D_W B_U & D_W(D_U D_f + B_U \Lambda_a^{(-1)} C_{af}) \end{array} \right]. \tag{6.28}$$

where $[U] = \{A_a^*, B_U, B_a^*, D_U\}$, $[W] = \{A_W, B_W, C_W, D_W\}$ are minimal realizations for U and W respectively and $\Lambda_a \in \mathcal{D}$ is the unique boundedly invertible solution for

$$A_a^* \Lambda_a^{(-1)} A_a + B_a^* B_a = \Lambda_a.$$

Realization $[G]$ is not necessarily observable. By straightforward inspection it shows that it is uniformly reachable with the AB-pair expressed entirely in the known terms. Hence, the Lyapunov-Stein equation

$$A^* \Lambda A + B^* B = \Lambda^{(-1)} \tag{6.29}$$

has the unique boundedly invertible solution $\Lambda \in \mathcal{D}$.

Lemma 6.11. *The solution Λ for the equation (6.29) is equal to $\Lambda = \begin{bmatrix} I & & & \\ & H_{W_m}^{-1} & & \\ & & \Lambda_a^{-1} & \\ & & & \Lambda_c \end{bmatrix}$*

where the s -dim sequence of the identity block corresponds to those of A_{W_f} from the unitary realization $[W_f]$ and H_{W_m} is the uniformly positive definite weight for the states corresponding to the minimal realization $[W_m] = \{A_{W_m}, B_{W_m}, C_{W_m}, D_{W_m}\}$ for W_m (i.e. $A_{W_m} H_{W_m}^{(-1)} A_{W_m}^* + C_{W_m} C_{W_m}^* = H_{W_m}$).

PROOF: Recall that $W = W_f W_m$ where the realization $[W_f]$ for the arbitrary inner W_f of an appropriate s -dim sequence is assumed unitary and W_m is defined as the essentially unique inner factor in the minimal factorization $W_m^* w_m = U u^*$ for another also essentially unique inner w . Hence, there is some uniformly positive definite Hermitian H_{W_m} for which $[W_m]$ is H_{W_m} -unitary u.e.s realization. These facts imply that

the block $\begin{bmatrix} I \\ H_{W_m}^{-1} \end{bmatrix}$ from Λ is the reachability Gramian corresponding to the causal

AB-pair (A_W, B_W) . The conclusion follows by straightforward substitution where we employ the fact that $[U] = \{A_a^*, B_U, B_a^*, D_U\}$ is A_a -coisometric, that is

$$[U]^* \begin{bmatrix} A_a^{-1} \\ I \end{bmatrix} [U] = \begin{bmatrix} A_a^{-(-1)} \\ I \end{bmatrix},$$

and the fact that $A_c^* A_c A_c + B_c^* B_c = A_a^{(-1)}$. This completes the proof. ■

Similarly let $[g^*] = \{a, b, c, \delta\}$ be u.e.s realization for G^{-1} i.e for $g^* = t_f u^* w^*$ introduced by (6.27) and obtained by brute force evaluation of the latter expression. For the sake of the latter consideration we evaluate this realization and obtain

$$[g^*] = \left[\begin{array}{cc|cc} a_a & c_a c_c^* & c_a D_u^* C_w^* & c_a D_u^* D_w^* \\ & a_c^* & B_u^* C_w^* & B_u^* D_w^* \\ & & A_w^* & B_w^* \\ \hline b_{a_f} d_f c_c^* + b_{c_f} \Pi_o^{(-1)} a_c^* & (d_f D_u^* + b_{c_f} \Pi_o^{(-1)} B_u^*) C_w^* & (d_f D_u^* + b_{c_f} \Pi_o^{(-1)} B_u^*) D_w^* & \end{array} \right] \quad (6.30)$$

where $[u] = \{a_c, B_u, c_c, D_u\}$, $[w] = \{A_w, B_w, C_w, D_w\}$ are minimal, u.e.s realizations for u and w respectively and the boundedly invertible Hermitian $\Omega_{o,c} \in \mathcal{D}$ is the bounded solution for

$$a_c \Omega_{o,c}^{(-1)} a_c^* + c_c c_c^* = \Omega_{o,c}.$$

For similar reasons as in the previous, the realization $[g^*]$ is not necessarily minimal. Its AC-pair is uniformly observable and expressed in the known terms. Hence, there is a boundedly invertible Hermitian $\Omega \in \mathcal{D}$ satisfying

$$a \Omega a^* + c c^* = \Omega^{(-1)}. \quad (6.31)$$

We omit the proof for the following statement because it is the dual to that from Lemma 6.11.

Lemma 6.12. *The solution Ω of the equation (6.31) is equal to $\Omega = \begin{bmatrix} \Omega_{o,a} & & & \\ & \Omega_{o,c}^{-1} & & \\ & & H_{w_m}^{-1} & \\ & & & I \end{bmatrix}$*

where the s -dim sequence of the identity block corresponds to those of A_{w_f} from the unitary realization $[w_f]$ and H_{w_m} is the uniformly positive definite weight for the states corresponding to the minimal realization $[w_m] = \{A_{w_m}, B_{w_m}, C_{w_m}, D_{w_m}\}$ for inner w_m (i.e $A_{w_m} H_{w_m}^{(-1)} A_{w_m}^* + C_{w_m} C_{w_m}^* = H_{w_m}$).

In the previous we have associated an invertible maximal-phase operator G to the operator T_f and showed how the inner factors can be chosen. Now we apply similar analysis to find a maximal-phase-star operator that corresponds to T_f .

Let us start from $T_f t_f = I$ again and introduce inner \hat{U} and \hat{u} by identifying them through $\mathcal{H}(\hat{U}) = \mathcal{D}_2(B_c Z(I - A_c Z)^{-1})^*$ and $\mathcal{H}_o(\hat{u}) = \mathcal{D}_2(I - a_a Z)^{-1} c_a$ where A_c, B_c

and a_a, c_a come from $[T_f]$ and $[t_f]$ respectively. Then, \hat{U}^*T_f and $t_f\hat{u}$ are respectively an anti-causal and a causal operator for which

$$\hat{U}^*T_f t_f \hat{u} = \hat{U}^* \hat{u} \quad (\in \mathcal{X}).$$

Due to possible cancellations in the unitary operator $\hat{U}^*\hat{u}$, there are inner \hat{W}_m, \hat{w}_m introducing no cancellations and such that $\hat{U}^*\hat{u} = \hat{W}_m \hat{w}_m^*$. Let \hat{W}_f be any inner factor of an appropriate s-dim sequence. Then

$$\hat{U}^*T_f t_f \hat{u} = \hat{W}_m \hat{w}_m^* = \hat{W}_m \hat{W}_f \hat{W}_f^* \hat{w}_m^*$$

or equivalently

$$(\hat{W}^* \hat{U}^* T_f)(t_f \hat{u} \hat{w}) = I$$

where $\hat{W} = \hat{W}_m \hat{W}_f$ and $\hat{w} = \hat{w}_m \hat{W}_f$. Define $\hat{G}^* := \hat{W}^* \hat{U}^* T_f \in \mathcal{L}$. Then $\hat{g} := (\hat{G}^*)^{-1} = t_f \hat{u} \hat{w}$ and is a causal operator. By the definition, the anti-causal operator \hat{G}^* is a maximal-phase-star operator.

U.e.s realizations $[\hat{G}^*] = \{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$, $[\hat{g}] = \{\hat{a}, \hat{b}, \hat{c}, \hat{d}\}$ for \hat{G}^* and its inverse \hat{g} can be obtained by evaluating products $\hat{W}^* \hat{U}^* T_f$ and $t_f \hat{u} \hat{w}$. By doing so, we obtain

$$[\hat{G}^*] = \left[\begin{array}{cc|c} A_{\hat{W}}^* & B_{\hat{W}}^* C_{\hat{U}}^* & B_{\hat{W}}^* D_{\hat{U}}^* B_a \\ & A_c^* & B_c^* B_a \\ \hline & A_a & C_a \end{array} \middle| \begin{array}{c} B_{\hat{W}}^* (D_{\hat{U}}^* D + C_{\hat{U}}^* A_c C_c) \\ B_c^* D + A_c^* A_c C_c \end{array} \right], \quad (6.32)$$

$$[\hat{g}] = \left[\begin{array}{cc|c} a_c & c_c c_a^* & c_c D_{\hat{u}} C_{\hat{w}} \\ & a_a^* & B_{\hat{u}} C_{\hat{w}} \\ \hline & A_{\hat{w}} & C_{\hat{w}} \end{array} \middle| \begin{array}{c} c_c D_{\hat{u}} D_{\hat{w}} \\ B_{\hat{w}} D_{\hat{w}} \\ C_{\hat{w}} \end{array} \right] \quad (6.33)$$

where $[\hat{W}] = \{A_{\hat{W}}, B_{\hat{W}}, C_{\hat{W}}, D_{\hat{W}}\}$, $[\hat{w}] = \{A_{\hat{w}}, B_{\hat{w}}, C_{\hat{w}}, D_{\hat{w}}\}$, $[\hat{U}] = \{A_c, B_c, C_{\hat{U}}, D_{\hat{U}}\}$ and $[\hat{u}] = \{a_a, c_a, B_{\hat{u}}, D_{\hat{u}}\}$ are minimal realizations for inner \hat{W} , \hat{w} , \hat{U} and \hat{u} . The realizations $[\hat{G}^*]$ and $[\hat{g}]$ are respectively uniformly reachable and uniformly observable. As a consequence, there will be boundedly invertible \hat{A} and $\hat{\Omega}$ satisfying

$$\hat{A}^* \hat{A}^{(-1)} \hat{A} + \hat{B}^* \hat{B} = \hat{A} \quad \text{and} \quad (6.34)$$

$$\hat{a} \hat{\Omega}^{(-1)} \hat{a}^* + \hat{c} \hat{c}^* = \hat{\Omega}. \quad (6.35)$$

In the following proposition we state how to relate cancellations between an invertible maximal-phase operator and its inverse through algebraic properties of their appropriate realizations.

Proposition 6.13. *Let a causal operator $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be given by a u.e.s realization $[G] = \{A, B, C, \Delta\}$ and let an anti-causal operator $g^* \in \mathcal{L}(\mathcal{N}, \mathcal{M})$ be given by a u.e.s realization $[g^*] = \{a, b, c, \delta\}$. Let $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{N}}$ be block diagonals having appropriate s-dim sequences and further arbitrary.*

Then, $G\eta_N g^* = \eta_M$ if and only if there is a bounded solution $\Gamma \in \mathcal{D}$ such that

$$\begin{bmatrix} A & C \\ B & \Delta \end{bmatrix} \begin{bmatrix} \Gamma^{(-1)} & \\ & \eta_N \end{bmatrix} \begin{bmatrix} a & c \\ b & \delta \end{bmatrix} = \begin{bmatrix} \Gamma & \\ & \eta_M \end{bmatrix}. \tag{6.36}$$

PROOF: (\Rightarrow) Assume $G\eta_N g = \eta_M$. Then,

$$\eta_M = (\Delta + BZ(I - AZ)^{-1}C)\eta_N(\delta + bZ^*(I - aZ^*)^{-1}c). \tag{6.37}$$

As $\ell_A < 1$, there exists a block diagonal Γ being the unique bounded solution for

$$A\Gamma^{(-1)}a + C\eta_N b = \Gamma. \tag{6.38}$$

By means of this quantity and by virtue of the mixed partial fraction lemma, the mixed causality term from the equation (6.37) allows for the additive decomposition into two operators of opposite causality and a constant term as

$$\begin{aligned} BZ(I - AZ)^{-1}C\eta_N bZ(I - aZ)^{-1}c &= BZ(I - AZ)^{-1}A\Gamma^{(-1)}c \\ &\quad + B\Gamma^{(-1)}c + B\Gamma^{(-1)}aZ^*(I - aZ^*)^{-1}c. \end{aligned}$$

Taking the projections on the upper, lower and the diagonal operator part of equation (6.37) together with Lyapunov-Stein equation (6.38), completes the proof for the necessity of the statement.

(\Leftarrow) Assume there is a bounded block diagonal Γ for which the equation (6.36) holds for the realizations $[G]$ and $[g^*]$ for a causal and an anti-causal operator respectively. Let us show that $G\eta_N g = \eta_M$. Indeed, viz. the equation (6.38) from (6.36) we have that

$$\begin{aligned} G\eta_N g^* &= (\Delta + BZ(I - AZ)^{-1}C)\eta_N(\delta + bZ^*(I - aZ^*)^{-1}c) \\ &= \Delta\eta_N\delta + B\Gamma^{(-1)}c \\ &\quad + (\Delta\eta_N b + B\Gamma^{(-1)}a)Z^*(I - aZ^*)^{-1}c \\ &\quad + BZ(I - AZ)^{-1}(A\Gamma^{(-1)}c + C\eta_N\delta). \end{aligned}$$

Viz. the equation (6.36) all expressions in the brackets from the last equality are zero and the term $\Delta\eta_N\delta + B\Gamma^{(-1)}c$ is equal to η_M . Hence, $G\eta_N g^* = \eta_M$ as we needed to show. This completes the proof. ■

The statement from Proposition 6.13 is valid for minimal realizations as well. When η_M and η_N are taken identity block diagonals, it gives us a recipe how to check if two minimal realizations correspond to respectively a maximal-phase operator and a maximal-phase-star operator which are partial inverses of each other. As a straightforward consequence of this fact, the necessary and sufficient conditions for $T_f t_f = I_M$ (or, dually, to have $t_f T_f = I_N$), can be expressed via the maximal-phase operators associated to T_f and t_f .

Therefore we have the following corollary.

Corollary 6.14. *Let*

$$[T_f] = \{A_c, B_c, C_{cf}, A_a, B_a, C_{af}, D_f\}$$

and

$$[t_f] = \{a_c, b_{cf}, c_c, a_a, b_{af}, c_a, d_f\}$$

as given before, be minimal, u.e.s realizations for bounded operators of mixed causality $T_f \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ and $t_f \in \mathcal{X}(\mathcal{N}, \mathcal{M})$ respectively. Let G and g^* be the maximal-phase and the maximal-phase-star operators associated to T_f and t_f as given before by the uniformly reachable and the uniformly observable realizations $[G] = \{A, B, C, \Delta\}$ and $[g^*] = \{a, b, c, \delta\}$ from (6.28) and (6.30) respectively.

Then, $T_f t_f = I_{\mathcal{M}}$ if and only if there is a bounded block diagonal Γ for which the equation (6.36) from Proposition 6.13 is fulfilled when $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{N}}$ are taken identity block diagonals.

Let us now write down the constitutive state-space equations from $Gg^* = I$ and $g^*G = I$ starting from the realizations $[G]$ and $[g^*]$. By virtue of Proposition 6.13 we have that

$$\begin{bmatrix} A & C \\ B & \Delta \end{bmatrix} \begin{bmatrix} \Gamma^{(-1)} & \\ & I \end{bmatrix} \begin{bmatrix} a & c \\ b & \delta \end{bmatrix} = \begin{bmatrix} \Gamma & \\ & I \end{bmatrix} \quad (\Leftrightarrow Gg^* = I), \quad (6.39)$$

$$\begin{bmatrix} a & c \\ b & \delta \end{bmatrix} \begin{bmatrix} P & \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ B & \Delta \end{bmatrix} = \begin{bmatrix} P^{(-1)} & \\ & I \end{bmatrix} \quad (\Leftrightarrow g^*G = I). \quad (6.40)$$

for some bounded block diagonals Γ and P . Note that P can always be evaluated by finding the fixed point solution of its corresponding Lyapunov-Stein equation. The Lyapunov-Stein equation defining Γ however is still given in the unknown terms C and b .

The identities $\hat{G}^* \hat{g} = I, \hat{g} \hat{G}^* = I$ and the realizations $[\hat{G}^*]$ and $[\hat{g}]$ are related through

$$\begin{bmatrix} \hat{A} & \hat{C} \\ \hat{B} & \hat{\Delta} \end{bmatrix} \begin{bmatrix} \hat{\Gamma} & \\ & I \end{bmatrix} \begin{bmatrix} \hat{a} & \hat{c} \\ \hat{b} & \hat{\delta} \end{bmatrix} = \begin{bmatrix} \hat{\Gamma}^{(-1)} & \\ & I \end{bmatrix} \quad (\Leftrightarrow \hat{G}^* \hat{g} = I), \quad (6.41)$$

$$\begin{bmatrix} \hat{a} & \hat{c} \\ \hat{b} & \hat{\delta} \end{bmatrix} \begin{bmatrix} \hat{P}^{(-1)} & \\ & I \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{C} \\ \hat{B} & \hat{\Delta} \end{bmatrix} = \begin{bmatrix} \hat{P} & \\ & I \end{bmatrix} \quad (\Leftrightarrow \hat{g} \hat{G}^* = I) \quad (6.42)$$

for some bounded block diagonals $\hat{\Gamma}$ and \hat{P} .

In the following proposition we state that the block diagonals P and Γ as defined by (6.39) and (6.40) are the inverses of each other. We exploit this property for purpose of determination of the unknown terms from $[T_f]$ and $[t_f]$ as stressed in the beginning.

Proposition 6.15. *For the realizations $[T_f]$, $[t_f]$, $[G]$ and $[g^*]$ as defined above it holds that $P, \Gamma \in \mathcal{D}$ satisfying (6.39)-(6.40) are locally square invertible and are such that $P\Gamma = I$.*

PROOF: By combining (6.39) and (6.40) we find that

$$\begin{bmatrix} \Gamma P \\ I \end{bmatrix} \begin{bmatrix} A & C \\ B & \Delta \end{bmatrix} = \begin{bmatrix} A & C \\ B & \Delta \end{bmatrix} \begin{bmatrix} (\Gamma P)^{(-1)} \\ I \end{bmatrix} \quad \text{and} \quad (6.43)$$

$$\begin{bmatrix} (P\Gamma)^{(-1)} \\ I \end{bmatrix} \begin{bmatrix} a & c \\ b & \delta \end{bmatrix} = \begin{bmatrix} a & c \\ b & \delta \end{bmatrix} \begin{bmatrix} P\Gamma \\ I \end{bmatrix}. \quad (6.44)$$

From the first equation from the previous set and the equation (6.29) we conclude that Γ is such that

$$A^*(P^*\Gamma^*\Lambda\Gamma P)A + B^*B = (P^*\Gamma^*\Lambda\Gamma P)^{(-1)}.$$

Hence,

$$A = P^*\Gamma^*\Lambda\Gamma P. \quad (6.45)$$

Similarly, from the second equation from the previous set and the equation (6.31) we find that

$$a(P\Gamma\Omega\Gamma^*P^*)a^* + cc^* = (P\Gamma\Omega\Gamma^*P^*)^{(-1)}.$$

Hence,

$$\Omega = P\Gamma\Omega\Gamma^*P^*. \quad (6.46)$$

The equations (6.45) and (6.46) together with the fact that Λ and Ω are invertible imply respectively that $P\Gamma$ and ΓP are locally square invertible. Hence, Γ and P are square invertible on their own.

Let us show that $P\Gamma = I$. Consider the equation (6.40) and premultiply it with $\begin{bmatrix} P^{(-1)} \\ I \end{bmatrix}$. We find that

$$\begin{bmatrix} P^{(-1)}aP & P^{(-1)}c \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = I. \quad (6.47)$$

From (6.43) we find

$$\begin{bmatrix} (\Gamma P)A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} (\Gamma P)^{(-1)}.$$

Let us premultiply the latter equation with $\begin{bmatrix} P^{(-1)}aP & P^{(-1)}c \end{bmatrix}$ and employ (6.47) into the result. We obtain that $(P^{(-1)}aP)(\Gamma P)A + P^{(-1)}cB = (\Gamma P)^{(-1)}$ or equivalently

$$a(P\Gamma P)A + cB = (P\Gamma P)^{(-1)}.$$

But (6.40) implies that $aPA + cB = P^{(-1)}$ which, after subtraction from the latter equation, yields the trivial Lyapunov-Stein equation

$$a(P\Gamma P - P)A = (P\Gamma P - P)^{(-1)}.$$

As $\ell_a, \ell_A < 1$, we have that a zero block diagonal is the unique bounded solution for the previous equation. As P is invertible we conclude that $P\Gamma = I$ (or $\Gamma P = I$). This completes the proof. ■

The following corollary is an immediate consequence of Proposition 6.15.

Corollary 6.16. *Realizations $[G] = \{A, B, C, \Delta\}$ and $[g^*] = \{a, b, c, \delta\}$, given by respectively (6.28) and (6.30) and satisfying (6.39)-(6.40), are invertible.*

PROOF: By Proposition 6.15 the block diagonals Γ and P are boundedly invertible. This fact allows for rewriting (6.39)-(6.40) equivalently as

$$\left(\begin{bmatrix} \Gamma^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ B & \Delta \end{bmatrix} \begin{bmatrix} \Gamma^{(-1)} & \\ & I \end{bmatrix} \right) \begin{bmatrix} a & c \\ b & \delta \end{bmatrix} = \begin{bmatrix} I & \\ & I \end{bmatrix} \quad \text{and} \\ \begin{bmatrix} a & c \\ b & \delta \end{bmatrix} \left(\begin{bmatrix} P & \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ B & \Delta \end{bmatrix} \begin{bmatrix} P^{(-1)} & \\ & I \end{bmatrix} \right) = \begin{bmatrix} I & \\ & I \end{bmatrix}.$$

The products between the brackets from the latter equations obviously define the left- and the right inverse for $[g^*] = \begin{bmatrix} a & c \\ b & \delta \end{bmatrix}$. Hence, $[g^*]$ is an invertible realization. One shows that $[G] = \begin{bmatrix} A & C \\ B & \Delta \end{bmatrix}$ is invertible in a similar way. This completes the proof. ■

We now show that, without loss of generality, G and g^* may be assumed having zero constant terms. That is we may assume that $\Delta = 0$ and $\delta = 0$. This case occurs, for instance, if premultiply G with Z . Simple algebraic manipulations then show that

$$[ZG] = \begin{bmatrix} 0 & B & \Delta \\ 0 & A & C \\ I & 0 & 0 \end{bmatrix} \quad \text{and} \quad [g^*Z^*] = \begin{bmatrix} 0 & 0 & I \\ c & a & 0 \\ \delta & b & 0 \end{bmatrix}$$

are u.e.s realizations for ZG and its inverse. The latter realizations are respectively uniformly reachable en observable. Note also that $[ZG]$ and $[g^*Z^*]$ are both invertible because $[G]$ and $[g^*]$ are so. Recall that the reachability and the observability assumptions were required in the proof for Proposition 6.15. Equations (6.39) and (6.40) can now be adjusted to incorporate Z . In fact, there are new block diagonals Γ and P both appropriately partitioned and such that

$$\begin{bmatrix} 0 & B & \Delta \\ 0 & A & C \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{11}^{(-1)} & \Gamma_{12}^{(-1)} \\ \Gamma_{21}^{(-1)} & \Gamma_{22}^{(-1)} \\ & & I \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ c & a & 0 \\ \delta & b & 0 \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \\ & & I \end{bmatrix} \quad (\Leftrightarrow (ZG)(g^*Z^*) = I)$$

and

$$\left[\begin{array}{cc|c} 0 & 0 & I \\ c & a & 0 \\ \delta & b & 0 \end{array} \right] \left[\begin{array}{cc|c} P_{11} & P_{12} & \\ \hline P_{21} & P_{22} & \\ \hline & & I \end{array} \right] \left[\begin{array}{ccc|c} 0 & B & \Delta & \\ \hline 0 & A & C & \\ \hline I & 0 & 0 & \end{array} \right] = \left[\begin{array}{cc|c} P_{11}^{(-1)} & P_{12}^{(-1)} & \\ \hline P_{21}^{(-1)} & P_{22}^{(-1)} & \\ \hline & & I \end{array} \right] \quad (\Leftrightarrow (g^* Z^*)(ZG) = I).$$

In the following proposition we prove that it is possible to evaluate C and b from realizations $[G]$ and $[g^*]$ by means of UL-decomposition.

As shown previously, we are allowed to assume that Δ and δ from $[G]$ and $[g^*]$ are zero block diagonals. This case occurs if we fix, for instance, $W_f = Z$.

Proposition 6.17. *Let G, g^* be as defined above in which the inner parameter W_f is chosen equal Z . Let $[G]$ and $[g^*]$, as given by (6.28) and (6.30) (with $\Delta = 0$ and $\delta = 0$), be their respective u.e.s realizations. Let $[V] := \{A, B, E, F\}$ and $[v^*] := \{a, e, c, f\}$ be respectively Λ -unitary and Ω -unitary realizations for an inner V and an anti-inner v^* . Let also, for some properly partitioned invertible x and y , the factorization*

$$\begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & \\ y_{21} & y_{22} \end{bmatrix} = [V]^{-1} \begin{bmatrix} P^{-1} & \\ & I \end{bmatrix} [v^*]^{-1},$$

be a UL-decomposition where the invertible P is the bounded solution for $aPA + cB = P^{(-1)}$.

Then, $x_{ii}, y_{ii}, i = 1, 2$ are locally square invertible and all possible candidates for C and b from $[G]$ and $[g^*]$ are

$$C = [A \ E] \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} k, \quad b = k^{-1} [y_{21} \ y_{22}] \begin{bmatrix} a \\ e \end{bmatrix}$$

whereby k is arbitrary invertible block diagonal constant of appropriate s -dim sequence.

In addition,

(1) the causal pair (A, C) is uniformly observable if and only if the bounded solution Ψ_o for

$$A\Psi_o^{(-1)}A^* + P^{-1}\Omega(\Omega^{-1} - a^*\Omega^{(-1)}a)\Omega P^{-*} = \Psi_o$$

is uniformly positive definite,

(2) the anti-causal pair (a, b) is uniformly reachable if and only if the bounded solution Φ for

$$a^*\Phi^{(-1)}a + P^{-*}A(\Lambda^{-1} - A\Lambda^{(-1)}A^*)AP^{-1} = \Phi$$

is uniformly positive definite.

PROOF: By Proposition 6.15, we have that P and Γ from (6.39)-(6.40) are the inverses of each other. By fixing $W_f = Z$, operators G and g^* from the consideration can be adjusted to have zero block diagonals as their constant terms. Hence, Δ and δ from

such realizations $[G]$ and $[g^*]$ are zero block diagonals. Since the identities (6.39) – (6.40) imply that $[G]$ and $[g^*]$ are invertible matrices, we represent the matrices by RQ-decompositions. For A -unitary $[V]$ and Ω -unitary $[v^*]$ as defined in the statement we find

$$\begin{bmatrix} A & C \\ B & 0 \end{bmatrix} = \begin{bmatrix} A & E \\ B & F \end{bmatrix} \begin{bmatrix} I & A^{(-1)}A^*AC \\ & E^*AC \end{bmatrix} \quad \text{and} \quad (6.48)$$

$$\begin{bmatrix} a & c \\ b & 0 \end{bmatrix} = \begin{bmatrix} I \\ b\Omega a^* \Omega^{(-1)} b\Omega e^* \end{bmatrix} \begin{bmatrix} a & c \\ e & f \end{bmatrix} \quad (6.49)$$

from which we deduce that E^*AC and $b\Omega e^*$ are locally square invertible block diagonals. Let us now substitute these representations into (6.39) in which we assume that $\Delta = 0, \delta = 0$. After a little algebra we obtain

$$\begin{bmatrix} P^{(-1)} & A^{(-1)}A^*AC \\ & E^*AC \end{bmatrix} \begin{bmatrix} I \\ b\Omega a^* \Omega^{(-1)} b\Omega e^* \end{bmatrix} = [V]^{-1} \begin{bmatrix} P^{-1} \\ I \end{bmatrix} [v]^{-1}$$

which is clearly a UL-decomposition. We complete the proof by showing that the factors from this decomposition are essentially the factors from the UL-decomposition from the statement. Indeed, viz. the two UL-decompositions there must exist some properly partitioned block diagonal $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ satisfying

$$xK = \begin{bmatrix} P^{(-1)} & A^{(-1)}A^*AC \\ & E^*AC \end{bmatrix} \quad \text{and} \quad K^{-1}y = \begin{bmatrix} I \\ b\Omega a^* \Omega^{(-1)} b\Omega e^* \end{bmatrix}.$$

Because of the invertibility of E^*AC and $b\Omega e^*$ we have that the (2-1) block from K and the (1-2) block from K^{-1} are zero blocks. Hence, K is block lower having a block upper inverse. This is only possible if $K = \begin{bmatrix} K_{11} & \\ & K_{22} \end{bmatrix}$ with $K_{ii}, i = 1, 2$ square invertible. This fact also implies that x_{11}, x_{22}, y_{11} and y_{22} are all square invertible block diagonals. The variables C and b from the statement then easily follow from

$$C = [A \ E] \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} k, \quad b = k^{-1} [y_{21} \ y_{22}] \begin{bmatrix} a \\ e \end{bmatrix}$$

in which we defined $k := K_{22}$. Since the UL-decomposition from the statement is assumed arbitrary, all possible UL-decompositions can be obtained by varying the parameter k . This proves the completeness and ends the proof for the first statement.

Let us now prove the first additional statement. Consider the pair (A, C) with C expressed through the UL-factorization terms. Note that

$$\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = [V]^{-1} \begin{bmatrix} P^{-1} \\ I \end{bmatrix} [v^*]^{-1} \begin{bmatrix} 0 \\ y_{22}^{-1} \end{bmatrix}$$

and that

$$[V]^{-1} = \begin{bmatrix} \Lambda^{(-1)} & \\ & I \end{bmatrix} [V]^* \begin{bmatrix} \Lambda & \\ & I \end{bmatrix}$$

and

$$[v^*]^{-1} = \begin{bmatrix} \Omega & \\ & I \end{bmatrix} [v^*]^* \begin{bmatrix} \Omega^{(-1)} & \\ & I \end{bmatrix}.$$

Hence,

$$C = [A \ E] \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} k = P^{-1} \Omega e^* y_{22}^{-1} k.$$

Because $y_{22}^{-1} k$ is invertible, by virtue of Lemma 6.8 we have that (A, C) is uniformly observable if and only if $(A, Ck^{-1}y_{22})$ is so. Equivalently said, (A, C) is uniformly observable if and only if the Gramian Ψ_o being the unique bounded solution for

$$A\Psi_o^{(-1)}A^* + (Ck^{-1}y_{22})(y_{22}^*k^{-*}C^*) = \Psi_o.$$

is uniformly positive definite. Substitution of $C = P^{-1}\Omega e^* y_{22}^{-1} k$ into this Lyapunov-Stein equation, the use of the identity $e^*e = \Omega^{-1} - a^*\Omega^{(-1)}a$ turns the previous Lyapunov-Stein equation into its equivalent form

$$A\Psi_o^{(-1)}A^* + P^{-1}\Omega(\Omega^{-1} - a^*\Omega^{(-1)}a)\Omega P^{-*} = \Psi_o,$$

as we needed to show. To prove the second additional statement, we first find from the UL-decomposition that $b = k^{-1}x_{22}^{-1}E^*AP^{-1}$. Since $x_{22}k$ is invertible, we have by Lemma 6.8 that (a, b) is uniformly reachable if and only if $(a, x_{22}kb)$ is so. Recall that $EE^* = \Lambda^{-1} - A\Lambda^{(-1)}A^*$. Then, the latter is equivalent to inspecting the Hermitian Φ being the bounded solution of

$$a^*\Phi^{(-1)}a + P^{-*}\Lambda(\Lambda^{-1} - A\Lambda^{(-1)}A^*)AP^{-1} = \Phi$$

on the uniform positive definiteness. This makes the proof complete. ■

In the following proposition we state the opposite to what claimed in Proposition 6.17.

Proposition 6.18. *Let $P \in \mathcal{D}$, the bounded solution for $aPA + cB = P^{(-1)}$ be invertible. Let $[V] := \{A, B, E, F\}$ and $[v^*] := \{a, e, c, f\}$ be respectively Λ -unitary and Ω -unitary realizations for an inner V and an anti-inner v^* . Let also, for some properly partitioned invertible $x, y \in \mathcal{D}^{2 \times 2}$ having square invertible x_{ii}, y_{ii} , $i = 1, 2$, the factorization*

$$\begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & \\ y_{21} & y_{22} \end{bmatrix} = [V]^{-1} \begin{bmatrix} P^{-1} & \\ & I_N \end{bmatrix} [v^*]^{-1} \tag{6.50}$$

be a UL-factorization.

Then, for an arbitrary invertible $k \in \mathcal{D}$, u.e.s realizations

$$[G] := \left[\begin{array}{c|c} A \left[\begin{array}{c} A \ E \\ B \ F \end{array} \right] \left[\begin{array}{c} x_{12} \\ x_{22} \end{array} \right] \\ \hline k \end{array} \right] \left(= \left[\begin{array}{c|c} A \left| P^{-1} \Omega e^* y_{22}^{-1} k \right. \\ \hline B \left| f^* y_{22}^{-1} k \right. \end{array} \right] \right) \text{ and}$$

$$[g^*] := \left[\begin{array}{c|c} a & c \\ \hline k^{-1} \left[\begin{array}{c} y_{21} \ y_{22} \end{array} \right] \left[\begin{array}{c} a \\ e \end{array} \right] & k^{-1} \left[\begin{array}{c} y_{21} \ y_{22} \end{array} \right] \left[\begin{array}{c} c \\ f \end{array} \right] \end{array} \right]$$

$$\left(= \left[\begin{array}{c|c} a & c \\ \hline k^{-1} x_{22}^{-1} E^* A P^{-1} & k^{-1} x_{22}^{-1} F^* \end{array} \right] \right)$$

define a causal operator $G \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ and its anti-causal inverse $g^* \in \mathcal{L}(\mathcal{N}, \mathcal{M})$.

PROOF: By Proposition 6.13 we have that $Gg^* = I_{\mathcal{M}}$ with realizations $[G]$ and $[g^*]$ from the statement if and only if there exists some bounded $\Gamma \in \mathcal{D}$ such that

$$[G] \left[\begin{array}{c} \Gamma^{(-1)} \\ \hline I_{\mathcal{N}} \end{array} \right] [g^*] = \left[\begin{array}{c} \Gamma \\ \hline I_{\mathcal{M}} \end{array} \right]. \quad (6.51)$$

We complete the proof by showing that the presumed realizations fulfill the previous equation and that $\Gamma = P^{-1}$. We start from (6.50) and pre- and postmultiply it with $[V]$ and $[v^*]$ respectively. We obtain

$$([V] \left[\begin{array}{c} I \ x_{12} \\ \hline x_{22} \end{array} \right]) \left[\begin{array}{c} x_{11} y_{11} \\ \hline I_{\mathcal{M}} \end{array} \right] \left(\left[\begin{array}{c} I \\ \hline y_{21} \ y_{22} \end{array} \right] [v^*] \right) = \left[\begin{array}{c} P^{-1} \\ \hline I_{\mathcal{N}} \end{array} \right] \quad (6.52)$$

and note that $[V] \left[\begin{array}{c} I \ x_{12} \\ \hline x_{22} \end{array} \right] = [G]$ and that $\left[\begin{array}{c} I \\ \hline y_{21} \ y_{22} \end{array} \right] [v^*] = [g^*]$ in which we took $k = I$ for simplicity reasons. Note also that $[G]$ and $[g^*]$ are invertible matrices.

Let us now invert both sides of (6.50). We obtain

$$\left[\begin{array}{c} y_{11}^{-1} \\ \hline -y_{22}^{-1} y_{21} y_{11}^{-1} \ y_{22}^{-1} \end{array} \right] \left[\begin{array}{c} x_{11}^{-1} \ -x_{11}^{-1} x_{12} x_{22}^{-1} \\ \hline x_{22}^{-1} \end{array} \right] = [v^*] \left[\begin{array}{c} P \\ \hline I_{\mathcal{M}} \end{array} \right] [V]$$

and, since $aPA + cB = P^{(-1)}$, we conclude that $x_{11} y_{11} = P^{(-1)}$ by inspection of the (1-1)-block from the previous equation. Hence, (6.51) is fulfilled for $\Gamma = P^{-1}$. This proves that $Gg^* = I_{\mathcal{M}}$.

To complete the proof we need to show that $g^*G = I_{\mathcal{N}}$ too. We start from (6.51) and postmultiply it with $\left[\begin{array}{c} \Gamma \\ \hline I_{\mathcal{M}} \end{array} \right] [G]$. Then, we premultiply the resulting equation with $\left[\begin{array}{c} \Gamma^{(-1)} \\ \hline I_{\mathcal{N}} \end{array} \right] [G]^{-1}$ and obtain

$$[g^*] \left[\begin{array}{c} \Gamma^{-1} \\ \hline I_{\mathcal{M}} \end{array} \right] [G] = \left[\begin{array}{c} \Gamma^{(-1)} \\ \hline I_{\mathcal{N}} \end{array} \right], \quad (\Gamma^{-1} = P).$$

By virtue of Proposition 6.13, the previous identity implies that $g^*G = I_{\mathcal{N}}$. This makes the proof complete. \blacksquare

6.3.2 Evaluation of the unknown data from $[T_f]$ and $[t_f]$

In this section we take the opposite course to what we did previously. We complete the realizations $[T_f]$ and $[t_f]$ by expressing their unknown terms in the realization terms for the inner and maximal-phase factors introduced in the previous section.

By virtue of Proposition 6.17 it is possible to produce the solution to the original problem as follows. The u.e.s realization $[t_f]$ for the sought $t_f \in \mathcal{X}$ we started from, can easily be completed analytically through the anti-causal/causal product

$$t_f = (g^* Z^*)(Zwu) \quad \text{where} \quad w = W_f w_m. \tag{6.53}$$

Similarly, the realization $[T_f]$ can be completed through

$$T_f = (ZWU)^*(ZG) \quad \text{where} \quad W = W_f W_m. \tag{6.54}$$

which is an anti-causal/causal product too. Starting from the original data from $[G]$ and $[g^*]$ given respectively by (6.28) and (6.30), the statement from the proposition is applicable on ZG and its inverse $g^* Z^*$. Realizations for these operators are respectively

$$[ZG] = \left[\begin{array}{cc|c} 0 & B & \Delta \\ 0 & A & C \\ \hline I & 0 & 0 \end{array} \right] \quad \left(=: \left[\begin{array}{c|c} A' & C' \\ \hline B' & 0 \end{array} \right] \right)$$

and

$$[g^* Z^*] = \left[\begin{array}{cc|c} 0 & 0 & I \\ c & a & 0 \\ \hline \delta & b & 0 \end{array} \right] \quad \left(=: \left[\begin{array}{c|c} a' & c' \\ \hline b' & 0 \end{array} \right] \right).$$

It shows by straightforward substitution that the reachability Gramian for $[ZG]$ is $A' := \begin{bmatrix} I \\ A \end{bmatrix}$ and the observability Gramian for $[g^* Z^*]$ equals $\Omega' := \begin{bmatrix} I \\ \Omega \end{bmatrix}$. Also, for the unique bounded solution $P' \in \mathcal{D}$ for

$$a' P' A' + c' B' = P'^{(-1)}$$

we have that $P' = \begin{bmatrix} I \\ P \end{bmatrix}$. Hence, P' is locally square invertible.

Let e', f', E' and F' be block diagonals for which the realizations $[V'] = \{A', B', E', F'\}$ and $[v'^*] = \{a', e', c', f'\}$ are respectively A' and Ω' -unitary realizations for an inner V' and an anti-inner v'^* . Note that then $F' = 0$ and $f' = 0$.

By virtue of Proposition 6.17, the UL-decomposition

$$\begin{bmatrix} x'_{11} & x'_{12} \\ & x'_{22} \end{bmatrix} \begin{bmatrix} y'_{11} & \\ y'_{21} & y'_{22} \end{bmatrix} = [V']^{-1} \begin{bmatrix} P'^{-1} & \\ & I \end{bmatrix} [v'^*]^{-1}, \tag{6.55}$$

for some properly partitioned invertibles $x', y' \in \mathcal{D}$, is such that

$$[ZG] = \left[\begin{array}{c|cc} A' & [A' E'] & \begin{bmatrix} x'_{12} \\ x'_{22} \end{bmatrix} \\ \hline B' & & 0 \end{array} \right] k \tag{6.56}$$

and

$$[g^* Z^*] = \left[\begin{array}{c|cc} & a' & c' \\ \hline k^{-1} [y'_{21} y'_{22}] & \begin{bmatrix} a' \\ e' \end{bmatrix} & 0 \end{array} \right] \tag{6.57}$$

where k is an arbitrary invertible block diagonal.

Lemma 6.19. *For the quantities $x'_{11}, x'_{12}, y'_{11}$ and y'_{21} from the UL-decomposition (6.55) it holds that there are some $x_{12}, y_{21} \in \mathcal{D}$ and invertibles $x_{11}, y_{11} \in \mathcal{D}$ for which*

$$x'_{11} = \begin{bmatrix} I \\ x_{11} \end{bmatrix}, \quad x'_{12} = \begin{bmatrix} 0 \\ x_{12} \end{bmatrix}, \quad y'_{11} = \begin{bmatrix} I \\ y_{11} \end{bmatrix}, \quad y'_{21} = [0 \ y_{21}]$$

where the partitioning is induced by the partitioning in $P' = \begin{bmatrix} I \\ P \end{bmatrix}$.

PROOF: Consider the realizations $[V'] = \{A', B', E', F'\}$ and $[v^*] = \{a', e', c', f'\}$ from (6.55) and note that the top row block from $[V]^{-1}$ is equal to $[0 \ I]$. Similarly, the left column block of $[v^*]^{-1}$ is equal to $\begin{bmatrix} 0 \\ I \end{bmatrix}$. Since also $P' = \begin{bmatrix} I \\ P \end{bmatrix}$, the conclusion immediately follows. This completes the proof. ■

Recall that a minimal realization for $Zwu (= ZW_f w_m u)$ can be obtained in the form

$$[Zwu] = \left[\begin{array}{ccc|c} 0 & D_w B_u & B_w & D_w D_u \\ 0 & a_c & 0 & c_c \\ 0 & C_w B_u & A_w & C_w D_u \\ \hline I & 0 & 0 & 0 \end{array} \right].$$

Define $N \in \mathcal{D}_2^{4 \times 3}$ to be the bounded solution for

$$a' N \begin{bmatrix} 0 & D_w B_u & B_w \\ 0 & a_c & 0 \\ 0 & C_w B_u & A_w \end{bmatrix} + c' [I \ 0 \ 0] = N^{(-1)}. \tag{6.58}$$

Lemma 6.20. *The bounded solution for (6.58) is equal to $N = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Omega_{o,c}^{-1} & 0 \\ 0 & 0 & H_w^{-1} \end{bmatrix}$ where*

H_w is the Hermitian weight for the states corresponding to the realization $[w]$ (i.e. $A_w H_w^{(-1)} A_w^* + C_w C_w^* = H_w$).

PROOF: The proof follows by straightforward substitution of N from the statement into (6.58), exploiting the equations from

$$\begin{bmatrix} A_w^* & B_w^* \\ C_w^* & D_w^* \end{bmatrix} \begin{bmatrix} H_w^{-1} \\ I \end{bmatrix} \begin{bmatrix} A_w & C_w \\ B_w & D_w \end{bmatrix} = \begin{bmatrix} H_w^{-(-1)} \\ I \end{bmatrix},$$

$$\begin{bmatrix} a_c^* & B_u^* \\ c_c^* & D_u^* \end{bmatrix} \begin{bmatrix} \Omega_{o,c}^{-1} \\ I \end{bmatrix} \begin{bmatrix} a_c & c_c \\ B_u & D_u \end{bmatrix} = \begin{bmatrix} \Omega_{o,c}^{-(-1)} \\ I \end{bmatrix}$$

and using the fact that the unitary realization $[Z^*] = \{0, I, I, 0\}$ for Z^* is used to construct the realization $[g^* Z^*]$. This completes the proof. ■

Evaluation of (6.53) yields the following partially nonminimal additive realization

$$[t_f]_{n,m} = \left[\begin{array}{c|cc|c} a_c & & & c_c \\ \hline C_w B_u & A_w & & C_w D_u \\ \hline & a_a & c_a c_c^* & c_a D_u^* C_w^* \\ & & a_c^* & B_u^* C_w^* \\ & & & A_w^* \\ \hline b_{cf} & 0 & b_{af} & * & * & d_f \end{array} \right]$$

where

$$b_{cf} := k^{-1} \begin{bmatrix} y'_{21} & y'_{22} \end{bmatrix} \begin{bmatrix} a' \\ e' \end{bmatrix} N \begin{bmatrix} D_w B_u \\ a_c \\ C_w B_u \end{bmatrix},$$

$$b_{af} := k^{-1} \begin{bmatrix} y'_{21} & y'_{22} \end{bmatrix} \begin{bmatrix} a' \\ e' \end{bmatrix} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix},$$

$$d_f := k^{-1} \begin{bmatrix} y'_{21} & y'_{22} \end{bmatrix} \begin{bmatrix} a' \\ e' \end{bmatrix} N \begin{bmatrix} D_w D_u \\ c_c \\ C_w D_u \end{bmatrix}.$$

Due to the unreachable and unobservable modes associated to the zero blocks from $[t_f]_{n,m}$, this realization reduces to the parameterized class of u.e.s realizations

$$[t_f] = \left[\begin{array}{c|c} a_c & c_c \\ \hline a_a & c_a \\ \hline b_{cf} & b_{af} & d_f \end{array} \right] \tag{6.59}$$

containing all possible completions for the partially specified inverse t_f .

By repeating the procedure above, one evaluates the unknown terms from $[T_f]$ in similar way. Recall that $ZW = ZW_f W_m$ with minimal, u.e.s realization

$$[ZWU] = \left[\begin{array}{ccc|c} 0 & B_W & D_W B_U & D_W D_U \\ 0 & A_W & C_W B_U & C_W D_U \\ 0 & 0 & A_a^* & B_a^* \\ \hline I & 0 & 0 & 0 \end{array} \right].$$

Define $M \in \mathcal{D}^{3 \times 4}$ to be the bounded solution for

$$\begin{bmatrix} 0 & 0 & 0 \\ B_W^* & A_W^* & 0 \\ B_U^* D_W^* & B_U^* C_W^* & A_a \end{bmatrix} M A' + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} B' = M^{(-1)}. \tag{6.60}$$

Then we have the following lemma. Its proof is similar to the proof of Lemma 6.21 and hence omitted.

Lemma 6.21. *The bounded solution for (6.60) is equal to $M = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & H_W^{-1} & 0 & 0 \\ 0 & 0 & A_a^{-1} & 0 \end{bmatrix}$*

where H_W is the Hermitian weight for the states corresponding to $[W]$ (i.e. $A_W H_W^{(-1)} A_W^* + C_W C_W^* = H_W$).

Now, evaluation of (6.54) involves the quantity M from Lemma 6.21 and produces the following partially nonminimal additive realization

$$[T_f]_{n,m} = \left[\begin{array}{ccc|cc|c} A_W & C_W B_U & C_W D_U B_c & & & * \\ & A_a^* & B_a^* B_c & & & * \\ & & A_c & & & C_{cf} \\ \hline & & & A_W^* & & 0 \\ & & & B_U^* C_W^* & A_a & C_{af} \\ \hline 0 & 0 & B_c & D_U^* C_W^* & B_a & D_f \end{array} \right]$$

where the sought C_{cf} , C_{af} and D_f are

$$\begin{aligned} C_{cf} &:= [0 \ 0 \ 0 \ I] [A' \ E'] \begin{bmatrix} x'_{12} \\ x'_{22} \end{bmatrix} k, \\ C_{af} &:= [B_U^* D_W^* \ B_U^* C_W^* \ A_a] M [A' \ E'] \begin{bmatrix} x'_{12} \\ x'_{22} \end{bmatrix} k, \\ D_f &:= [D_U^* D_W^* \ D_U^* C_W^* \ B_a] M [A' \ E'] \begin{bmatrix} x'_{12} \\ x'_{22} \end{bmatrix} k. \end{aligned}$$

Because of the block zeroes from $[T_f]$, this realization reduction producing a class of u.e.s, additive realizations of the form

$$[T_f] = \left[\begin{array}{c|cc} A_c & C_{cf} \\ \hline & A_a & C_{af} \\ \hline B_c & B_a & D_f \end{array} \right] \tag{6.61}$$

parameterized by an invertible k and an inner W_f and complete, by the construction, the proof for the following theorem. It states the main result of this section.

Theorem 6.22. *U.e.s realizations $[t_f]$ and $[T_f]$ given respectively by (6.59) and (6.61) and both parameterized by the invertible block diagonal k and the unitary realization $[W_f]$ for an inner factor of appropriate s -dim sequence, generate all mixed causality invertible operators t_f and T_f which are the inverses of each other and solve the completion problem specified by Definition 6.9.*

What remarkable is in Theorem 6.22 is the fact that the computed realizations $[T_f]$ and $[f_f]$ need not be minimal in order to give rise to t_f and T_f which are such that $t_f T_f = I$. Extraction of eventual nonminimal modes from the realizations, reduces the presumed degrees of operators. Minimal u.e.s realizations for T' and t' are then obtained after the reduction is computed. Operators T' and t' are possible candidates for factors in a minimal factorization for T and t as we encounter in the following section.

Note that a complete parameterization of T' and t' can not be provided by the invertible block diagonal k . By virtue of Lemma 6.8, we conclude that this parameter can not introduce new unobservable or new unreachable modes in respectively $[T_f]$ and $[t_f]$ and, hence, is not capable of generating solutions of different degrees for the completion problem.

Actually, the only control parameter that can give rise to the unreachable and the unobservable modes is the inner parameter W_f . By means of this factor it is possible to replace determination of the unitary state-space transformations Q and q as introduced in the very beginning of this chapter, by finding a candidate for W_f which gives rise to degree reduction in $[T_f]$ and $[t_f]$.

6.4 A recipe on how to perform the search

In this section we establish how to perform the search for factors in a minimal factorization for an invertible operator T . In the first section of this chapter and for minimal factorization purposes, we have explained the need for four properly partitioned unitary state-space transformations that bring the state transition matrices from $[T]$ and $[t]$ ($t := T^{-1}$) in appropriate block triangular forms. In the previous section a single inner parameter W_f is introduced to generate all essentially different solutions to the completion problem specified by Definition 6.9. In the present section we continue to work on the parameter W_f and show how finding an invertible factor of T depends on W_f .

Let

$$[T] = \left[\begin{array}{c|c} A_c & C_c \\ \hline A_a & C_a \\ \hline B_c & B_a \\ \hline & D \end{array} \right] \quad \text{and} \quad [t] = \left[\begin{array}{c|c} a_c & c_c \\ \hline a_a & c_a \\ \hline b_c & b_a \\ \hline & d \end{array} \right]$$

be u.e.s minimal additive realizations for $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ and its inverse $t := T^{-1} \in \mathcal{X}(\mathcal{N}, \mathcal{M})$. Since $tT = I$ and both $[t]$ and $[T]$ are minimal realizations, the completion problem as specified by Definition 6.9 can be set on the AB- and the AC-data from respectively $[T]$ and $[t]$ to produce alternative realizations for T and t when initialized by an appropriate inner operator (see Theorem 6.22).

In the previous section we have calculated u.e.s realizations (6.61) and (6.59) for t_f and T_f when the above completion problem is initialized by an arbitrary inner W_f of appropriate s-dim sequence. These realization were found in the forms

$$[T_f] = \left[\begin{array}{c|c} A_c & C_{cf} \\ \hline A_a & C_{af} \\ \hline B_c & B_a \\ \hline & D_f \end{array} \right] \quad \text{and} \quad [t_f] = \left[\begin{array}{c|c} a_c & c_c \\ \hline a_a & c_a \\ \hline b_{cf} & b_{af} \\ \hline & d_f \end{array} \right]$$

with $C_{cf}, C_{af}, D_f, b_{cf}, b_{af}$ and d_f expressed in the given terms and the terms from realizations for the inner factors that were instrumental in construction of appropriate maximal-phase operators.

To find a candidate for a factor in a minimal factorization for T and its inverse t it is then necessary to find W_f for which the realization $[t_f]$ is partially nonminimal. It is clear that for the same W_f the realization $[T_f]$ is no longer minimal too.

A candidate for W_f need be found such that at least one of the Lyapunov-Stein equations in Ψ_{cf} and Ψ_{af}

$$\begin{aligned} a_c^* \Psi_{cf} a_c + b_{cf}^* b_{cf} &= \Psi_{cf}^{(-1)}, \\ a_a^* \Psi_{af}^{(-1)} a_a + b_{af}^* b_{af} &= \Psi_{af} \end{aligned}$$

has a singular bounded solution where b_{cf} and b_{af} are as given by (6.59). Such a W_f is guaranteed to exist but it may give rise to completely nonminimal realizations because the completion problem as specified by Definition 6.9 always has a trivial solution. Determination of W_f by inspection of the above Lyapunov-Stein equations however is a non-linear problem.

Once W_f is found that gives rise to neither a nontrivial solution nor a maximal degree solution, there are unitary state-space transformations that bring (a_c, b_{cf}) or (a_a, b_{af}) into their respective Kalman canonical forms. If $q_{cf}, q_{af} \in \mathcal{D}^{2 \times 1}$ are such transformations then their application on $[t_f]$ produces

$$[t_f]_{q_f} := \left[\begin{array}{c|c|c} a''_{cf} & a_{cf,12} & c_{cf,1} \\ & a'_{cf} & c'_{cf} \\ \hline & a''_{af} & a_{af,12} & c_{af,1} \\ & & a'_{af} & c'_{af} \\ \hline \underline{0} & b_{cf,2} & \underline{0} & b_{af,2} & \underline{0} & d \end{array} \right]$$

for some properly chosen block diagonals from this realization and where the underlined zero blocks are of maximal rank. For similar reason there will be another unitary state-space transformations $Q_{cf}, Q_{af} \in \mathcal{D}^{2 \times 1}$ which bring the realization $[T_f]$ into

$$[T_f]_{Q_f} := \left[\begin{array}{cc|c} A'_{cf} & A_{cf,12} & C_{cf,1} \\ & A''_{cf} & \underline{0} \\ \hline & A'_{af} & A_{af,12} \\ & & A''_{af} \\ \hline B'_{cf} & B_{cf,2} & B'_{af} & B_{af,2} & D_f \end{array} \right]$$

for some properly chosen block diagonals from this realization and where the underlined zero blocks are of maximal rank.

Minimal, u.e.s realizations for a candidate and its inverse for a right invertible factor for T are then

$$[T'_f]_{Q_f, min} := \left[\begin{array}{cc|c} A'_{cf} & C'_{cf,1} \\ & A'_{af} & C'_{af,1} \\ \hline B'_{cf} & B'_{af,2} & D_f \end{array} \right] \quad \text{and} \quad [t'_f]_{q_f, min} := \left[\begin{array}{cc|c} a'_{cf} & c'_{cf} \\ & a'_{af} & c'_{af} \\ \hline b_{cf,2} & b_{af,2} & d_f \end{array} \right].$$

Before invoking Proposition 6.7 we transform $[T]$ by Q_f too. This produces an alternative minimal realization for T of the form

$$[T]_{Q_f} := \left[\begin{array}{cc|c} A'_{cf} & A_{cf,12} & \gamma_{cf,1} \\ & A''_{cf} & \gamma_{cf,2} \\ \hline & A'_{af} & A_{af,12} \\ & & A''_{af} \\ \hline B'_{cf} & B_{cf,2} & B'_{af} & B_{af,2} & D \end{array} \right]$$

for some properly chosen block diagonals $\gamma_{cf,i}, \gamma_{af,i} \ i = 1, 2$.

All we need to do now in order to check out if T'_f is a left factor of T is to apply the statement from Proposition 6.7 on $[T]_{Q_f}, [T'_f]_{Q_f, min}$ and $[t'_f]_{Q_f, min}$. This consists of checking out if the Hermitians $H_{o,c}, H_{o,a}$ from the proposition have singular parts of dimensions $s\text{-dim}(a'_{cf})$ and $s\text{-dim}(a'_{af})$ respectively.

It is worth noting that neither $H_{o,c}$ nor $H_{o,a}$ depend on the completion terms $C'_{cf,1}, C'_{af,1}, D_f, b_{cf,2}, b_{af,2}, d_f$.

6.5 Minimal causal/anti-causal factorization of J-unitary operators

The theory from the previous sections is applicable in finding necessary and sufficient conditions for minimal causal/anti-causal and/or anti-causal/causal factorizations of

mixed causality invertible operators. As the theory claims, an appropriate partitioning of the states in u.e.s minimal additive realizations for both the given operator and its inverse is needed in order that the Lyapunov-Stein equations in $H_{o,c}$ and $H_{o,a}$ introduced in Proposition 6.7 have singular solutions of appropriate dimensions. If we know that the operator admits the minimal causal/anti-causal and/or anti-causal/causal factorization, then the theory can be slightly reduced since we no longer need to run the search for Q . The search for q however, still need to be done, because the inverse of the presumed factors of the original operator are, in general, of mixed causality. An interesting question arises: what are the mixed causality operators which allow for a minimal maximal-phase/maximal-phase-star factorization? Clearly the inverse of such an operator admits a minimal maximal-phase-star/maximal-phase factorization with the known degrees for the factors involved. As an example, we mention unitary operators from \mathcal{X} . They admit minimal inner/anti-inner factorizations and their inverse admit anti-inner/inner factorizations since inner operators are special class of maximal-phase operators.

We address this section to finding necessary and sufficient conditions for, so called, J-unitary operators to have a minimal causal/anti-causal factorization.

Definition 6.23. Let $\Theta \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be a properly partitioned operator mapping between non-uniform spaces. Let $J_i \in \mathcal{D}^{2 \times 2}$, $i = 1, 2$ be signatures associated to the spaces \mathcal{M} and \mathcal{N} respectively.

Θ is called

- (J_1, J_2) -isometric if $\Theta J_2 \Theta = J_1$,
- (J_1, J_2) -coisometric if $\Theta^* J_1 \Theta = J_2$ and
- (J_1, J_2) -unitary if it is both (J_1, J_2) -isometric and (J_1, J_2) -coisometric.

As a consequence of the above definition, a (J_1, J_2) -unitary Θ is invertible with the inverse $\Theta^{-1} = J_2 \Theta^* J_1$. Let

$$[\Theta] = \left[\begin{array}{c|c} A_c & C_c \\ \hline A_a & C_a \\ \hline B_c & B_a \\ \hline & D \end{array} \right] \quad \text{and} \quad [\Theta^{-1}] = \left[\begin{array}{c|c} A_a^* & B_a^* J_1 \\ \hline & B_c^* J_1 \\ \hline J_2 C_a^* & J_2 C_c^* \\ \hline & J_2 D^* J_1 \end{array} \right] \quad (6.62)$$

be minimal, u.e.s additive realizations for J-unitary Θ and its inverse Θ^{-1} . Assume we are looking for necessary and sufficient conditions under which Θ represents as

$$\Theta = T_c T_a, \quad \begin{cases} T_c \in \mathcal{U}(\mathcal{M}, \mathcal{M}'), & \mathcal{H}(T_c) = \mathcal{D}_2 B_c Z (I - A_c Z)^{-1}, \\ T_a \in \mathcal{L}(\mathcal{M}', \mathcal{N}), & \mathcal{H}_o(T_a) = \mathcal{D}_2 (I - A_a Z^*)^{-1} C_a \end{cases} \quad (6.63)$$

for some intermediate space \mathcal{M}' where, in addition, T_c and T_a are assumed invertible. As we can see, the unitary Q that introduce partitioning into the state transition matrices from $[\Theta]$ is no longer needed. The unitary q that triangularize the

state transition matrices from $[\Theta^{-1}]$ however is necessary to be taken into account because the inverses T_c^{-1} and T_a^{-1} are, in general, of mixed causality. Hence, let $q_c = [q_{c,1} \ q_{c,2}]$, $q_a = [q_{a,1} \ q_{a,2}] \in \mathcal{D}$ be properly partitioned unitary state-space transformations acting on the causal and the anti-causal part of $[\Theta^{-1}]$ respectively.

Let

$$[\Theta^{-1}]_q = \left[\begin{array}{cc|cc} a''_c & a_{c,12} & c_{c,1} & \\ & a'_c & c'_c & \\ & & a''_a & a_{a,12} \\ & & a'_a & c'_{a,1} \\ \hline b''_c & b_{c,2} & b'_a & b_{a,2} \end{array} \middle| \begin{array}{c} J_2 D^* J_1 \end{array} \right] \tag{6.64}$$

where

$$\begin{cases} a'_c = q_{c,2}^* A_a^* q_{c,2}^{(-1)}, & c'_c = q_{c,2}^* B_a^* J_1, \\ a'_a = q_{a,2}^{*(-1)} A_c^* q_{a,2}, & c'_a = q_{a,2}^{*(-1)} B_c^* J_1 \end{cases} \tag{6.65}$$

be the alternative minimal u.e.s realization for Θ^{-1} obtained after applying q on $[\Theta^{-1}]$. The original problem then translates to finding necessary and sufficient conditions for q_c and q_a under which the completion problem

$$\begin{cases} [T_c] = \begin{bmatrix} A_c & C_{T_c} \\ B_c & D_{T_c} \end{bmatrix}, & [t'] = \left[\begin{array}{cc|c} a'_c & a'_a & c'_c \\ & b'_a & d' \\ \hline b'_c & b'_a & d' \end{array} \right], & T_c^{-1} = t', \\ [T_a] = \begin{bmatrix} A_a & C_a \\ B_{T_a} & D_{T_a} \end{bmatrix}, & [t''] = \left[\begin{array}{cc|c} a''_c & a''_a & c''_c \\ & b''_a & d'' \\ \hline b''_c & b''_a & d'' \end{array} \right], & T_a^{-1} = t'' \end{cases} \tag{6.66}$$

has the solution for some bounded C_{T_c} , D_{T_c} , b'_c , b'_a , d' , B_{T_a} , D_{T_a} , c''_c , c''_a and d'' .

As it has been done in the previous section, we first define inner u by identifying it through $\mathcal{H}_o(u) = \mathcal{D}_2(I - a'_c Z)^{-1} c'_c$. Let $\Omega'_{o,c}$ be the boundedly invertible solution to $a'_c \Omega'_{o,c} a'^*_c + c'_c c'^*_c = \Omega'_{o,c}$ and let B_u, D_u be such that $[u] = \{a'_c, B_u, c'_c, D_u\}$ is a minimal, u.e.s $\Omega_{o,c}$ -unitary realization for the inner u . Then

$$G := uT_c \in \mathcal{U}, \quad [G] = \left[\begin{array}{cc|c} a'_c & c'_c B_c & c'_c D_{T_c} \\ & A_c & C_{T_c} \\ \hline B_u & D_u B_c & D_u D_{T_c} \end{array} \right] \quad \left(=: \left[\begin{array}{c|c} A|C \\ \hline B|D \end{array} \right] \right)$$

and

$$g^* := t' u^* \in \mathcal{L}, \quad [g^*] = \left[\begin{array}{cc|c} a'_a & c'_a c'^*_c & c'_a D^*_u \\ & a'^*_c & B^*_u \\ \hline b'_a & d' c'^*_c + b'_c \Omega'^{(-1)} a'^*_c & d' D^*_u + b'_c \Omega'^{(-1)} B^*_u \end{array} \right] \quad \left(=: \left[\begin{array}{c|c} a|c \\ \hline b|d \end{array} \right] \right)$$

are the inverses of each other for appropriate C_{T_c} , D_{T_c} , b'_c , b'_a and d' if and only if (by Proposition 6.18) the bounded locally square $P \in \mathcal{D}^{2 \times 2}$ satisfying

$$\begin{bmatrix} a'_a & c'_a c'^*_c \\ & a'^*_c \end{bmatrix} P \begin{bmatrix} a'_c & c'_c B_c \\ & A_c \end{bmatrix} + \begin{bmatrix} c'_a D^*_u \\ B^*_u \end{bmatrix} [B_u \ D_u B_c] = P^{(-1)} \tag{6.67}$$

is invertible.

We also repeat the previous consideration for T_a . Define the inner \hat{u} by identifying it through $\mathcal{H}_o(u) = \mathcal{D}_2(I - a''^*_a Z)^{-1} b''^*_a$. Let $[\hat{u}] = \{a''^*_a, B_{\hat{u}}, b''^*_a, D_{\hat{u}}\}$ be a A''_a -unitary realization where $a''^*_a A''^{(-1)}_a a''_a + b''^*_a b''_a = A''_a$. Then

$$\hat{G}^* := T_a \hat{u}^* \in \mathcal{L}, \quad [\hat{G}^*] = \left[\begin{array}{c|c} A_a & C_a b''^*_a \\ \hline a''_a & B^*_{\hat{u}} \\ \hline B_{T_a} & D_{T_a} b''^*_a \end{array} \middle| \begin{array}{c} C_a D^*_{\hat{u}} \\ B^*_{\hat{u}} \\ D_{T_a} D^*_{\hat{u}} \end{array} \right] \quad \left(=: \left[\begin{array}{c|c} \hat{A} & \hat{C} \\ \hline \hat{B} & \hat{D} \end{array} \right] \right)$$

and

$$\hat{g} := \hat{u} t'' \in \mathcal{U}, \quad [\hat{g}] = \left[\begin{array}{c|c} a''^*_a & b''^*_a b''_c \\ \hline a''_c & b''^*_a d'' + a''^*_a A''^{(-1)}_a c''_a \\ \hline B_{\hat{u}} & D_{\hat{u}} b''_c \end{array} \middle| \begin{array}{c} b''^*_a d'' + a''^*_a A''^{(-1)}_a c''_a \\ c''_c \\ D_{\hat{u}} d'' + B_{\hat{u}} A''^{(-1)}_a c''_a \end{array} \right] \quad \left(=: \left[\begin{array}{c|c} \hat{a} & \hat{c} \\ \hline \hat{b} & \hat{d} \end{array} \right] \right)$$

define a maximal-phase \hat{g} and its inverse $\hat{G}^* = \hat{g}^{-1}$ for appropriate block diagonals $B_{T_a}, D_{T_a}, c''_c, c''_a$ and d'' if and only if the bounded locally square $\hat{P} \in \mathcal{D}^{2 \times 2}$ satisfying

$$\begin{bmatrix} A_a & C_a b''^*_a \\ & a''_a \end{bmatrix} \hat{P} \begin{bmatrix} a''^*_a & b''^*_a b''_c \\ & a''_c \end{bmatrix} + \begin{bmatrix} C_a D^*_{\hat{u}} \\ B^*_{\hat{u}} \end{bmatrix} [B_{\hat{u}} \ D_{\hat{u}} b''_c] = \hat{P}^{(-1)} \tag{6.68}$$

is invertible.

Lemma 6.24. *The bounded solutions $P, \hat{P} \in \mathcal{D}^{2 \times 2}$ for respectively (6.67) and (6.68) are*

$$P = \begin{bmatrix} 0 & P_{12} \\ \Omega'^{-1}_{o,c} & 0 \end{bmatrix} \text{ and } \hat{P} = \begin{bmatrix} 0 & \hat{P}_{12} \\ A''^{-1}_a & 0 \end{bmatrix}$$

where P_{12} and \hat{P}_{12} are the bounded solution for

$$a'_a P_{12} A_c + c'_a B_c = P_{12}^{(-1)} \tag{6.69}$$

and

$$A_a \hat{P}_{12} a''_c + C_a b''_c = \hat{P}_{12}^{(-1)}. \tag{6.70}$$

PROOF: The proof for P follows by straightforward substitution and using the identity

$$\begin{bmatrix} a'^*_c & B^*_u \\ c'^*_c & D^*_u \end{bmatrix} \begin{bmatrix} \Omega'^{-1}_{o,c} & \\ & I \end{bmatrix} \begin{bmatrix} a'_c & c'_c \\ B_u & D_u \end{bmatrix} = \begin{bmatrix} \Omega'^{-(-1)}_{o,c} & \\ & I \end{bmatrix}.$$

In a similar vein, the proof for \hat{P} follows after substitution of the identity

$$\begin{bmatrix} a''_a & B^*_{\hat{u}} \\ b''_a & D^*_{\hat{u}} \end{bmatrix} \begin{bmatrix} A''^{-1}_a & \\ & I \end{bmatrix} \begin{bmatrix} a''^*_a & b''^*_a \\ B_{\hat{u}} & D_{\hat{u}} \end{bmatrix} = \begin{bmatrix} A''^{-(-1)}_a & \\ & I \end{bmatrix}$$

into (6.68). This completes the proof. ■

We also find that the observability and the reachability Gramian corresponding to respectively the anti-causal AC-pair (a, c) and the causal AB-pair (\hat{a}, \hat{b}) are $\Omega := \begin{bmatrix} \Omega'_{o,a} & \\ & \Omega'_{o,c} \end{bmatrix}$ and $\hat{A} := \begin{bmatrix} A''_a^{-1} & \\ & A''_c \end{bmatrix}$.

Consider now (6.69) from Lemma 6.24 and express a'_a and c'_a in the original terms by recalling (6.65). We have that

$$q_{a,2}^{*(-1)} \left(A_c^*(q_{a,2}P_{12})A_c + B_c^*J_1B_c \right) = P_{12}^{(-1)} \tag{6.71}$$

In order $P_{12} \in \mathcal{D}$ to be invertible, it is necessary to be locally square.

Lemma 6.25. *Consider the equation (6.71) in which $q_{a,2}$ is a coisometric block diagonal.*

It holds that P_{12} is locally square if and only if $q_{a,2}$ is unitary.

PROOF: Note that $Y := A_c^*(q_{a,2}P_{12})A_c + B_c^*J_1B_c$ is a locally square block diagonal. Consider $q_{a,2}^{*(-1)}Y = P_{12}^{(-1)}$. We conclude that $q_{a,2}^{*(-1)}$ has the same local degree as $P_{12}^{(-1)}$. This completes the proof. ■

By Lemma 6.25, we see that the local squareness for P_{12} is sustainable if and only if the coisometry $q_{a,2}$ is a unitary block diagonal which, in turn, may always be taken the identity. This means that if (J_1, J_2) -unitary $\Theta \in \mathcal{X}$ admits the minimal causal/anti-causal factorization then it is necessary that the equation

$$A_c^*P_{12}A_c + B_c^*J_1B_c = P_{12}^{(-1)}$$

has invertible (Hermitian) solution P_{12} .

By repeating the above consideration for T_a one produces similar result regarding the pair (A_a, C_a) and concludes that it is necessary for $q_{c,2}$ to be unitary in order to have a solution for the completion problem (6.66).

This consideration provides the proof for the main statement of the following result.

Theorem 6.26. *Let (J_1, J_2) -unitary $\Theta \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be given by the additive minimal, u.e.s realization $[\Theta]$. Then, the necessary condition for Θ to admit the (minimal) causal/anti-causal factorization $\Theta = T_cT_a$ with T_c, T_a as specified by (6.63) is that the bounded solution $A_c^J \in \mathcal{D}$ and $-A_{o,a}^J \in \mathcal{D}$ satisfying*

$$A_c^*A_c^JA_c + B_c^*J_1B_c = A_c^{J(-1)} \tag{6.72}$$

and

$$-A_a A_{o,a}^J A_a^* + C_a J_2 C_a^* = -A_{o,a}^{J(-1)} \tag{6.73}$$

are both invertible.

In addition, if the factorization exists then $\Theta_c^{-1} \in \mathcal{L}$ and $\Theta_a^{-1} \in \mathcal{U}$.

PROOF: We prove the additional statement. Assume the factorization exists and let T_c and T_a be the factors. By virtue of Lemma 6.25, the invertibility argument for T_c implies that P_{12} from the lemma is invertible. This implies that the s-dim of the coisometry $q_{a,2}$ from the unitary state-space transformation $q_a = [q_{a,1} \ q_{a,2}]$ vanishes. Hence $q_{a,1}$ may be chosen identity. As a consequence, $t' := T_c^{-1}$ will be such that $\mathcal{H}_a(\mathbf{P}_{\mathcal{L}}(t')) = J_1 C_c^* Z^* (I - A_c^* Z^*)^{-1} \mathcal{D}_2$. Dually, the invertibility argument for T_a implies that the coisometric $q_{c,1}$ from the unitary state-space transformation $q_c = [q_{c,1} \ q_{c,2}]$ vanishes forcing $q_{c,2}$ to be unitary itself and, for simplicity, may be chosen identity. As a consequence, $t'' := T_a^{-1}$ will be such that $\mathcal{H}_c(\mathbf{P}(t'')) = J_1 C_a^* Z (I - A_a^* Z)^{-1} \mathcal{D}_2$. But since $T_c T_a$ is a minimal factorization, the factorization $T_a^{-1} T_c^{-1}$ ($= t'' t'$) is minimal too and such that $\text{s-dim}(t'' t') = \text{s-dim}(A_a^*) + \text{s-dim}(A_c^*)$. But this means that $\mathbf{P}_{\mathcal{U}\mathcal{Z}}(t') = 0$ and $\mathbf{P}_{\mathcal{L}\mathcal{Z}^{-1}}(t'') = 0$. Hence, t' is anti-causal and t'' is causal as we needed to show. This completes the proof. ■

By means of the UL-decomposition scheme from Proposition 6.18, we are capable of constructing the matrices 'C' and 'D' from $[G]$ and the matrices 'B' and 'D' from $[\hat{G}^*]$ as follows. Let, for some block diagonals $E, F, e, f, \hat{E}, \hat{F}, \hat{e}$ and \hat{f} realizations $[V] = \{A, B, E, F\}$, $[v^*] = \{a, e, c, f\}$, $[\hat{V}^*] = \{\hat{A}, \hat{E}, \hat{C}, \hat{F}\}$ and $[\hat{v}] = \{\hat{a}, \hat{b}, \hat{e}, \hat{f}\}$ be respectively Λ -unitary, Ω -unitary, $\hat{\Omega}$ -unitary and $\hat{\Lambda}$ -unitary realization for an inner V , anti-inner v^* , anti-inner \hat{V}^* and an inner \hat{v} . Let properly partitioned block diagonals x, y, \hat{x} and \hat{y} be such that

$$\begin{bmatrix} x_{11} & x_{12} \\ & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & \\ y_{21} & y_{22} \end{bmatrix} = [V]^{-1} \begin{bmatrix} P^{-1} & \\ & I \end{bmatrix} [v^*]^{-1} \text{ and} \tag{6.74}$$

$$\begin{bmatrix} \hat{y}_{11} & \hat{y}_{12} \\ & \hat{y}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_{11} & \\ \hat{x}_{21} & \hat{x}_{22} \end{bmatrix} = [\hat{v}]^{-1} \begin{bmatrix} \hat{P}^{-1} & \\ & I \end{bmatrix} [\hat{V}^*]^{-1} \tag{6.75}$$

are UL-factorizations where the block diagonals P and \hat{P} are boundedly invertible solutions for

$$aPA + cB = P^{(-1)} \text{ and } \hat{A}\hat{P}\hat{a} + \hat{C}\hat{b} = \hat{P}^{(-1)}$$

respectively. By virtue of Proposition 6.18 we then have that

$$[G] = \begin{bmatrix} A & P^{-1} \Omega e^* y_{22}^{-1} \\ B & f^* y_{22}^{-1} \end{bmatrix} \text{ and } [\hat{G}^*] = \begin{bmatrix} \hat{A} & \hat{C} \\ \hat{y}_{22}^{-1} \hat{e} \hat{\Lambda} \hat{P}^{-1} & \hat{y}_{22}^{-1} \hat{f} \end{bmatrix}$$

are respectively a uniformly reachable and a uniformly observable realization for the maximal-phase operator G and the maximal-phase-star operator \hat{G}^* as introduced

previously. In these expressions we used $P = \begin{bmatrix} & P_{12} \\ \Omega'_{o,c}{}^{-1} & \end{bmatrix}$ where $P_{12} = \Lambda_c^J$ as stated in Lemma 6.24. Now, a u.e.s additive realization for T_c follows after evaluation of $T_c = u^*G$ and yields nonminimal additive realization

$$[T_c]_{n,m} = \left[\begin{array}{c|c} A & P^{-1}\Omega e^*y_{22}^{-1} \\ \hline a'_c{}^* & (B_u^*f^* + a'_c{}^*KP^{-1}\Omega e^*)y_{22}^{-1} \\ \hline D_u^*B + c'_c{}^*KA & c'_c{}^* \end{array} \middle| \begin{array}{c} \\ \hline (D_u^*f^* + c'_c{}^*KP^{-1}\Omega e^*)y_{22}^{-1} \\ \hline \end{array} \right]$$

where $K = [\Omega'_{o,c}{}^{-1} \ 0]$ is the bounded solution for

$$a'_c{}^*KA + B_u^*B = K^{(-1)}.$$

When expressed in the original terms, the previous realization reads

$$[T_c]_{n,m} = \left[\begin{array}{c|c} \begin{array}{c} a'_c \ c'_c B_c \\ \hline A_c \end{array} & \begin{array}{c} [0 \ \Omega'_{o,c}] \Omega e^* y_{22}^{-1} \\ [\Lambda_c^{J-1} \ 0] \Omega e^* y_{22}^{-1} \end{array} \\ \hline \begin{array}{c} a'_c{}^* \\ \hline 0 \ B_c \end{array} & \begin{array}{c} (B_u^*f^* + a'_c{}^* \Omega'_{o,c}{}^{-1} [\Omega_{o,a} \ I] e^*) y_{22}^{-1} \\ \hline (D_u^*f^* + c'_c{}^* \Omega'_{o,c}{}^{-1} [\Omega_{o,a} \ I] e^*) y_{22}^{-1} \end{array} \end{array} \right].$$

It holds that $X := B_u^*f^* + a'_c{}^* \Omega'_{o,c}{}^{-1} [\Omega_{o,a} \ I] e^* = 0$. This conclusion follows after evaluation of XX^* in which we employ the identity

$$\begin{bmatrix} e^* \\ f^* \end{bmatrix} [e \ f] = \begin{bmatrix} \Omega^{-1} \\ I \end{bmatrix} - \begin{bmatrix} a^* \\ c^* \end{bmatrix} \Omega^{(-1)} [a \ c].$$

Then, $[T_c]_{n,m}$ reduces to

$$[T_c] = \left[\begin{array}{c|c} A_c & [\Lambda_c^{J-1} \Omega'_{o,a} \ 0] e^* y_{22}^{-1} \\ \hline B_c & (D_u^*f^* + c'_c{}^* [0 \ \Omega'_{o,c}{}^{-1}] e^*) y_{22}^{-1} \end{array} \right]. \tag{6.76}$$

In a similar way we evaluate a realization for T_a by starting from the product $T_a = \hat{G}\hat{u}$. We obtain

$$[T_a] = \left[\begin{array}{c|c} A_a & C_a \\ \hline \hat{y}_{22}^{-1} \hat{e} \begin{bmatrix} 0 \\ -\Lambda_c'' \Lambda_{o,a}^{J-1} \end{bmatrix} & \hat{y}_{22}^{-1} (\hat{f}D_{\hat{u}} + \hat{e} \begin{bmatrix} A''_a{}^{-1} \\ 0 \end{bmatrix} b''_a{}^*) \end{array} \right]. \tag{6.77}$$

As we have seen, realizations $[T_c]$ and $[T_a]$ are, by construction, such that $T_c T_a = \Theta$. Hence, they are not only minimal but they also generate all possible solutions for the problem.

The previous consideration provides the proof for the following result.

Theorem 6.27. *Let (J_1, J_2) -unitary $\Theta \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be given by the additive minimal, u.e.s realization $[\Theta]$.*

Then, the condition that both equations (6.72)-(6.73) from Theorem 6.26 have boundedly invertible solutions is sufficient for Θ to admit the (minimal) causal/anti-causal factorization $\Theta = T_c T_a$ with T_c, T_a specified by (6.63). In addition, the factorization is essentially unique. It is also a maximal-phase/maximal-phase-star factorization. Minimal, u.e.s realizations for the factors are given by (6.76) and (6.77).

Importance of Theorem 6.27 is the completeness of the factorizations it states. As in the case of minimal factorizations of unitary operators being characterized as $U = T_o G_o^*$ for some outer T_o, G_o , it is also possible to have minimal factorizations of J-unitary operators involving no J-unitary factors.

6.6 Extraction of intrinsic factors from a J-unitary operator

In this section we state a simple algorithm for extraction of intrinsic factors from a J-unitary operator $\Theta \in \mathcal{X}$ given by a minimal, additive u.e.s realization $[\Theta]$. In the previous section we have obtained necessary and sufficient conditions for such a Θ to admit a minimal causal/anti-causal factorization. Let us concentrate on the factors. As we concluded, the completeness part of Theorem 6.27 states that these factors are of maximal-phase though not necessarily of a minimal degree. This, in turn, means that if

$$T' = T_{o,\Theta} U_\Theta$$

is the outer/inner factorization for a causal invertible minimal factor in the minimal factorization $\Theta = T' T''$ then, in general, it holds that $\text{s-dim}(T') \geq \text{s-dim}(T_{o,\Theta})$. Hence, one might expect that U_Θ contains intrinsic factors with respect to $T_{o,\Theta}$. Assume U_Θ is such an inner factor. Then there is a minimal factorization into inner factors $U_\Theta = U_m U_i$ for which the inner U_m is such that $T_{o,\Theta} U_m$ is a maximal-phase factor of minimal degree. The remaining inner U_i is such that

$$\text{s-dim}(T') = \text{s-dim}(T_{o,\Theta} U_m) + \text{s-dim}(U_i).$$

Finally, if T' is a causal J-unitary operator then the existence of U_i implies that this factor is J-unitary as well. It is obvious that factors of the form

$$U_i := \begin{bmatrix} W \\ V \end{bmatrix}$$

are the only intrinsic factors which are both inner and causal J-unitary. As a consequence, extraction of the intrinsic factors for a causal J-unitary operator reduces to the extraction of the inner intrinsic factors from its outer/inner factorization. The

dual conclusion can be made by considering the extraction at the input port of the operator.

A simple algorithm for the extraction at the output port is given in Figure 6.1.

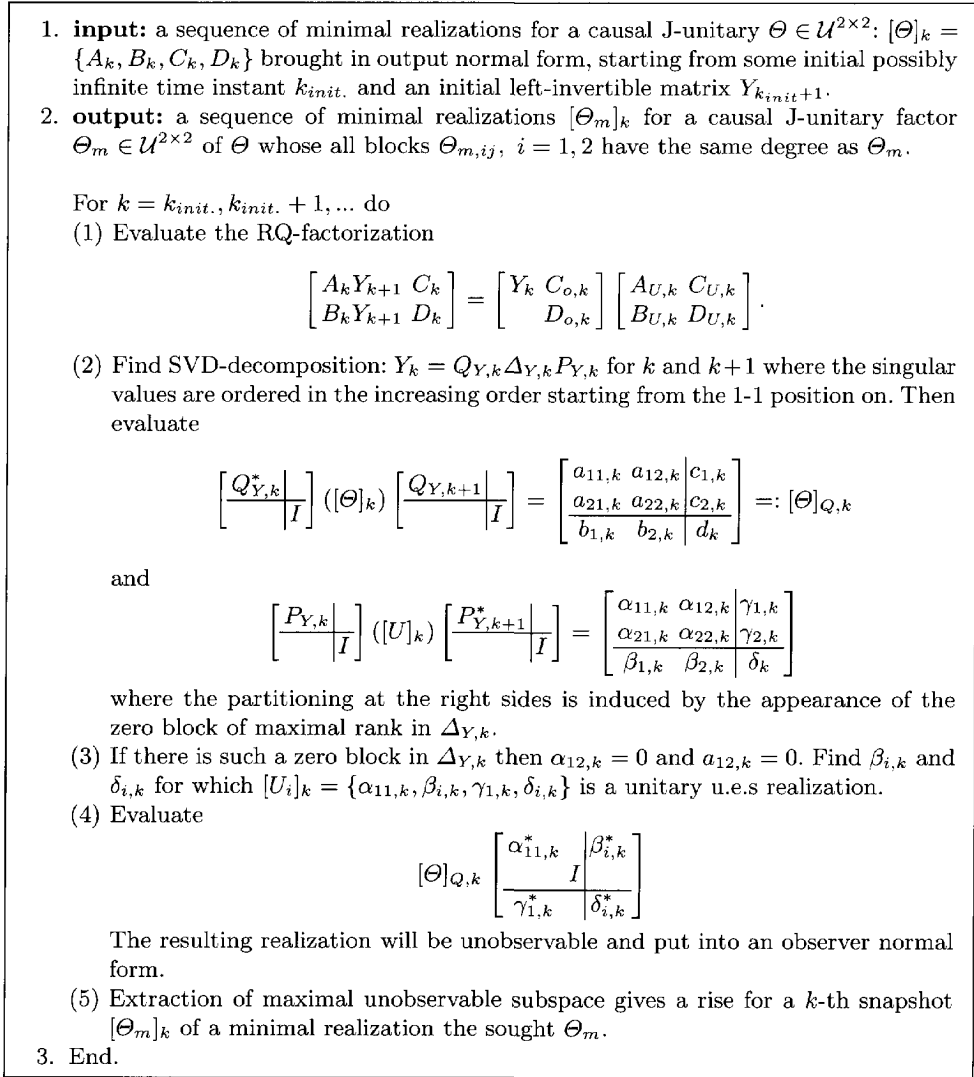


Fig. 6.1. Algorithm for extraction of intrinsic inner factors from Θ that are in the same time J-unitary.

6.7 Conclusions

In this chapter we have considered the minimal factorization problem for locally finite boundedly invertible operators of mixed causality. This problem is a search problem, and, even if the solution exists, in general, it is not necessarily unique. The search is conducted by means of unitary state-space transformations whose goal is to bring minimal realizations for a mixed causality operator and its inverse in alternative forms whose state transition operators have block triangular forms. If and only if the alternative realizations fulfill the algebraic conditions specified in Proposition 6.7 then the operator in question and its inverse admit a minimal factorization into two factors of mixed causality. The problem of constructing the factors is closely related to the problem of completion of the partially specified operator when its inverse is partially known too (see Definition 6.9). We have shown that this problem can be formulated as a Cholesky factorization problem and found all its solution. The solutions are parameterized by a single inner operator W_f . We have also shown that the problem of finding the unitary state-space transformations can be replaced by performing the search for appropriate W_f .

In particular, a minimal causal/anti-causal factorization of a J-unitary operator is considered and all causal/anti-causal factorizations involving invertible factors are given.

Finally, we have shown (see the algorithm from Figure 6.1) how to extract all intrinsic inner factors from a causal J-unitary operator by applying an outer/inner factorization on the operator.

7

POLAR DECOMPOSITION IN PONTYAGIN SPACES

Preliminary work on indefinite spectral factorizations of rational matrix valued functions appears to be due to [Cop72], [Jac70], [Jjub78]. In the recent engineering literature on time-invariant H_∞ -control theory, indefinite spectral factorization is known as J-spectral factorization. For example, in the H_∞ -control problem [DGKF87], the design of controllers essentially reduces to study of two indefinite Lyapunov-Stein equations. Their solutions correspond to: (1) the state-feedback gain and (2) the state-estimator gain. A direct J-spectral approach to this problem is investigated in [Green92], [GrGiLiDo90]. In these applications, for a given causal operator T and signature $J = \text{diag}(I_p, -I_q)$, the spectral density TJT^* is usually proper with no zeros on the imaginary axis and is invertible at infinity. These problems are mostly canonical, i.e, the spectral factor G such that $TJT^* = GJ_xG^*$ for some signature J_x of as minimal as possible rank, is aimed to be outer i.e. both $G(= G_o)$ and its inverse are analytic. The more difficult non-canonical cases such as relaxing the outerness and allowing J and/or J_x to be singular are usually ruled out by the problem assumptions.

In the present chapter we study a factorization that is closely related to the indefinite spectral factorization. By stipulating T to represent a causal time-varying linear boundedly left invertible operator and J to be an invertible indefinite signature, we investigate necessary and sufficient conditions that there is a causal (J, J_x) -unitary Θ for which $T = G_{ol}\Theta$ where G_{ol} is left outer. Such a factorization is called *indefinite polar decomposition*. Unlike standard outer/inner factorization which, under a mild

technical conditions posed on T always exists, the indefinite factorization does not necessarily exist. But, for T as mentioned above, it easily shows that the existence of the indefinite spectral factorization is equivalent to the existence of the indefinite polar decomposition and vice versa since aiming that Θ is causal (J, J_x)-unitary is then equivalent to aiming that $\Theta = T^\dagger G$ for an appropriate bounded left inverse T^\dagger for T .

7.1 Introduction

In the subsequent section we introduce the notion of *indefinite inner product space* and study locally finite operators acting between them. In particular, we are interested in so-called *left-outer/causal-J-isometric* and *causal-J-coisometric/right-outer* factorization on these spaces. The formal definition of these concepts will be given in the subsequent sections. In [Xi96], necessary and sufficient conditions for such an operator to admit the factorization are stated in operator-space terms. From the algorithmic point of view, these results may be regarded to as having a limited significance since we prefer an algebraic reformulation - i.e. in terms of state-space bases associated to a minimal realization for the operator at hand.

In formulating the algebraic condition we use standard inner/outer factorization¹ framework as we did in the previous chapters. By looking at the operator from two perspectives: (1) as actions between indefinite (Krein) spaces and (2) as actions between Hilbert spaces, we show that it is possible to replace the conditions from [Xi96] by equivalent conditions stated in diagonal algebra terms. As a causal J-isometric factor in the factorization is also a maximal-phase operator, the algebraic properties needed for studying the representation are easily understood and derived by analyzing this operator by means of standard inner/outer factorization framework.

7.2 Preliminaries

Let us recall some basic facts from the indefinite inner product theory. The interested reader is addressed to [Bog74] for an elementary- and to [AY89] for an extensive research on this subject. The research there has been done on uniform inner product spaces.

Let ℓ_2^M and ℓ_2^N be two non-uniform ℓ_2 -spaces. Formal definition of these spaces is given in Chapter 2. Let I_{M+}, I_{N+}, I_{M-} and I_{N-} be identity block diagonals defined

¹More precisely, it is left-outer/isometric or coisometric/right-outer. We write 'inner/outer' for short.

on subspaces $\ell_2^{\mathcal{M}+}, \ell_2^{\mathcal{M}-}$ of $\ell_2^{\mathcal{M}}$ and subspaces $\ell_2^{\mathcal{N}+}, \ell_2^{\mathcal{N}-}$ of $\ell_2^{\mathcal{N}}$ such that the invertible indefinite Hermitian operators $J_{\mathcal{M}}, J_{\mathcal{N}} \in \mathcal{D}$ defined as the maps

$$\cdot J_{\mathcal{M}}: \ell_2^{\mathcal{M}} \rightarrow \ell_2^{\mathcal{M}} \quad \text{and} \quad \cdot J_{\mathcal{N}}: \ell_2^{\mathcal{N}} \rightarrow \ell_2^{\mathcal{N}}$$

and having the corresponding matrix representations in the standard bases from $\ell_2^{\mathcal{M}}$ and $\ell_2^{\mathcal{N}}$ in the form

$$J_{\mathcal{M}} := \begin{bmatrix} I_{\mathcal{M}+} & \\ & -I_{\mathcal{M}-} \end{bmatrix} \quad \text{and} \quad J_{\mathcal{N}} := \begin{bmatrix} I_{\mathcal{N}+} & \\ & -I_{\mathcal{N}-} \end{bmatrix}, \tag{7.1}$$

share the s-dim sequences with $\ell_2^{\mathcal{M}}$ and $\ell_2^{\mathcal{N}}$ respectively. Operators $J_{\mathcal{M}}$ and $J_{\mathcal{N}}$ are clearly bounded. They however do not belong to \mathcal{D}_2 .

The local dimensions of the identity operators constituting $J_{\mathcal{M}}$ and $J_{\mathcal{N}}$ define their *inertia* the formal definition of which follows.

Definition 7.1. Let $X \in \mathcal{D}$ be a Hermitian. Then, at each snapshot X_k of X we define

- $\iota_+(X_k) :=$ the number of strictly positive eigenvalues of X_k ,
- $\iota_0(X_k) :=$ the number of the eigenvalues of X_k equal to zero,
- $\iota_-(X_k) :=$ the number of strictly negative eigenvalues of X_k .

Definition 7.2. For a non-uniform space $\ell_2^{\mathcal{M}}$, the operator $J_{\mathcal{M}}$ defined by (7.1) as the map $\cdot J_{\mathcal{M}}: \ell_2^{\mathcal{M}} \rightarrow \ell_2^{\mathcal{M}}$ is called an (invertible) signature operator associated to $\ell_2^{\mathcal{M}}$.

Similar definition can be given for $J_{\mathcal{N}}$. We omit it for simplicity.

In the sequel, the signature matrices $J_{\mathcal{M}}$ and $J_{\mathcal{N}}$ shall always be chosen invertible hence the $\iota_0(J_{\mathcal{M},k}) = 0$ for each k . However, as it may occur that the fixed point solution of some Lyapunov-Stein equations² is singular, a part of the inertia matrix of the solution might contain a certain number of zeros.

Let $\ell_2^{\mathcal{M}+}, \ell_2^{\mathcal{M}-}$ and $\ell_2^{\mathcal{N}+}, \ell_2^{\mathcal{N}-}$ be pairs of the subspaces of $\ell_2^{\mathcal{M}}$ and $\ell_2^{\mathcal{N}}$ associated with respectively the positive and the negative inertia matrices as introduced previously. They decompose $\ell_2^{\mathcal{M}}$ and $\ell_2^{\mathcal{N}}$ in the standard³ orthogonal fashion, i.e

$$\begin{cases} \ell_2^{\mathcal{M}} = \ell_2^{\mathcal{M}+} \oplus \ell_2^{\mathcal{M}-} \\ \ell_2^{\mathcal{N}} = \ell_2^{\mathcal{N}+} \oplus \ell_2^{\mathcal{N}-} \end{cases} \tag{7.2}$$

²such as $A^J \in \mathcal{D}$ of $A^* A^J A + B^* J_{\mathcal{M}} B = A^{J(-1)}$ for some u.e.s causal AB-pair (A, B)

³Hilbert space orthogonal fashion as introduced in Chapter 2

Definition 7.3. Let $\ell_2^{\mathcal{M}}$ be a non-uniform signal space to which an invertible signature operator $J_{\mathcal{M}}$ is associated. Let $x = [x_1 \ x_2]$ and $y = [y_1 \ y_2]$ be elements from $\ell_2^{\mathcal{M}}$ partitioned according to the s -dim sequences of $I_{\mathcal{M}+}$ and $I_{\mathcal{M}-}$ from $J_{\mathcal{M}}$. The scalar valued map

$$\begin{aligned} \langle \cdot, \cdot \rangle_{J_{\mathcal{M}}} &: \ell_2^{\mathcal{M}} \times \ell_2^{\mathcal{M}} \rightarrow \mathbb{C}, \\ \langle x, y \rangle_{J_{\mathcal{M}}} &= x J_{\mathcal{M}} y^* = x_1 y_1^* - x_2 y_2^* \end{aligned}$$

is called the scalar $J_{\mathcal{M}}$ -inner product on $\ell_2^{\mathcal{M}}$.

Notice that the $J_{\mathcal{M}}$ -inner product as introduced in Definition 7.3 does not satisfy the axioms of an inner product map on Hilbert spaces as stated in Chapter 2. A simple check reveals that

$$\langle x, x \rangle_{J_{\mathcal{M}}} = 0 \not\Rightarrow x = 0$$

for $x = [1 \ 1]$ from some $\ell_2^{\mathcal{M}}$ on which $J_{\mathcal{M}} := \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$.

This example motivate us to introduce the following definition.

Definition 7.4. Let x be an element of an non-uniform indefinite inner product space $(\mathcal{S}, \langle \cdot, \cdot \rangle_{J_{\mathcal{M}}})$. As $\langle x, x \rangle_{J_{\mathcal{M}}} = \overline{\langle x, x \rangle_{J_{\mathcal{M}}}}$, we have that: $\langle x, x \rangle_{J_{\mathcal{M}}} < 0$, $\langle x, x \rangle_{J_{\mathcal{M}}} = 0$ or $\langle x, x \rangle_{J_{\mathcal{M}}} > 0$ and refer to x as $J_{\mathcal{M}}$ -negative, $J_{\mathcal{M}}$ -neutral or $J_{\mathcal{M}}$ -positive element of \mathcal{S} respectively.

A mixed causality linear operator T acting between indefinite spaces as $\cdot T: \ell_2^{\mathcal{M}} \rightarrow \ell_2^{\mathcal{N}}$ has a partitioned matrix representation due to the decomposition (7.2) in the form

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : \ell_2^{\mathcal{M}+} \oplus \ell_2^{\mathcal{M}-} \rightarrow \ell_2^{\mathcal{N}+} \oplus \ell_2^{\mathcal{N}-}. \quad (7.3)$$

An important locally finite operator defined on an indefinite inner product space is introduced by the following definition.

Definition 7.5. Let $\ell_2^{\mathcal{M}}$ and $\ell_2^{\mathcal{N}}$ be non-uniform spaces on which the corresponding invertible signatures $J_{\mathcal{M}}$ and $J_{\mathcal{N}}$ are defined. An operator T acting between the spaces as given by (7.3) is called:

(1) $(J_{\mathcal{M}}, J_{\mathcal{N}})$ -isometric if

$$\langle uT, vT \rangle_{J_{\mathcal{N}}} = \langle u, v \rangle_{J_{\mathcal{M}}} \quad \forall u, v \in \ell_2^{\mathcal{M}},$$

(2) $(J_{\mathcal{M}}, J_{\mathcal{N}})$ -coisometric if its adjoint⁴ T^* is $(J_{\mathcal{N}}, J_{\mathcal{M}})$ -isometric,

⁴see Chapter 2.

(3) (J_M, J_N) -unitary if it is both (J_M, J_N) -isometric and (J_M, J_N) -coisometric - in this case the operator⁵

$$\Sigma := \begin{bmatrix} T_{11} - T_{12}T_{22}^{-1}T_{21} & -T_{12}T_{22}^{-1} \\ T_{22}^{-1}T_{21} & T_{22}^{-1} \end{bmatrix} : \ell_2^{\mathcal{M}+} \oplus \ell_2^{\mathcal{N}-} \rightarrow \ell_2^{\mathcal{N}+} \oplus \ell_2^{\mathcal{M}-} \quad (7.4)$$

exists as a unitary operator,

(4) (J_M, J_N) -inner if it is (J_M, J_N) -unitary, and Σ given by (7.4) is causal (hence inner).

The proof for the following lemma is obvious and is a direct consequence of Definition 7.5 and the definition of the adjoint operator on Hilbert space.

Lemma 7.6. *Let $\ell_2^{\mathcal{M}}$ and $\ell_2^{\mathcal{N}}$ be non-uniform spaces to which invertible signatures J_M and J_N are associated correspondingly. An operator $T: \ell_2^{\mathcal{M}} \rightarrow \ell_2^{\mathcal{N}}$ is (J_M, J_N) -isometric if and only if $TJ_N T^* = J_M$.*

In Definition 7.5, in the equation (7.4), we used the fact that T_{22} is boundedly invertible. This however is automatically fulfilled whenever T is (J_M, J_N) -unitary since by then

$$\begin{aligned} T_{22}T_{22}^* &= I + T_{21}T_{21}^* \gg 0 \quad \text{and} \\ T_{22}^*T_{22} &= I + T_{12}^*T_{12} \gg 0. \end{aligned}$$

For a detailed⁶ proof see [DvdV98].

Another remark is that if T is (J_M, J_N) -unitary, then T is boundedly⁷ invertible. Its inverse follows after invoking Lemma 7.6 twice. The inverse is then found as

$$T^{-1} = J_N T^* J_M.$$

In Chapter 2, stacking all sequences from $\ell_2^{\mathcal{M}}$ onto each other, made it possible to study the action of a bounded operator and those bounded in Hilbert-Schmidt norm e.g $\mathcal{X}_2(\mathcal{M}, \mathcal{N})$ on all sequences from $\ell_2^{\mathcal{M}}$ at once. To such spaces we associated a \mathcal{D} -valued inner product. In this section we introduce a similar map associated to the signature of the underlying indefinite spaces.

Definition 7.7. *Let $\mathcal{X}_2^{\mathcal{M}+} \times \mathcal{X}_2^{\mathcal{M}-}$ be a non-uniform inner product space to which a signature operator $J_M \in \mathcal{D}$ is associated. The \mathcal{D} -valued map*

⁵it is called a scattering operator associated to the chain-scattering operator T .

⁶ T_{22} has closed range and trivial kernel. The proof then completes after invoking the closed graph theorem.

⁷if T is bonded then T^* is bounded as well (see Chapter 2).

$$\begin{aligned} \{\cdot, \cdot\}_{J_{\mathcal{M}}} &: \mathcal{X}_2^{M_+} \times \mathcal{X}_2^{M_-} \rightarrow \mathcal{D}, \\ \{x, y\}_{J_{\mathcal{M}}} &:= \mathbf{P}_0(xJy^*) \quad \forall x, y \in \mathcal{X}_2^{M_+} \times \mathcal{X}_2^{M_-} \end{aligned}$$

is called the diagonal $J_{\mathcal{M}}$ -inner product.

The definitions and the statements as introduced in the previous section extend easily and in full generality to the space \mathcal{X}_2 as well.

In that sense, and by the definitions 7.3 and 7.7, we extend the definition of the scalar $J_{\mathcal{M}}$ -inner product space on ℓ_2^M to the spaces of Hilbert-Schmidt operators by taking

$$\langle x, y \rangle_{J_{\mathcal{M}}} := \text{trace}\{x, y\}_{J_{\mathcal{M}}} \quad \forall x, y \in \mathcal{X}_2^{M_+} \times \mathcal{X}_2^{M_-}.$$

To avoid complicated notation for the spaces between which T acts, we write from now on $T \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ or even $T \in \mathcal{X}^{2 \times 2}$ for short.

Definition 7.8. For some subspace \mathcal{S} of $\mathcal{X}_2^{M_+} \times \mathcal{X}_2^{M_-}$ to which a signature $J_{\mathcal{M}}$ is associated, we say that

- (1) \mathcal{S} is $J_{\mathcal{M}}$ -positive if for all $x \in \mathcal{S}$, $\{x, x\}_{J_{\mathcal{M}}} \geq 0$,
- (2) \mathcal{S} is uniformly $J_{\mathcal{M}}$ -positive if there is a real $\epsilon > 0$ such that $\forall x \in \mathcal{S}$, $\{x, x\}_{J_{\mathcal{M}}} \geq \epsilon\{x, x\}$,
- (3) \mathcal{S} is $J_{\mathcal{M}}$ -neutral if $\forall x \in \mathcal{S}$, $\{x, x\}_{J_{\mathcal{M}}} = 0$,
- (4) \mathcal{S} is $J_{\mathcal{M}}$ -negative if $\forall x \in \mathcal{S}$, $\{x, x\}_{J_{\mathcal{M}}} \leq 0$,
- (5) \mathcal{S} is uniformly $J_{\mathcal{M}}$ -negative if there is a real $\epsilon > 0$ such that $\forall x \in \mathcal{S}$, $-\{x, x\}_{J_{\mathcal{M}}} \geq \epsilon\{x, x\}$.

Definition 7.9. Let \mathcal{S}' be a subspace of some $J_{\mathcal{M}}$ -inner product space $\mathcal{S} = \mathcal{X}_2^{M_+} \times \mathcal{X}_2^{M_-}$. The space

$$\mathcal{S}'^{[\perp]} := \{x \in \mathcal{S} : \forall y \in \mathcal{S}', \{x, y\}_{J_{\mathcal{M}}} = 0\}$$

is called the $J_{\mathcal{M}}$ -orthogonal complement of \mathcal{S}' .

It is important to note that Definition 7.9 implies that the operation $(\cdot)^{[\perp]}$, when applied twice on a subspace does not necessarily produce the subspace. Actually, $\mathcal{S}' \subseteq (\mathcal{S}'^{[\perp]})^{[\perp]}$ with the equality sign if and only if $J_{\mathcal{M}}$ is invertible.

Let us recall the notion of direct sum $\mathcal{S}' \dot{+} \mathcal{S}''$ between Hilbert spaces \mathcal{S}' and \mathcal{S}'' :

$$\mathcal{S}' \dot{+} \mathcal{S}'' := \{x + y \text{ where } x \in \mathcal{S}', y \in \mathcal{S}''\}.$$

In the following definition we introduce the concept of the J -orthogonality of two spaces with respect to each other (see [DvdV98],[Bog74]).

Definition 7.10. Let \mathcal{S}' and \mathcal{S}'' be two non-uniform spaces. Let J be a signature operator defined on these spaces. We say that \mathcal{S}' is J -orthogonal to \mathcal{S}'' and write

$$\mathcal{S}'[\perp]_J \mathcal{S}''$$

if for all $x \in \mathcal{S}', y \in \mathcal{S}''$ it holds that $\mathbf{P}_0(xJy^*) = 0$.

An obvious consequence of Definition 7.10 is that

$$\mathcal{S}'[\perp]_J \mathcal{S}'' = \mathcal{S}' \perp (\mathcal{S}'' J). \tag{7.5}$$

Recall Definition 7.9 where we introduced the J -orthogonal complement of a subspace \mathcal{S}' of some larger space \mathcal{S} . The spaces \mathcal{S}' and $\mathcal{S}'^{[\perp]}$ as introduced in Definition 7.9 not necessarily span the entire space \mathcal{S} . On the other side, they also may have nonempty intersection. The latter case occurs if \mathcal{S}' contain a so called *neutral subspace*. A neutral subspace is a space consisting of vectors which are J -orthogonal on themselves and on the subspace itself. The maximal neutral subspace of a space \mathcal{S} will be denoted with \mathcal{S}_n .

It is obvious that if \mathcal{S}' is a subspace of \mathcal{S} which contains the maximal neutral subspace \mathcal{S}'_n , then

$$\mathcal{S}' \cap \mathcal{S}'^{[\perp]} = \mathcal{S}'_n. \tag{7.6}$$

Hence, the check if \mathcal{S}' has a neutral subspace relies upon checking if the intersection (7.6) is nontrivial.

Definition 7.11. Let \mathcal{S} be a space on which a signature operator J_2 is defined. If there exist subspaces $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$ which are mutually J -orthogonal and such that for each $s \in \mathcal{S}$ it holds that

$$s = s' + s''$$

for some $s' \in \mathcal{S}'$ and $s'' \in \mathcal{S}''$ then we write

$$\mathcal{S} = \mathcal{S}' \boxplus \mathcal{S}'' \tag{7.7}$$

to denote that \mathcal{S} admits a J -orthogonal decomposition.

To introduce Krein space decomposition, it is necessary to know that \mathcal{S}' and \mathcal{S}'' are indeed subspaces of a given \mathcal{S} . This however is not enough.

Definition 7.12. The space \mathcal{S} to which a signature operator J is associated, is called a *Krein space* if it splits in a J -orthogonal sum of an uniformly J -positive \mathcal{S}' and uniformly J -negative \mathcal{S}'' (Hilbert) spaces:

$$\mathcal{S} = \mathcal{S}' \boxplus \mathcal{S}'' \text{ such that } \begin{cases} \mathcal{S}' \subseteq \mathcal{S}, \\ \mathcal{S}'' \subseteq \mathcal{S} \end{cases}$$

Concurrently to '⊖' as introduced in Chapter 2 we introduce the composition '⊞' between two spaces by the following definition.

Definition 7.13. Let \mathcal{S} and \mathcal{S}' denote respectively a space and its subspace. Let J be a signature operator defined on \mathcal{S} . $\mathcal{S} \boxplus \mathcal{S}'$ denotes a subspace of \mathcal{S} defined as

$$\mathcal{S} \boxplus \mathcal{S}' := \{x \in \mathcal{S} : \forall y \in \mathcal{S}' \quad \mathbf{P}_0(xJy^*) = 0\}.$$

Unlike the difference '⊖' between a Hilbert space and its subspace which is always $\supseteq \{0\}$, the J -difference $\mathcal{S} \boxplus \mathcal{S}'$ may be equal $\{0\}$ even if \mathcal{S}' is a nontrivial subspace of \mathcal{S} . In particular, if \mathcal{S}' is the neutral subspace of \mathcal{S} then $\mathcal{S} \boxplus \mathcal{S}' = \mathcal{S}'$.

7.3 State-space properties of causal chain-scattering operators

In this section we discuss some relationships between Gramians and J -Gramians associated to a u.e.s realization of a chain-scattering operator T . In Chapter 2, we have introduced the notion of a minimal, u.e.s additive realization and the notion of bases for state-spaces of operators.

Let a mixed causality operator

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$$

$$[y_+ \ y_-] = [u_+ \ u_-] T$$

to which the input and the output port signatures

$$J_{\mathcal{M}} = \begin{bmatrix} I_{\mathcal{M}_+} & \\ & -I_{\mathcal{M}_-} \end{bmatrix} \quad \text{and} \quad J_{\mathcal{N}} = \begin{bmatrix} I_{\mathcal{N}_+} & \\ & -I_{\mathcal{N}_-} \end{bmatrix}$$

are associated, be given by a minimal, additive u.e.s realization

$$[T] = \begin{bmatrix} A_c & & C_{c1} & C_{c2} \\ & A_a & C_{a1} & C_{a2} \\ B_{c1} & B_{a1} & D_{11} & D_{12} \\ B_{c2} & B_{a2} & D_{21} & D_{22} \end{bmatrix} \tag{7.8}$$

$$[x_c^{(-1)} \ x_a | y_+ \ y_-] = [x_c \ x_a^{(-1)} | u_+ \ u_-] [T] \tag{7.9}$$

where

$$x_c \in \ell_2^{\mathcal{B}_c} \quad \text{and} \quad x_a \in \ell_2^{\mathcal{B}_a}$$

are the state-space variables defined respectively on state-spaces $\ell_2^{\mathcal{B}_c}$ and $\ell_2^{\mathcal{B}_a}$ and having s -dim sequences \mathcal{B}_c and \mathcal{B}_a . For the formal definition of these spaces see Chapter 2.

Figure 7.1 illustrates the realization $[T]$ corresponding to a causal chain scattering operator T . The quantities $u = [u_+ \ u_-]$, $y = [y_+ \ y_-]$ are the variables from the input and the output port of T and are partitioned accordingly the inertias of \mathcal{M} and \mathcal{N} . Variables $x = [x_+ \ x_0 \ x_-]$ and $x^{(-1)} = [x_+^{(-1)} \ x_0^{(-1)} \ x_-^{(-1)}]$ are respectively the state-space variable and its delayed counterpart partitioned accordingly the inertias of \mathcal{B} and $\mathcal{B}^{(-1)}$. The stiped diagonal line divides the realization according to the realization's inputs and outputs.

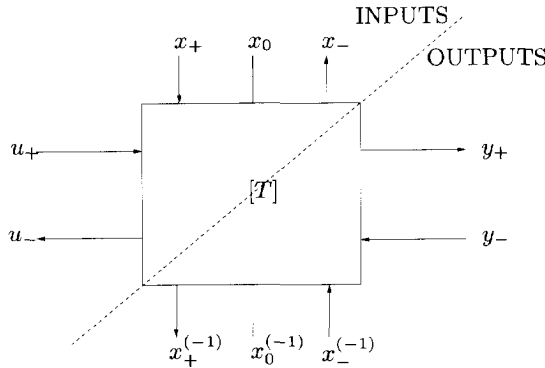


Fig. 7.1. A realization $[T]$ for a causal chain scattering operator T .

Recall from Chapter 2 that \mathbf{F}_c represents a sliced basis for a input state space $\mathcal{H}_c(T)$ of a causal operator part from T . We will also use the fact from [DvdV98] that if \mathcal{H} is a locally finite D -invariant subspace equipped with the signature J_B , then it has a strong basis representation in the form $\mathcal{H}_c = \mathcal{D}_2^B \mathbf{F}_c$ where \mathbf{F}_c is the basis for the input state-space \mathcal{H}_c of the causal operator part from T as introduced in Chapter 2. \mathbf{F}_c is then said to be a J_B -basis. The formal definition for a J_B -(sliced)-base follows.

Definition 7.14. Let $\mathbf{F} = \mathcal{D}_2 \left(BZ(I - AZ)^{-1} \right)^*$ be a sliced basis representation defined on the u.e.s AB -pair (A, B) . The block diagonal quantity

$$A_{\mathbf{F}}^J := \mathbf{P}_o(\mathbf{F}J\mathbf{F}^*) \in \mathcal{D}(\mathcal{B}, \mathcal{B})$$

is called the J -Gram operator. If $A_{\mathbf{F}}^J = J_B$, then the basis \mathbf{F} is called a J_B -orthogonal sliced basis representation.

Definition 7.15. Let (A, B) be a causal AB -pair and let J be a signature of an appropriate s -dim sequence. The equation

$$A^* \Lambda^J A + B^* J B = \Lambda^{J(-1)} \tag{7.10}$$

is called an indefinite Lyapunov-Stein equation in $\Lambda^J \in \mathcal{D}$.

Definition 7.16. Let the pair (A, C) with $\ell_A < 1$ be as introduced in Definition 7.15. It is called

- (1) partially J -unreachable if Λ^J is singular,
- (2) J -reachable if Λ^J is invertible.

Notion of the J -observability for some causal (A, C) -pair is defined similarly. It is worth noting that the reachability in the usual sense does not necessarily imply the J -reachability and vice versa. In the particular case if $\Lambda^J \gg 0$ then the pair (A, B) is reachable as the following lemma states.

Lemma 7.17. Let a causal AB -pair (A, B) with $\ell_A < 1$ be such that (7.10) has the unique strictly positive definite bounded solution Λ^J for some invertible signature $J_{\mathcal{M}}$. Then (A, B) is a uniformly reachable pair.

PROOF: According to the partitioning in the signature $J_{\mathcal{M}} = \begin{bmatrix} I_{\mathcal{M}_+} & \\ & -I_{\mathcal{M}_-} \end{bmatrix}$ we introduce the partitioning into $B = \begin{bmatrix} B_+ \\ B_- \end{bmatrix}$ conformably. Let $\Lambda \in \mathcal{D}$ be the unique boundedly invertible Gramian corresponding to (A, B) . Then Λ satisfies

$$A^* \Lambda A + B_+^* B_+ + B_-^* B_- = \Lambda^{(-1)}$$

whereby $\Lambda \geq 0$. The equation (7.10) rewrites equivalently as

$$A^* \Lambda^J A + B_+^* B_+ - B_-^* B_- = \Lambda^{J(-1)}.$$

Let us add the latter two equations to each other. This results in the Lyapunov-Stein equation

$$A^*(\Lambda + \Lambda^J)A + 2B_+^* B_+ = (\Lambda + \Lambda^J)^{(-1)}$$

the bounded solution of which is such that $\Lambda + \Lambda^J \geq 0$. Since Λ^J is assumed strictly positive definite and $\Lambda \geq 0$, we conclude that $\Lambda + \Lambda^J > 0$ and hence that (A, B_+) is a reachable pair. Then $(A, \begin{bmatrix} B_+ \\ B_- \end{bmatrix})$ is a uniformly reachable pair too. This completes the proof. ■

Definition 7.18. \mathcal{H} is called a regular subspace if $\Lambda_{\mathbb{F}}^J$ for $J = I$ is boundedly invertible.

If T is a mapping between indefinite spaces then its causal $\ell_2^{\mathcal{B}_c}$ and anti-causal state-space $\ell_2^{\mathcal{B}_a}$ associated to an additive, u.e.s realization $[T]$ for T are, in general, indefinite too. Moreover, for some appropriately chosen identity block diagonals $I_{\mathcal{B}_{c,+}}, I_{\mathcal{B}_{c,-}}, I_{\mathcal{B}_{a,+}}, I_{\mathcal{B}_{a,-}}$ and zero blocks $0_{\mathcal{B}_{c,0}}, 0_{\mathcal{B}_{a,0}}$ the block diagonals

$$J_{\mathcal{B}_c} := \begin{bmatrix} I_{\mathcal{B}_{c,+}} & & \\ & 0_{\mathcal{B}_{c,0}} & \\ & & -I_{\mathcal{B}_{c,-}} \end{bmatrix} \quad \text{and} \quad J_{\mathcal{B}_a} := \begin{bmatrix} I_{\mathcal{B}_{a,+}} & & \\ & 0_{\mathcal{B}_{a,0}} & \\ & & -I_{\mathcal{B}_{a,-}} \end{bmatrix}$$

are signature operators associated to \mathcal{B}_c and \mathcal{B}_a respectively.

In the case T is causal $(J_{\mathcal{M}}, J_{\mathcal{N}})$ -unitary where these signatures are invertible and $[T] = \{A, B, C, D\}$ is minimal and u.e.s, it can be shown (see [DvdV98]) that the s -dim sequence for $0_{\mathcal{B}_{c,0}}$ vanishes whereas the following *inertia conditions* hold

$$\iota_+(J_{\mathcal{M}}) + \iota_+(J_{\mathcal{B}_c}) = \iota_+(J_{\mathcal{N}}) + \iota_+(J_{\mathcal{B}_c}^{(-1)}), \tag{7.11}$$

$$\iota_-(J_{\mathcal{M}}) + \iota_-(J_{\mathcal{B}_c}) = \iota_-(J_{\mathcal{N}}) + \iota_-(J_{\mathcal{B}_c}^{(-1)}). \tag{7.12}$$

In that case, there is a relationship between the J-observability and the J-reachability Gramians: A_o^J and A^J corresponding to $[T]$. They are the inverses of each other. The signatures corresponding to these Gramians are the same since constrained by the inertia law.

If $[T] = \{A, B, C, D\}$ is a u.e.s realization for an arbitrary causal chain-scattering operator, then the signatures of possibly singular indefinite A^J and A_o^J are not necessarily related to each other - not even if $[T]$ is minimal.

In a standard inner product space, extraction of maximal unobservable or unreachable portions of a space specified by an AC- or an AB-pair is easily done by considering unique bounded solutions of Lyapunov-Stein equations defined on later pairs. Since the norm in a Hilbert space is always definite, singularity in the solutions gives rise to the ability that A-matrix can be brought into upper or lower block echelon forms. For some classes of operators, these forms can be exploited for minimal factorization purposes.

In an indefinite inner product space this property does not necessarily hold. Therefore, in order to obtain a block triangular form for a state transition operator, we shall encounter an indefinite Cholesky factorization problem.

Let us state the following simple lemma first. Therein we make use of $|J| := J \times J$ and include zero subspaces in \mathcal{M} and \mathcal{N} .

Lemma 7.19. *Let $T \in \mathcal{X}^{3 \times 3}$ be a unitary operator for which $TJ_{\mathcal{N}}T^* = J_{\mathcal{M}}$ for some signatures $J_{\mathcal{M}} = \text{diag}(I_{\mathcal{M},+}, 0_{\mathcal{M},0}, -I_{\mathcal{M},-}) \in \mathcal{D}^{3 \times 3}$ $J_{\mathcal{N}} = \text{diag}(I_{\mathcal{N},+}, 0_{\mathcal{N},0}, -I_{\mathcal{N},-}) \in \mathcal{D}^{3 \times 3}$ having non-vanishing s -dim($I_{\mathcal{M},+}$), s -dim($I_{\mathcal{M},-}$), s -dim($I_{\mathcal{M},-}$) and s -dim($I_{\mathcal{N},-}$). Then $T = \text{diag}(T_{11}, T_{22}, T_{33})$ with unitary $T_{ii}, i = 1, 2, 3$.*

PROOF: Let us start from $TJ_{\mathcal{N}}T^* = J_{\mathcal{M}}$ and $T^*J_{\mathcal{M}}T = J_{\mathcal{N}}$ and post- and premultiply them with their respective transposes. This produces the first line of

$$\begin{aligned} T|J_{\mathcal{N}}|T^* &= |J_{\mathcal{M}}|, & T^*|J_{\mathcal{M}}|T &= |J_{\mathcal{N}}|, \\ TJ_{\mathcal{N}}T^* &= J_{\mathcal{M}}, & T^*J_{\mathcal{M}}T &= J_{\mathcal{N}}. \end{aligned}$$

Let us add and subtract the pair equations on the left and do the same with the equations on the right. This produces

$$\begin{aligned} T(J_{\mathcal{N}} + |J_{\mathcal{N}}|)T^* &= J_{\mathcal{M}} + |J_{\mathcal{M}}|, & T^*(J_{\mathcal{M}} + |J_{\mathcal{M}}|)T &= J_{\mathcal{N}} + |J_{\mathcal{N}}|, \\ T(J_{\mathcal{N}} - |J_{\mathcal{N}}|)T^* &= J_{\mathcal{M}} - |J_{\mathcal{M}}|, & T^*(J_{\mathcal{M}} - |J_{\mathcal{M}}|)T &= J_{\mathcal{N}} - |J_{\mathcal{N}}|. \end{aligned}$$

Now, the pair equations on the left, when written out, implies that T_{11} and T_{33} are isometric whileas the pair equations on the right implies that T_{11} and T_{33} are coisometric independently of $s\text{-dim}(I_{\mathcal{M},0})$ or $s\text{-dim}(I_{\mathcal{N},0})$. Hence, T_{11} and T_{33} are unitary and because T is unitary, this implies that T_{22} is unitary too. All other blocks from T are then zero. This completes the proof. ■

7.4 Left-outer/isometric characterization

In this section we first attempt at characterizing causal and boundedly invertible chain-scattering operators by means of inner/outer factorization framework. Recall the definitions for the input and the output state-spaces associated to a minimal u.e.s additive realization $[T]$ for $T \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ as introduced in Chapter 2. Since we will assume T causal, we omit the index 'c' for simplicity and reintroduce the definition for the spaces as

$$\mathcal{K}(T) = \ker(H_T) = \{L \in \mathcal{L}_2^{2 \times 2} Z^{-1} : \mathbf{P}(LT) = 0\} \tag{7.13}$$

$$\mathcal{H}(T) = \text{ran}(H_T^*) = \mathbf{P}'(\mathcal{U}_2^{2 \times 2} T^*) \tag{7.14}$$

$$\mathcal{H}_o(T) = \text{ran}(H_T) = \mathbf{P}(\mathcal{L}_2^{2 \times 2} Z^{-1} T) \tag{7.15}$$

$$\mathcal{K}_o(T) = \ker(H_T^*) = \{Y \in \mathcal{U}_2^{2 \times 2} : \mathbf{P}'(YT^*) = 0\}. \tag{7.16}$$

To keep notations as simple as possible we write J_1 and J_2 instead of $J_{\mathcal{M}}$ and $J_{\mathcal{N}}$ to denote respectively the input- and the output port signatures for T .

Because a J-isometric causal operator Θ is the subject of study, let us investigate the spaces related to its Hankel operator. We start with the following simple lemma.

Lemma 7.20. *Let U be a causal isometry a minimal realization of which is u.e.s. Let U^\perp be another causal isometry such that $\begin{bmatrix} U \\ U^\perp \end{bmatrix}$ is inner. Then*

$$\ker(\cdot U^*|_{\mathcal{U}_2}) = \mathcal{U}_2 U^\perp. \tag{7.17}$$

PROOF: By straightforward substitution we convince ourselves that $\mathcal{U}_2 U^\perp \subseteq \ker(\cdot U^*|_{\mathcal{U}_2})$. Indeed, for $x \in \mathcal{U}_2 U^\perp$, there is a causal x' such that $x = x' U^\perp$ because $\cdot U^\perp$ has closed⁸ range. Because $\begin{bmatrix} U \\ U^\perp \end{bmatrix}$ is unitary we have that $U^\perp U^* = 0$. Hence,

⁸This is due to the u.e.s property of a minimal realization for U^\perp .

$$x'U^\perp U^* = 0$$

which means that $\mathcal{U}_2 U^\perp \subseteq \ker(\cdot U^*|_{\mathcal{U}_2})$.

To show that ' \subseteq ' reduces to the equality, it suffices to show that $\mathcal{U}_2 U^\perp \ominus \ker(\cdot U^*|_{\mathcal{U}_2}) = \{0\}$. By the definition of ' \ominus ' as introduced in Chapter 2 we have that

$$\mathcal{U}_2 U^\perp \ominus \ker(\cdot U^*|_{\mathcal{U}_2}) = \{x \in \mathcal{U}_2 U^\perp : \forall y \in \ker(\cdot U^*|_{\mathcal{U}_2}) \quad \mathbf{P}_0(xy^*) = 0\}.$$

Obviously $\mathbf{P}_0(xy^*) = 0$ for an anti-causal y^* if and only if $x \in \mathcal{L}_2 Z^{-1}$. But $x \in \mathcal{U}_2 U^\perp$ and is therefore causal. The only possible x belongs then to

$$\mathcal{U}_2 U^\perp \cap \mathcal{L}_2 Z^{-1}.$$

Since 0 is the only element of this intersection, we conclude that

$$\mathcal{U}_2 U^\perp \ominus \ker(\cdot U^*|_{\mathcal{U}_2}) = \{0\}$$

as needed. This completes the proof. ■

In the following proposition we evaluate the space \mathcal{K}_o . Therein we make use of the outer/isometric characterization for a causal J-isometric Θ .

Proposition 7.21. *Let $J_i, i = 1, 2$ be invertible signatures and $\Theta \in \mathcal{U}^{2 \times 2}(\mathcal{M}, \mathcal{N})$, be (J_1, J_2) -isometric. Then,*

$$\mathcal{K}_o(\Theta) = \overline{\mathcal{U}_2^{2 \times 2} \Theta J_2} \boxplus \ker(\cdot \Theta^*|_{\mathcal{U}_2^{2 \times 2}}). \tag{7.18}$$

PROOF: By the definition for \mathcal{K}_o , we have that

$$\mathcal{K}_o(\Theta) = \{Y \in \mathcal{U}_2^{2 \times 2} : \mathbf{P}'(Y\Theta^*) = 0\}.$$

By replacing Y with the subspace $\overline{\mathcal{U}_2^{2 \times 2} \Theta J_2}$ from the statement, we obtain identity since

$$\mathbf{P}'(\overline{\mathcal{U}_2^{2 \times 2} \Theta J_2} \Theta^*) = \mathbf{P}'(\mathcal{U}_2^{2 \times 2} J_1) = 0.$$

Hence, $\overline{\mathcal{U}_2^{2 \times 2} \Theta J_2}$ is indeed a candidate subspace of $\mathcal{K}_o(\Theta)$.

Let us evaluate the J-orthogonal complement for $\overline{\mathcal{U}_2^{2 \times 2} \Theta J_2}$ in $\mathcal{K}_o(\Theta)$. First, we need to show that $\ker(\cdot \Theta^*|_{\mathcal{U}_2^{2 \times 2}})$ is a subspace of $\mathcal{K}_o(\Theta)$. Let $y \in \mathcal{U}_2^{2 \times 2}$ be arbitrary element from the kernel subspace. Then $y\Theta^* = 0$. We also have that $\mathbf{P}'(y\Theta^*) = 0$ which due to the definition for $\mathcal{K}_o(\Theta)$ and the causality of y implies that $y \in \mathcal{K}_o(\Theta)$. Because y was arbitrary, it holds that $\ker(\cdot \Theta^*|_{\mathcal{U}_2^{2 \times 2}}) \subseteq \mathcal{K}_o(\Theta)$ as claimed. It holds then that

$$\mathcal{K}_o(\Theta) = \overline{\mathcal{U}_2^{2 \times 2} \Theta J_2} \dot{+} \ker(\cdot \Theta^*|_{\mathcal{U}_2^{2 \times 2}}). \tag{7.19}$$

To complete the proof, we need to show that ' $\dot{+}$ ' can be replaced by ' \boxplus '. Indeed, since Θ is causal boundedly right invertible, it admits an outer/isometric factorization

$\Theta = T_{o,\Theta}U_\Theta$ for some outer $T_{o,\Theta}$ and a causal isometric U_Θ that has a u.e.s realization. By Lemma 7.20, we then have that

$$\ker(\cdot\Theta^*|_{\mathcal{U}_2^{2\times 2}}) = \mathcal{U}_2^{2\times 2}U_\Theta^\perp$$

where U_Θ^\perp is a causal isometry for which $\begin{bmatrix} U_\Theta \\ U_\Theta^\perp \end{bmatrix}$ is inner. Let causal y_1 and y_2 be such that $y_2\Theta^* = 0$ and $y_1 := \overline{u_1\Theta J_2}$ for a causal u_1 . Note that such a u_1 exists, since J_2 is invertible and Θ is boundedly right invertible. Consider now $\mathbf{P}_0(y_1 J_2 y_2^*)$. We have that

$$\mathbf{P}_0(y_1 J_2 y_2^*) = \mathbf{P}_0(u_1 U_\Theta U_\Theta^\perp y_2^*) = 0.$$

Hence, $y_1[\perp]_{J_2}y_2$ for arbitrary elements y_1 and y_2 as introduced above. This is equivalent to saying that

$$\overline{\mathcal{U}_2^{2\times 2}\Theta J_2[\perp]_{J_2} \ker(\cdot\Theta^*|_{\mathcal{U}_2^{2\times 2}})}$$

and, together with (7.19) and the definition for '⊕' produces the truth for the statement. This completes the proof. ■

In the proof of Proposition 7.21 we needed to show that the composition '⊕' holds between the subspaces $\overline{\mathcal{U}_2^{2\times 2}\Theta J_2}$ and $\ker(\cdot\Theta^*|_{\mathcal{U}_2^{2\times 2}})$ of $\mathcal{K}_o(\Theta)$. An alternative proof to this can be given if we consider the space $\mathcal{K}_o(\Theta) \boxplus \overline{\mathcal{U}_2^{2\times 2}\Theta J_2}$ instead. Viz. Definition 7.13 for '⊕' we have that

$$\begin{aligned} \mathcal{K}_o(\Theta) \boxplus \overline{\mathcal{U}_2^{2\times 2}\Theta J_2} &= \{x \in \mathcal{K}_o(\Theta) : \forall y \in \overline{\mathcal{U}_2^{2\times 2}\Theta J_2} \quad \mathbf{P}_0(x J_2 y^*) = 0\} \\ &= \{x \in \mathcal{U}_2^{2\times 2} : \forall l \in \mathcal{L}_2^{2\times 2} \quad \mathbf{P}'(x\Theta^*) = 0 \wedge \mathbf{P}_0(x\Theta^*l) = 0\}. \end{aligned}$$

Now, $\mathbf{P}_0(x\Theta^*l) = 0$ for all $l \in \mathcal{L}_2^{2\times 2}$ if $x \in \mathcal{U}_2^{2\times 2}$ is such that $x\Theta^* \in \mathcal{L}_2^{2\times 2}Z^{-1}$. But then the condition $\mathbf{P}'(x\Theta^*) = 0$ reduces to the equivalent but simpler condition $x\Theta^* = 0$. This of course leads to

$$\begin{aligned} \mathcal{K}_o(\Theta) \boxplus \overline{\mathcal{U}_2^{2\times 2}\Theta J_2} &= \{x \in \mathcal{U}_2^{2\times 2} : x\Theta^* = 0\} \\ &= \ker(\cdot\Theta^*|_{\mathcal{U}_2^{2\times 2}}). \end{aligned}$$

Because of $\ker(\cdot\Theta^*|_{\mathcal{U}_2^{2\times 2}}) \subseteq \mathcal{K}_o(\Theta)$ the latter equation implies the truth for

$$\mathcal{K}_o(\Theta) = \overline{\mathcal{U}_2^{2\times 2}\Theta J_2} \boxplus \ker(\cdot\Theta^*|_{\mathcal{U}_2^{2\times 2}})$$

as needed in showing that this alternative proof is valid if J_2 is invertible.

Proposition 7.22. *Let $\Theta \in \mathcal{U}^{2\times 2}(\mathcal{M}, \mathcal{N})$ be (J_1, J_2) -isometric and let for some causal isometric U_Θ that has a u.e.s realization and some outer $T_{o,\Theta}$, the factorization $\Theta = T_{o,\Theta}U_\Theta$ be the outer/isometric representation for Θ . Let U_Θ^\perp be another causal isometry for which $\begin{bmatrix} U_\Theta \\ U_\Theta^\perp \end{bmatrix}$ is inner. Then*

$$\mathcal{K}_o(\Theta) = \overline{\mathcal{U}_2^{2 \times 2} U_\Theta J_2} \boxplus \overline{\mathcal{U}_2^{2 \times 2} U_\Theta^\perp}. \quad (7.20)$$

PROOF: Viz. the factorization $\Theta = T_{o,\Theta} U_\Theta$ we have that

$$\begin{aligned} \overline{\mathcal{U}_2^{2 \times 2} \Theta J_2} &= \overline{\mathcal{U}_2^{2 \times 2} T_{o,\Theta} U_\Theta J_2} \\ &= \overline{\mathcal{U}_2^{2 \times 2} T_{o,\Theta} U_\Theta J_2} \\ &= \overline{\mathcal{U}_2^{2 \times 2} U_\Theta J_2}. \end{aligned}$$

When substituted in the first subspace on the right side of the equation (7.18) from Proposition 7.21, the latter identity implies that

$$\mathcal{K}_o(\Theta) = \overline{\mathcal{U}_2^{2 \times 2} U_\Theta J_2} \boxplus \ker(\cdot \Theta^* |_{\mathcal{U}_2^{2 \times 2}}).$$

Consider the kernel subspace from the latter equation. We have that

$$\ker(\cdot \Theta^* |_{\mathcal{U}_2^{2 \times 2}}) = \ker(\cdot U_\Theta^* T_{o,\Theta}^* |_{\mathcal{U}_2^{2 \times 2}}).$$

Because $\overline{\mathcal{U}_2 T_{o,\Theta}} = \mathcal{U}_2$, we have that for $x \in \ker(\cdot U_\Theta^* T_{o,\Theta}^* |_{\mathcal{U}_2^{2 \times 2}})$ it holds that

$$x U_\Theta^* T_{o,\Theta}^* = 0 \Leftrightarrow x U_\Theta^* = 0.$$

Hence $x \in \ker(\cdot U_\Theta^* |_{\mathcal{U}_2^{2 \times 2}})$ as needed to be shown. The characterization of the latter space by means of the orthonormal complement U_Θ^\perp is given by Lemma 7.20. Hence, all such x belong to the set

$$\overline{\mathcal{U}_2^{2 \times 2} U_\Theta^\perp}.$$

This completes the proof. ■

Corollary 7.23. *Let J_2 be an invertible signature. If J_1 is an invertible signature then the subspaces of the space $\mathcal{K}_o(\Theta)$ for a causal (J_1, J_2) -isometry Θ from Proposition 7.21 or Proposition 7.22 are such that*

$$\overline{\mathcal{U}_2^{2 \times 2} U_\Theta J_2} \cap \overline{\mathcal{U}_2^{2 \times 2} U_\Theta^\perp} = \{0\}. \quad (7.21)$$

PROOF: Consider (7.20) from Proposition 7.22 and recall that the subspaces $\overline{\mathcal{U}_2^{2 \times 2} U_\Theta J_2}$ and $\overline{\mathcal{U}_2^{2 \times 2} U_\Theta^\perp}$ are J-orthogonal which readily shows by applying Definition 7.10 and the identity $U_\Theta^\perp U_\Theta^* = 0$. Let us show that (7.21) holds true. First recall that, because both U_Θ and U_Θ^\perp have closed⁹ ranges, it holds that $u_1 = 0$ if and only if $u_2 = 0$ in $u_1 U_\Theta J_2 = u_2 U_\Theta^\perp$.

We complete the proof by applying the contradiction. Assume that $u_i \in \mathcal{U}_2^{2 \times 2}$, $i = 1, 2$ where $u_1 \neq 0$ can be found such that

⁹Minimal realizations of both are u.e.s

$$u_1 U_\Theta J_2 = u_2 U_\Theta^\perp.$$

Postmultiply this equation with U_Θ^* . Because $U_\Theta^\perp U_\Theta^* = 0$, we have that

$$u_1 U_\Theta J_2 U_\Theta^* = 0. \quad (7.22)$$

But $U_\Theta J_2 U_\Theta^*$ is an invertible operator. This fact follows from the decomposition $\Theta = T_{o,\Theta} U_\Theta$, the invertibility of J_1 and $T_{o,\Theta}$ and the indefinite spectral factorization

$$T_{o,\Theta}^{-1} J_1 T_{o,\Theta}^{-*} = U_\Theta J_2 U_\Theta^*.$$

Then (7.22) implies that $u_1 = 0$ and, hence, yields the contradiction and completes the proof that

$$\overline{\mathcal{U}_2^{2 \times 2} U_\Theta J_2} \cap \overline{\mathcal{U}_2^{2 \times 2} U_\Theta^\perp} = \{0\}.$$

This completes the proof. ■

The following proposition proves in a similar way as Proposition 7.21 (see also [DvdV98]).

Proposition 7.24. *Let $U \in \mathcal{U}(\mathcal{M}, \mathcal{N})$, be isometric. Then,*

$$\mathcal{K}_o(U) = \overline{\mathcal{U}_2 \bar{U}} \oplus \ker(\cdot U^* |_{\mathcal{U}_2}). \quad (7.23)$$

Corollary 7.25. *For the causal isometry U from Proposition 7.24 and an invertible signature J_2 of appropriate s -dim sequence, it holds that*

$$\mathcal{K}_o(U) = \overline{\mathcal{U}_2 \bar{U}} \boxplus \{\ker(\cdot U^* |_{\mathcal{U}_2}) J_2\}. \quad (7.24)$$

In particular, if there is another causal isometry U^\perp with a u.e.s realization and such that $\begin{bmatrix} U_\Theta \\ U_\Theta^\perp \end{bmatrix}$ is unitary, then

$$\mathcal{K}_o(U) = \overline{\mathcal{U}_2 \bar{U}} \boxplus \mathcal{U}_2 U^\perp J_2. \quad (7.25)$$

PROOF: The proof follows after recalling that $\mathbf{P}_0(xy^*) = 0$ where $x \in \overline{\mathcal{U}_2 \bar{U}}$ and $y \in \ker(\cdot U^* |_{\mathcal{U}_2})$ can equivalently¹⁰ be written as $\mathbf{P}_0(x J_2 (y J_2)^*) = 0$. The latter condition implies that

$$\overline{\mathcal{U}_2 \bar{U}} [\perp]_{J_2} \{\ker(\cdot U^* |_{\mathcal{U}_2}) J_2\}.$$

The proof for the additional statement follows immediately after recalling the equation (7.17) from Lemma 7.20. This completes the proof. ■

¹⁰When J_2 is invertible.

As we saw, outer/inner characterization of causal J-isometric Θ appears to be very useful in characterization of the subspace $\mathcal{K}(\Theta)$. Evaluation of $\mathcal{H}_o(\Theta)$ then can be done by using the identity $\mathcal{H}_o(\Theta) = \mathcal{U}_2^{2 \times 2} \ominus \mathcal{K}_o(\Theta)$. We take different course and evaluate \mathcal{H}_o by starting from the definition of this space and a minimal external factorization for Θ . In fact, it easily shows that, the outer $T_{o,\Theta}$ and the causal isometry U_Θ appear in such an external factorization.

Viz. the decomposition $\Theta = T_{o,\Theta}U_\Theta$ and the fact that Θ is boundedly right invertible - a bounded anti-causal inverse of it is equal to $J_2\Theta^*J_1$ - we find from

$$J_2\Theta^*J_1 = U_\Theta^*T_{o,\Theta}$$

that

$$\Theta = J_1T_{o,\Theta}^*U_\Theta J_2. \tag{7.26}$$

By using the definition for $\mathcal{H}_o(\Theta)$ as given in the beginning of this section and the characterization (7.26) we evaluate that

$$\begin{aligned} \mathcal{H}_o(\Theta) &= \mathbf{P}(\mathcal{L}_2^{2 \times 2}Z^{-1}\Theta) \\ &= \mathbf{P}(\mathcal{L}_2^{2 \times 2}Z^{-1}J_1T_{o,\Theta}^*U_\Theta J_2) \\ &= \mathbf{P}(\mathcal{L}_2^{2 \times 2}Z^{-1}U_\Theta)J_2 \\ &= \mathcal{H}_o(U_\Theta)J_2. \end{aligned}$$

where in the second line we used the fact that $\overline{T_{o,\Theta}J_1U_2} = U_2$ and, in the third one, that $J_2 \in \mathcal{D}$. We conclude that $\text{s-dim}(U_\Theta) = \text{s-dim}(\Theta)$. Hence, the factorization (7.26) is indeed a minimal external factorization the external factor of which; $J_1T_{o,\Theta}^*$, is anti-outer.

Unlike the outer/inner factorization for an invertible operator T which, because of DZ-invariantness of \mathcal{K}_o and absence of *defect spaces* [DvdV98] related to the inner factor, is always guaranteed to exist, similar factorization outer/causal-J-unitary does not necessarily exist. The formal definition for this factorization follows.

Definition 7.26. *Let $T \in \mathcal{X}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ be a chain scattering operator. Let J_1 and J_2 be invertible scattering operators associated to its input and the output port. If there exist an intermediate space \mathcal{M}' with invertible signature J_x , a left outer G_{ol} (i.e. $\overline{\mathcal{U}_2^{2 \times 2}G_{ol}} = \mathcal{U}_2^{2 \times 2}$) and a (J_x, J_2) -isometric Θ such that*

$$T = G_{ol}\Theta$$

then we say that T admits a left-outer/causal-J-isometric factorization.

In particular, if T from Definition 7.26 is invertible and the left-outer/causal-J-isometric factorization exists, then G_{ol} is both left- and right outer and Θ is causal (J_x, J_2) -unitary. We than say that T admits an *outer/causal-J-unitary* factorization.

To find necessary and sufficient conditions for the existence of an outer/causal-J-unitary factorization, let us recall the properties of the terms involved in the factorization for a bounded $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$. In [DvdV98] it is shown that

$$T = T_{ol}U$$

is a left-outer/isometric representation for T if and only if the isometric U is such that $\overline{\mathcal{U}_2 T} = \mathcal{U}_2 U$.

Let us adjoin an invertible signature J_2 to the output port of T . In [Xi96] it is shown that a left-outer/causal-J-isometric representation

$$T = G_{ol}\Theta$$

exists if and only if the causal-J-isometric Θ is such that $\overline{\mathcal{U}_2 T J_2} = \mathcal{U}_2 \Theta J_2$.

In particular, if T is causal isometric a minimal realization of which is u.e.s, then the proof for the following proposition is obvious.

Proposition 7.27. *Let $U \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be isometric having a u.e.s realization and let J_2 be an invertible signature of appropriate s -dim sequence. Then, there are: an invertible signature J_x , an outer (both left- and right) $T_{o,\Theta}$ and a causal (J_x, J_2) -isometric Θ having a u.e.s J -isometric realization such that*

$$U = T_{o,\Theta}\Theta \tag{7.27}$$

is the outer/causal-J-isometric factorization for U if and only if the J -Gramian corresponding to the standard basis of the space $\mathcal{U}_2 U J_2$ is boundedly invertible. In particular, it then holds that

$$\overline{\mathcal{U}_2 U J_2} = \mathcal{U}_2 \Theta J_2. \tag{7.28}$$

The following results are straightforward consequences of Proposition 7.27.

Corollary 7.28. *Let U and J_2 be as in Proposition 7.27. Let a minimal realization of U be u.e.s.*

Then, there are: an invertible signature J_x , an outer (both left- and right) $T_{o,\Theta}$ and a causal (J_x, J_2) -isometric Θ such that the factorization (7.27) holds if and only if Θ is such that

$$\mathcal{K}_o(U)J_2 \boxplus \mathcal{U}_2 U^\perp J_2 = \mathcal{K}_o(\Theta) \boxplus \ker(\cdot\Theta^*|_{\mathcal{U}_2}) \tag{7.29}$$

where U^\perp is a causal isometry for which $\begin{bmatrix} U \\ U^\perp \end{bmatrix}$ is inner.

PROOF: The proof completes after bringing together the equations (7.18) and (7.25) from respectively Proposition 7.21 and Proposition 7.25 by virtue of the property (7.28) from Proposition 7.27. ■

Lemma 7.29. *Let U be causal isometric having a u.e.s realization. Let J_2 and J_x be invertible signature and Θ a causal bounded operator such that $\Theta J_2 \Theta^* = J_x$. Then*

$$\mathcal{K}_o(U)J_2 = \mathcal{K}_o(\Theta) \Leftrightarrow \mathcal{H}_o(U)J_2 = \mathcal{H}_o(\Theta). \tag{7.30}$$

PROOF: (\Rightarrow) The operator-space identities for causal bounded operators as introduced in Chapter 2 imply that $\mathcal{K}_o(U) \oplus \mathcal{H}_o(U) = \mathcal{U}_2$ and $\mathcal{K}_o(\Theta) \oplus \mathcal{H}_o(\Theta) = \mathcal{U}_2$. Then, we have from the assumption $\mathcal{K}_o(U)J_2 = \mathcal{K}_o(\Theta)$ that

$$\begin{aligned} \mathcal{H}_o(U)J_2 &= (\mathcal{U}_2^{2 \times 2} \ominus \mathcal{K}_o(U))J_2 \\ &= \mathcal{U}_2^{2 \times 2}J_2 \ominus (\mathcal{K}_o(U)J_2) \\ &= \mathcal{U}_2^{2 \times 2} \ominus \mathcal{K}_o(\Theta) \\ &=: \mathcal{H}_o(\Theta) \end{aligned}$$

where, in the second line we used the fact that J_2 is unitary and in the third one the fact that J_2 is outer i.e. $\mathcal{U}_2^{2 \times 2}J_2 \ominus \mathcal{U}_2^{2 \times 2} = \{0\}$. Hence,

$$\mathcal{H}_o(U)J_2 = \mathcal{H}_o(\Theta)$$

which completes one direction of the proof.

(\Leftarrow) The proof that $\mathcal{H}_o(U)J_2 = \mathcal{H}_o(\Theta)$ implies $\mathcal{K}_o(U)J_2 = \mathcal{K}_o(\Theta)$ goes in similar vein. This completes the proof. ■

Corollary 7.30. *Let U from Proposition 7.27 be inner. Then, U admits the outer/causal- J -unitary factorization (7.27) if and only if*

$$\mathcal{K}_o(U)J_2 = \mathcal{K}_o(\Theta). \tag{7.31}$$

PROOF: (\Rightarrow) If U is inner and the factorization (7.28) holds then Θ is invertible. Recall Propositions 7.21 and 7.24. Because both U and Θ are invertible the kernels from the right sides of (7.18) and (7.23) contain the zero element only. Comparison of (7.18) and (7.23) then yields the equation (7.31).

(\Leftarrow) If a causal (J_x, J_2) -unitary Θ can be found such that (7.31) holds for an inner U then the proof completes if we succeed to show that $U\Theta^{-1}$ is outer. To do this we have to show that both $U\Theta^{-1}$ and its inverse ΘU^* are causal. By applying J -external factorization, it is not hard to see that

$$\mathbf{P}'(UJ_2\Theta^*J_x) = 0 \text{ if } \Theta \text{ is such that } \mathcal{H}_o(U)J_2 \supseteq \mathcal{H}_o(\Theta) \tag{7.32}$$

$$\mathbf{P}'(\Theta U^*) = 0 \text{ if } \Theta \text{ is such that } \mathcal{H}_o(\Theta) \supseteq \mathcal{H}_o(U)J_2. \tag{7.33}$$

But Lemma 7.29 claims that (7.31) is equivalent to $\mathcal{H}_o(\Theta) = \mathcal{H}_o(U)J_2$. Hence (7.32) and (7.33) are both fulfilled and $T_{o,\Theta} := U\Theta^{-1}$ is indeed the outer factor needed for the factorization (7.27) to exist. This completes the proof. ■

An alternative and more direct proof to the statement from Corollary 7.30 can be obtained after considering (7.29) from Corollary 7.28. In the case U is inner, the s -dim sequence of U^\perp vanishes and implies that the subspace $\mathcal{U}_2 U^\perp J_2$ consists of the zero element only. The invertibility for U implies the invertibility for Θ if the factorization $U = T_{o,\Theta} \Theta$ exists. Hence, $\ker(\cdot \Theta^* |_{\mathcal{U}_2}) = \{0\}$ if the factorization exists. This will be the topic of the subsequent section.

Finally, we have the following consequence of Proposition 7.27.

Theorem 7.31. *Let U from Proposition 7.27 be inner. Then U admits the outer/causal- J -unitary factorization (7.27) if and only if*

$$\mathcal{H}_o(U)J_2 = \mathcal{H}_o(\Theta). \quad (7.34)$$

PROOF: By Corollary 7.30 we have that the factorization (7.27) exists if and only if (7.31) holds true. Since by Lemma 7.29 the identity (7.31) holds true if and only if (7.34) holds true, the proof is complete. ■

Recall that our goal was to find necessary and sufficient conditions for an indefinite factorization for a causal boundedly invertible T ; $T = G_o \Theta$. Since T admits outer/inner factorization $T = T_o U_\Theta$ if and only if $\mathcal{H}_o(T) = \mathcal{H}_o(U_\Theta)$ and U_Θ , in turn, admits outer/causal- (J_x, J_2) -unitary factorization $U_\Theta = T_{o,\Theta} \Theta$ if and only if Θ is such that $\mathcal{H}_o(\Theta) = \mathcal{H}_o(U_\Theta)J_2$, it is then straightforward to bring the things together and formulate the conditions for T to admit an indefinite factorization. All what we need to do then is to avoid the formulation that involve the terms from Θ . In the following section we show how to do this.

7.5 Outer/inner factorization for a causal J -unitary operator

In this section we assume that the factor Θ as mentioned in the previous section, is causal- J -unitary and study the relationship between Θ and the factors from its outer/inner factorization.

Let henceforth $\Theta \in \mathcal{U}^{2 \times 2}$ be (J_x, J_2) -unitary for some invertible signature J_x, J_2 and let $[\Theta] = \{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ be a u.e.s, minimal, J -unitary realization for it. Since invertible, there will be an outer $T_{o,\Theta}$ and an inner U for which $\Theta = T_{o,\Theta} U$. U.e.s realizations for these factors: $[T_{o,\Theta}] = \{A_\Theta, B_\Theta, C_{o,\Theta}, D_{o,\Theta}\}$ and unitary $[U] = \{A_U, B_U, C_U, D_U\}$ are then obtained from the RQ-decomposition

$$\begin{bmatrix} A_\Theta Y_\Theta^{(-1)} & C_\Theta \\ B_\Theta Y_\Theta^{(-1)} & D_\Theta \end{bmatrix} = \begin{bmatrix} Y_\Theta & C_{o,\Theta} \\ & D_{o,\Theta} \end{bmatrix} \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix} \quad (7.35)$$

for some left invertible $Y_\Theta \in \mathcal{D}$ of maximal column rank. The interested reader is addressed to [DvdV98] for the relationship between outer/inner factorization and RQ-decomposition such as (7.35).

By virtue of Corollary 7.31 we know the relationship between spaces $\mathcal{H}_o(U)$ and $\mathcal{H}_o(\Theta)$. From practical purposes as it already discussed in the previous section, we are frequently interested in an algebraic counterpart of this relationship. Let us first rewrite the previous equation equivalently as

$$[\Theta] \begin{bmatrix} Y_{\Theta}^{(-1)} & \\ & I \end{bmatrix} [U]^* = \begin{bmatrix} Y_{\Theta} C_{o,\Theta} \\ D_{o,\Theta} \end{bmatrix}. \tag{7.36}$$

Then, we premultiply this equation first with $\begin{bmatrix} J_B & \\ & J_x \end{bmatrix}$ in which J_B is an appropriate state-space signature and then with its transpose. After the identity

$$[\Theta]^* \begin{bmatrix} J_B & \\ & J_x \end{bmatrix} [\Theta] = \begin{bmatrix} J_B^{(-1)} & \\ & J_2 \end{bmatrix}$$

is employed into the result, we obtain

$$[U] \begin{bmatrix} (Y_{\Theta}^* J_B Y_{\Theta})^{(-1)} & \\ & J_2 \end{bmatrix} [U]^* = \begin{bmatrix} Y_{\Theta}^* J_B Y_{\Theta} & Y_{\Theta}^* J_B C_{o,\Theta} \\ C_{o,\Theta}^* J_B Y_{\Theta} & C_{o,\Theta}^* (Y_{\Theta}^* J_B Y_{\Theta}) C_{o,\Theta} + D_{o,\Theta}^* J_1 D_{o,\Theta} \end{bmatrix}. \tag{7.37}$$

From (7.37) we extract the (1-1)-term and obtain

$$A_U (Y_{\Theta}^* J_B Y_{\Theta})^{(-1)} A_U^* + C_U J_2 C_U^* = Y_{\Theta}^* J_B Y_{\Theta}.$$

As $\ell_{A_U} < 1$, the Hermitian

$$\Lambda_o^J := Y_{\Theta}^* J_B Y_{\Theta}$$

is the unique bounded solution for the previous Lyapunov-Stein equation.

When no right intrinsic J-unitary terms are in Θ - those are causal J-unitaries which are inner in the same time - we have that $s\text{-dim}(\Theta) = s\text{-dim}(T_{o,\Theta})$. In this case Y_{Θ} is locally square invertible. The Hermitian $Y_{\Theta}^* J_B Y_{\Theta}$ is then invertible if and only if the signature J_B is so. By virtue of Corollary 7.31, we conclude then that Y_{Θ} is the state-space transformation which relates $\mathcal{H}_o(U)$ and $\mathcal{H}_o(\Theta)$ giving rise to the following corollary.

Corollary 7.32. *Let $[\Theta] = \{A_{\Theta}, B_{\Theta}, C_{\Theta}, D_{\Theta}\}$ be minimal, u.e.s, J-unitary realization for causal (J_x, J_2) -unitary $\Theta \in \mathcal{U}^{2 \times 2}(\mathcal{M}, \mathcal{N})$ and having no right intrinsic causal J-unitary terms. Let $\Theta = T_{o,\Theta} U$ be an outer/inner factorization for some outer $T_{o,\Theta}$ and inner U minimal, u.e.s realizations of which are respectively $[T_{o,\Theta}] = \{A_{\Theta}, B_{\Theta}, C_{o,\Theta}, D_{o,\Theta}\}$ and unitary $[U] = \{A_U, B_U, C_U, D_U\}$. Let $\Lambda_o^J \in \mathcal{D}$ be the unique bounded solution for the Lyapunov-Stein equation*

$$A_U \Lambda_o^J A_U^* + C_U J_2 C_U^* = \Lambda_o^J \tag{7.38}$$

Let, in addition, a signature J_B , and a locally square invertible $Y_{\Theta} \in \mathcal{D}$ be such that $\Lambda_o^J = Y_{\Theta}^ J_B Y_{\Theta}$ is an eigenvalue decomposition of Λ_o^J .*

Then,

(1) J_B is the state signature corresponding to the realization $[\Theta]$,

(2) the realization $[T_{o,\Theta}]$ is equal to

$$[T_{o,\Theta}] = \left[\begin{array}{c|c} A_\Theta & A_\Theta Y_\Theta^{(-1)} B_U^* + C_\Theta D_U^* \\ \hline B_\Theta & B_\Theta Y_\Theta^{(-1)} B_U^* + D_\Theta D_U^* \end{array} \right],$$

(3) Y_Θ and the realizations $[T_{o,\Theta}]$ and $[U]$ give rise to an RQ-decomposition scheme of the form (7.35).

To this end we show that if the signature J_B is nonsingular then the outer factor $T_{o,\Theta}$ obtained from (7.35) can be expressed in the terms from U explicitly. We simply start from the opposite problem to the previous one. That is, we start from the algebraic test that tells us that, for a given signature J_2 and an inner U , there are another nonsingular signature J_x and a causal (J_x, J_2) -unitary Θ for which

$$U = T_{o,\Theta}^{-1} \Theta$$

for some boundedly invertible outer $T_{o,\Theta}$. Again, necessary and sufficient conditions for the factorization are already known, i.e. it must hold that $\mathcal{H}_o(U)J_2 = \mathcal{H}_o(\Theta)$. Given a minimal realization $[U]$ for U , a minimal realization $[T_{o,\Theta}]$ for $T_{o,\Theta}$ entirely expresses in the terms from $[U]$ after invoking the equation (7.37). This equation is, in fact, a state-space equivalent to the indefinite factorization

$$U J_2 U^* = T_{o,\Theta}^{-1} J_x T_{o,\Theta}^{-*} \quad (7.39)$$

for the inner U . Since U is unitary, we also have that $U J_2 U^* = (U J_2 U^*)^{-1}$. Hence,

$$U J_2 U^* = T_{o,\Theta}^* J_x T_{o,\Theta}. \quad (7.40)$$

and is equivalent to (7.37) as well. The outer $T_{o,\Theta}$ is then a special operator in the sense that

$$(T_{o,\Theta} T_{o,\Theta}^*) J_x (T_{o,\Theta} T_{o,\Theta}^*) = J_x.$$

But it is obvious that such an indefinite factorization exists for some $T_{o,\Theta}$ and J_x if and only if there is a (J_x, J_2) -unitary Θ that relates U and $T_{o,\Theta}$ from (7.40). Recall that J_B is assumed nonsingular. To evaluate $T_{o,\Theta}$ in terms from $[U]$ we start from (7.37) and evaluate $C_{o,\Theta}$ as

$$C_{o,\Theta} = J_B Y_\Theta^{-*} (A_U \Lambda_o^{J(-1)} B_U^* + C_U J_2 D_U^*) \quad (7.41)$$

On the other side, $D_{o,\Theta}$ is defined to be the invertible factor in an eigenvalue decomposition for the right side of

$$D_{o,\Theta}^* J_x D_{o,\Theta} = B_U \Lambda_o^{J(-1)} B_U^* + D_U J_2 D_U^* - C_{o,\Theta}^* \Lambda_o^J C_{o,\Theta} \quad (7.42)$$

for some invertible signature J_x . Note that the invertibility of J_x is guaranteed by the invertibility of J_B, J_2 and J-orthogonalization procedure performed on the appropriately renormalized AC-pair (A_U, C_U) . As a consequence, $U\Theta^{-1}$ and its inverse ΘU^* are both bounded. A simple inspection and using external factorization argument in which $s\text{-dim}(U) = s\text{-dim}(\Theta)$ show that the latter operators are both causal. Then, $D_{o,\Theta} := \mathbf{P}_0(U\Theta^{-1})$ is locally square invertible meaning that the invertibility for $D_{o,\Theta}$ defined by (7.42) is automatically guaranteed. The invertible $Y_\Theta \in \mathcal{D}$ and the invertible J_B mentioned in (7.41) and (7.42) constitute the eigenvalue decomposition $A_o^J = Y_\Theta^* J_B Y_\Theta$ for the unique bounded solution A_o^J for (7.38) as mentioned in the statement of Corollary 7.32. To determine the block diagonals 'A' and 'B' for $[T_{o,\Theta}]$ recall the realization $[T_{o,\Theta}]$ from the additional statement of the corollary and recall that

$$\begin{bmatrix} A_\Theta \\ B_\Theta \end{bmatrix} = \begin{bmatrix} Y_\Theta & C_{o,\Theta} \\ & D_{o,\Theta} \end{bmatrix} \begin{bmatrix} A_U \\ B_U \end{bmatrix} Y_\Theta^{-(-1)}. \tag{7.43}$$

Hence,

$$[T_{o,\Theta}] = \left[\begin{array}{c|c} (Y_\Theta A_U + C_{o,\Theta} B_U) Y_\Theta^{-(-1)} & C_{o,\Theta} \\ \hline D_{o,\Theta} B_U Y_\Theta^{-(-1)} & D_{o,\Theta} \end{array} \right] \tag{7.44}$$

with $C_{o,\Theta}$ and $D_{o,\Theta}$ as defined by (7.41) and (7.42). The latter realization is the same as that stated in Corollary 7.32. The only difference is that it is expressed in the data from $[U]$ rather than those from $[\Theta]$.

Finally, construction of a minimal realization for Θ can be done after completion of the $\left(\left[\begin{array}{c} (A_o^J)^{-1} \\ J_x \end{array} \right], J_2 \right)$ -coisometric column block given by (7.43) by first normalizing it by the state-space transformation Y_Θ and then applying J-orthogonalization procedure on it. This is nothing else then a state-space equivalent to the analytic formula $\Theta = T_{o,\Theta}^{-1} U$. Evaluation of Θ viz. the latter formula is possible as well but should be avoided since it introduces the product of two causal terms.

The algorithm stated in Figure 7.2 represents an one-pass procedure to find an indefinite factorization $U = T_{o,\Theta} \Theta$ for a given inner U to whose output space is an invertible signature J_2 associated and that contains no inner factors of the form $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ where the partitioning¹¹ in W is induced by the partitioning in J_2 .

We are now ready to state an one-pass algorithm that performs a check on if indefinite factorization for a left invertible causal operator T exists to whose output is an invertible signature J_2 associated and, similar as before, is assumed such that it contains no inner factors of the form $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$. We first make use of an

¹¹ $W_i, i = 1, 2$ are both inner.

1. **input:** Initial integer k_{init} ,
sequence of invertible signatures $J_{2,k}, k = k_{init}, k_{init} - 1, \dots$,
sequence of minimal realizations for inner U : $[U]_k = \{A_{U,k}, B_{U,k}, C_{U,k}, D_{U,k}\}, k = k_{init}, k_{init} - 1, \dots$,
initial invertible Hermitian $\Lambda_{o, k_{init}+1}^J, k = k_{init}, k_{init} - 1, \dots$
output: Sequence of minimal realizations $[T_{o,\theta}]_k, k = k_{init}, k_{init} - 1, \dots$ for outer $T_{o,\theta}$,
sequence of invertible signatures $J_{x,k}, k = k_{init}, k_{init} - 1, \dots$ and
sequence of minimal realizations $[\Theta]_k, k = k_{init}, k_{init} - 1, \dots$ for a causal (J_x, J_2) -unitary Θ for which $U = T_{o,\theta}\Theta$.

For $k = k_{init}, \dots, -\infty$ do

2. Find invertible $Y_{\theta,k}$ and signature $J_{B,k}$ for which $\Lambda_{o,k}^J = Y_{\theta,k}^* J_{B,k} Y_{\theta,k}$ is an eigenvalue decomposition.
3. Update $\Lambda_{o,k}^J$ via (7.38) as $\Lambda_{o,k}^J := A_{U,k} \Lambda_{o,k+1}^J A_{U,k}^* + C_{U,k} J_{2,k} C_{U,k}^*$ and find an eigenvalue decomposition for the updated $\Lambda_{o,k}^J$.
4. If $\Lambda_{o,k}^J$ is singular then break and display 'There is no indefinite factorization for U '.
5. Define $(C_{o,\theta})_k$ as given by (7.41):

$$(C_{o,\theta})_k := J_{B,k} Y_{\theta,k}^{-*} (A_{U,k} \Lambda_{o,k+1}^J B_{U,k}^* + C_{U,k} J_{2,k} D_{U,k}^*).$$

6. Define Hermitian H_k^J to be the right side of (7.42):

$$H_k^J := B_U \Lambda_o^{J(-1)} B_U^* + D_U J_2 D_U^* - C_{o,\theta}^* \Lambda_o^J C_{o,\theta}.$$

7. Find invertible $(D_{o,\theta})_k$ and signature $J_{x,k}$ for which

$$(D_{o,\theta})_k^* J_{x,k} (D_{o,\theta})_k = H_k^J$$

is an eigenvalue decomposition for H_k^J .

8. Define $[T_{o,\theta}]_k$ viz. (7.44) as

$$[T_{o,\theta}]_k = \left[\begin{array}{c} (Y_{\theta,k} A_{U,k} + (C_{o,\theta})_k B_{U,k}) Y_{\theta,k+1}^{-1} \\ (D_{o,\theta})_k B_{U,k} Y_{\theta,k+1}^{-1} \end{array} \middle| \begin{array}{c} (C_{o,\theta})_k \\ (D_{o,\theta})_k \end{array} \right].$$

9. Define $A_{\theta,k}$ and $B_{\theta,k}$ viz. (7.43) as

$$\begin{bmatrix} A_{\theta,k} \\ B_{\theta,k} \end{bmatrix} = \begin{bmatrix} Y_{\theta,k} & (C_{o,\theta})_k \\ & (D_{o,\theta})_k \end{bmatrix} \begin{bmatrix} A_{U,k} \\ B_{U,k} \end{bmatrix} Y_{\theta,k+1}^{-1}.$$

10. Find $C_{\theta,k}$ and $D_{\theta,k}$ for which $[\Theta]_k := \begin{bmatrix} A_{\theta,k} & C_{\theta,k} \\ B_{\theta,k} & D_{\theta,k} \end{bmatrix}$ is

a $\left(\begin{bmatrix} \Lambda_{o,k+1}^J & \\ & J_{2,k} \end{bmatrix}, \begin{bmatrix} \Lambda_{o,k}^J & \\ & J_{x,k} \end{bmatrix} \right)$ -unitary realization.

11. End.

Fig. 7.2. Algorithm for obtaining an indefinite factorization $U = T_{o,\theta}\Theta$ for an inner U to whose output is an indefinite invertible signature J_2 associated.

outer/inner factorization for T which, in the case T is left invertible, produces a left-outer factor T_{ol} and an inner U such that

$$T = T_{ol}U.$$

The remaining construction of the indefinite factorization depends on the indefinite factorizability of U as specified by the algorithm stated in Figure 7.2. If U admits the indefinite factorization with the factors as introduced in the algorithm then $U = T_{o,\theta}\Theta$. Hence,

$$T = G_{ol}\Theta \tag{7.45}$$

is the sought indefinite factorization where G_{ol} is left-outer and defined as $G_{ol} := T_{ol}T_{o,\theta}$.

An one-pass algorithm for finding realizations for factors on the right side of (7.45) is stated in Figure 7.3.

As defined locally Pontryagin spaces are spaces whose parts of the inner products that carry on 'negative' energy is finite dimensional. Krein spaces are restrictions of Pontryagin spaces in the sense that they allow a Pontryagin space to be represented by a J-direct sum of two Hilbert spaces. Though operators defined on Pontryagin spaces are extensively investigated in interpolation theory, Krein space operators have found their applications in lossless cascade filter design and in, recently introduced [Kim97], approach to H_∞ -control via outer/J-inner factorization. To recall, a J-inner operator is a bounded causal J-unitary operator minimal u.e.s realization of which has strictly positive definite Gramians¹². A J-anti-inner operator is defined similarly as a bounded causal J-unitary operator minimal, u.e.s realization of which has strictly negative definite Gramians.

If $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ is a properly partitioned causal (J_1, J_2) -inner operator then it is easy to derive that Θ_{ii} , $i = 1, 2$ are both expansive. Hence, their inverses exist and are, in turn, contractive operators. Moreover, $\Sigma := \begin{bmatrix} \Theta_{11}^{-*} & -\Theta_{12}\Theta_{22}^{-1} \\ \Theta_{21}\Theta_{11}^{-*} & \Theta_{22}^{-1} \end{bmatrix}$ is inner.

The test if a causal boundedly left invertible T allows for an indefinite factorization $T = G_{ol}\Theta$ where Θ is (J_x, J_2) -inner and G_{ol} is left outer can be conducted by the algorithm stated in Figure 7.3 if the latter is adjusted to take only strictly positive definite state-space signature J_B into the consideration.

7.6 Conclusions

In this chapter, causal bounded operators acting between locally Pontryagin spaces are studied by means of their state-space representations. In particular, we started the

¹²The Gramians are automatically the mutual inverses of each other

1. **input:** Initial integer k_{init} ,
sequence of invertible signatures $J_{2,k}, k = k_{init}, k_{init} - 1, \dots$,
sequence of minimal realizations for causal left invertible $T: [T]_k = \{A_k, B_k, C_k, D_k\}, k = k_{init}, k_{init} - 1, \dots$,
initial left invertible Y_k having maximal column rank,
invertible $\Lambda_{o, k_{init}+1}^J, k = k_{init}, k_{init} - 1, \dots$
output: Sequence of reachable but not necessarily observable realizations $[G_{ol}]_{n.n.o,k}, k = k_{init}, k_{init} - 1, \dots$ for a left outer G_{ol} ,
sequence of invertible signatures $J_{x,k}, k = k_{init}, k_{init} - 1, \dots$ and
sequence of minimal realizations $[\Theta]_k, k = k_{init}, k_{init} - 1, \dots$ for a causal (J_x, J_2) -unitary Θ for which $T = G_{ol}\Theta$.

For $k = k_{init}, \dots, -\infty$ do

2. Apply RQ-decomposition to the left side of

$$\left[\begin{array}{c|c} A_k Y_{k+1} & C_k \\ \hline B_k Y_{k+1} & D_k \end{array} \right] = \left[\begin{array}{c|c} Y_k & (C_{ol})_k \\ \hline & (D_{ol})_k \end{array} \right] \left[\begin{array}{c|c} A_{U,k} & C_{U,k} \\ \hline B_{U,k} & D_{U,k} \end{array} \right]$$

to find u.e.s realizations: unitary $[U]_k = \{A_{U,k}, B_{U,k}, C_{U,k}, D_{U,k}\}$ and not necessarily observable $[T_{ol}]_k = \{A_k, B_k, (C_{ol})_k, (D_{ol})_k\}$ for inner U and left outer T_{ol} for which $T = T_{ol}U$.

3. Apply the algorithm stated in Figure 7.2 on the realization $[U]_k$.
4. Break if the output is "There is no indefinite factorization for U " and display "There is no indefinite factorization for T ".
5. Display $[T_{o,\Theta}]_k = \{(A_{T_{o,\Theta}})_k, (B_{T_{o,\Theta}})_k, (C_{T_{o,\Theta}})_k, (D_{T_{o,\Theta}})_k\}$, $[\Theta]_k = \{A_{\Theta,k}, B_{\Theta,k}, C_{\Theta,k}, D_{\Theta,k}\}$ and $J_{x,k}$ as produced by the algorithm stated in Figure 7.2.
6. Define the reachable but possibly unobservable realization for the left outer G_{ol} as

$$[G_{ol}]_{n.n.o,k} := \left[\begin{array}{c|c} A_k & (C_{ol})_k \\ \hline I & \\ B_k & (D_{ol})_k \end{array} \right] \left[\begin{array}{c|c} I & \\ \hline (A_{T_{o,\Theta}})_k & (C_{T_{o,\Theta}})_k \\ (B_{T_{o,\Theta}})_k & (D_{T_{o,\Theta}})_k \end{array} \right].$$

7. End.

Fig. 7.3. Algorithm for obtaining an indefinite factorization $T = G_{ol}\Theta$ for a left invertible bounded causal T to whose output is an indefinite invertible signature J_2 associated.

study by investigating first a special class of such operators called causal J-unitary operators. All for indefinite factorization relevant properties of these operators are extracted by means of a standard outer/inner factorization setting. As a consequence, we have expressed necessary and sufficient condition for the indefinite factorization of an inner operator U through an algebraic test on invertibility of the J-Gramian defined in terms of bases for the observability space of U . Equivalently, a necessary condition states that the indefinite factorization $U = T_{o,\Theta}\Theta$ for which $s\text{-dim}(U) = s\text{-dim}(\Theta)$ exists if and only if Θ is such that $\mathcal{H}_o(U)J_2 = \mathcal{H}_o(\Theta)$. By considering this condition

in state-space terms, it was possible to eliminate Θ . This, in turn, gave a rise to the one-pass algorithm as stated in Figure 7.2. Since an arbitrary causal boundedly left-invertible operator T admits a left-outer/inner factorization, the conditions for the existence of an indefinite factorization for such an operator translate to the inner factor itself. This made possible to state a backward stable algorithm for the indefinite factorization of T as given in Figure 7.3. In both cases Figure 7.2 and Figure 7.3 it were assumed that U from the left-outer/inner factorization for T does not contain intrinsic factors at its right side that are J-unitary in the same time. For simplicity reasons this kind of factors should be found and extracted first in the way we stated at the end of the previous chapter in Figure 6.1.



8

CONCLUSIONS

8.1 Main results

In this thesis, algebraic properties of minimal and non-minimal realizations of locally finite mixed causality operators have been investigated for purpose of inversion, minimal factorizations and indefinite factorizations. Operators at hand were specified by sequences of additive realizations whereby the overall realization is presumed to have u.e.s property.

As is widely known, operator inversion problems frequently arise in both theory and practice. Therefore in Chapter 3 we have payed a special attention to the evaluation of inverse and its underlying algorithmic. In obtaining closed formulas for the inverse we were, in particular, interested in the one-pass character of expressions because this greatly limits the numerical complexity. In absence of intrinsic inner factors, a minimal realization for the inverse is derived in an one-pass fashion. After that we have also shown that the inverse can be stated in a non-one-pass fashion. In the latter case non-intrinsic inner factors were taken into calculations.

Another approach to evaluate the inverse was conducted in Chapter 4 where, contrary to the inner/outer approach taken in Chapter 3, we repeated the analysis by adopting another representation of the operator at hand namely maximal-phase/inner. Unlike an outer causal factor whose inverse is again causal, a maximal-phase operator is causal with an anti-causal inverse. This property, as shown in Chap-

ter 4, introduces nice algebraic properties of a minimal realization of such an operator. In this way, maximal-phase operators become interesting operators of their own.

In characterization of a mixed causality operators, isometric operators play an important role. In Chapter 5 we studied algebraic properties of minimal u.e.s additive realizations for isometric operators with the purpose of finding minimal factorization and unitary extension. In the obtained calculations solutions of Sylvester equations played a key role. Some of unitary operators give rise to so-called J-unitary operators. As the latter are naturally given in a partitioned form, this partitioning induces the partitioning in unitary operators and their realizations. This was the reason for considering the algebraic properties of a partitioned additive realization for a unitary operator. As a consequence an important relationship between solutions of Sylvester equations and Lyapunov-Stein equations arising from the partitioning is stated.

In Chapter 6 we have derived the conditions for minimal factorizability of a mixed causality operator. The maximal-phase operator approach introduced in Chapter 4 appeared to be fruitful in characterization of the factors. In particular, we focused on J-unitary operators of mixed causality and derived necessary and sufficient conditions for them to admit a minimal causal/anti-causal factorization. Moreover, we characterized all the factors from such a factorization.

Chapter 7 deals with indefinite polar decomposition of operators mapping between non-uniform Pontryagin spaces. Its importance is recognized in interpolation theory and robust control where, in the latter case, the decomposition is considered in the framework of Krein spaces. The key point in analysis of such operators has been made by characterizing the operators by standard polar decomposition (inner/outer factorization) and not by indefinite polar decomposition.

As a last remark, all results from the thesis have a purely algebraic flavor. This appears to be the straightforward consequence of results from realization theory which is mainly concerned about the algebra of projections. As it was the case in Chapter 7, the algebra of projections has provided better insights by avoiding long untractable expressions and led to clear results. However, in expressing closed formulas in original data it was often necessary to complete all algebraic manipulations on the data as we did in Chapter 5. This quest gave rise to long expressions but it was necessary in showing the one-pass character of the underlying mechanism.

8.2 Future research

In the following we enlist some problems which this thesis does not consider but are closely related to the obtained results.

Inversion

- (1) In Chapter 3 we dealt with inverses of stable operators and have assumed that the inverses are stable too. Evaluation of the inverses by means from realization theory naturally introduces block triangularization in the state transition operators for both the operator and its inverse. As the u.e.s property is captured in the entries positioned on the main diagonal of such a state transition matrix and the triangularization itself is induced by an appropriate boundedly invertible state space transformation, the question rises: How to determine this transformation in the case the inverse of a stable invertible operator is unbounded?
- (2) An one-pass evaluation of the inverse asks for two factorizations on the original (mixed causality) operator, an external and an inner/outer factorization. Algorithms for doing these factorizations are, in the thesis, formulated in a minimal (in the state-space sense) fashion. However, the state-space minimality does not guarantee the minimality in the number of flops needed for obtaining the realization for the inverse. Equivalently said, state-space minimality does not necessarily imply algebraic minimality of the involved calculations. The question is: Is it possible to merge some equations arising from external and inner/outer square-root schemes and to reduce the number of parameters involved?

Sylvester equation

In Chapter 5 we have studied the factorizability of mixed causality unitary operators. In many expressions from the chapter we encounter a time-varying Sylvester equations in Y_o and Y_c . Although the solvability of a time invariant Sylvester equation is well understood a little is known about its time-varying sibling. In the second half of Chapter 5 we derived a formula that relates the solutions of Sylvester equations and solutions of Lyapunov-Stein equations arising from a partitioned u.e.s realization for the unitary operator. The question is: given a Sylvester equation (5.18) from Theorem 5.6 in Y_o and specified by two u.e.s AC-pairs (A_c, C_c) and (A_a, C_a) , is it possible to find block diagonals B_c , B_a and D for which

$$\{A_c, B_c, C_c, A_a, B_a, C_a, D\}$$

is a minimal additive realization for the mixed causality unitary operator? If the answer is affirmative, then the equation (5.49) from Proposition 5.12 provides a stable way to evaluate Y_o .



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Index

- J*-orthogonal, 217
- J*-reachable, 220
- J*-unreachable, 220
- $J_{\mathcal{M}}$ -negative, 214, 216
- $J_{\mathcal{M}}$ -neutral, 214, 216
- $J_{\mathcal{M}}$ -orthogonal complement, 216
- $J_{\mathcal{M}}$ -positive, 214, 216

- additive realization, 8, 53, 81, 128, 138, 140, 150, 152, 167, 169, 173, 178, 181, 198
- adjoint, 16, 17, 28, 214, 215
- anti-causal, 5, 6, 8, 9, 17, 22, 23, 27–29, 39, 40, 56, 62, 63, 69, 70, 72, 75, 79, 86, 89, 91, 103, 106, 108, 119, 128, 140, 153, 161, 173–175, 177, 182, 183, 186, 187, 191, 194, 203, 205, 206, 220, 223
- anti-causal/anti-causal, 173
- anti-causal/causal, 146, 195, 201, 202
- anti-inner, 8, 37, 125, 126, 136, 162, 191, 193, 195, 206
- anti-inner/causal-coisometric, 139
- anti-inner/inner, 147, 168, 183, 202
- anti-inner/inner/outer, 126

- Banach, 14, 26
- Beurling-Lax, 34, 126
- block diagonal, 23, 29, 42, 46–49, 55, 74, 77, 83, 109, 117, 127, 181, 187, 188, 191, 192, 196, 199, 205
- block triangular, 59, 172, 176, 199

- canceling dynamics, 174
- Cauchy, 14
- Cauchy-Schwartz, 16
- causal, 5–10, 17, 21–23, 25–29, 34–39, 41, 42, 44, 45, 47, 50–52, 56, 59, 63, 71–73, 76, 77, 80, 81, 83, 86, 90–96, 99, 103, 105, 108–110, 113, 119, 121, 126, 147, 161, 164, 172–177, 181–184, 186, 187, 191, 194, 203, 206, 208–213, 215, 219–225, 228–231, 233, 235, 236
- causal-isometric/anti-inner, 136, 140, 144
- causal-*J*-coisometric/right-outer, 212
- causal/anti-causal, 8, 9, 119, 146, 172, 201, 202, 205, 208, 210
- causal/causal, 173
- Cholesky, 6, 44, 83, 88, 89, 99, 104, 118, 159, 210
- closed graph theorem, 215
- coisometric, 34, 35, 58, 79, 125, 127, 128, 134–136, 138, 139, 143, 145, 146, 149, 152, 160, 164, 168, 169, 205
- coisometric/right-outer, 34, 35, 212
- coisometry, 205, 206
- conjugation, 159, 160
- coprime, 126

- degree, 22, 35, 172, 183, 199, 205, 208, 209
- dichotomy, 69
- dimension sequence, 172

- eigenvalue decomposition, 40, 43
- external, 7, 8, 30, 34, 36, 37, 61–64, 67, 69–73, 85, 92, 103, 126, 161, 227

- fixed point, 26, 55, 84, 87, 89, 92, 94, 156, 188, 213
- Frobenius, 17

- Gramian, 27, 38, 39, 44, 68, 74, 77, 164, 175, 184, 193, 195, 205

- Hankel, 5, 6, 8, 20, 28, 30, 71, 121, 171, 222
- Hardy, 34
- Hausdorff, 14

- Hermitian, 25–27, 40, 43, 98, 104, 184, 185, 193, 196, 198, 205, 213
 Hilbert, 4, 5, 11, 15, 16, 18, 30, 168, 212–215, 217, 218, 221, 235
 Hilbert-Schmidt, 17, 18, 20, 215, 216

 indefinite, 211, 212, 214, 215, 221, 230, 232–237
 indefinite inner product, 212
 indefinite polar decomposition, 9, 211, 212
 inertia, 213, 221
 inner, 14–18, 30, 36, 37, 39–43, 45, 49, 51–54, 56, 63, 65, 67, 68, 172, 183–186, 191, 193, 195, 199, 202–204, 206, 208–210
 inner/anti-causal-isometric, 140, 145
 inner/anti-inner, 126, 147, 168, 202
 inner/outer, 7, 8, 10, 30, 34, 37, 39, 40, 51, 52, 55, 61, 67, 69, 74, 77, 81–83, 86, 87, 91, 92, 95, 96, 98, 99, 103, 108, 116, 119, 121, 126, 212, 222
 intrinsic, 9, 10, 45, 52, 54, 56, 58, 59, 62, 63, 67, 76, 89, 91, 95, 108–110, 172, 208, 210, 231, 237
 invariant-variant, 55
 inverse, 8–10, 33, 38–41, 45, 47, 49, 51, 52, 54, 60–63, 71–73, 76–78, 126, 171, 172, 178, 181, 182, 186, 190, 192, 194, 195, 197, 200, 202, 204, 210
 isometric, 85, 125, 127, 128, 132, 134, 136–141, 143–150, 160, 161, 168, 169
 isometric/anti-inner, 10, 125
 isomorphism, 125

 J-coisometric, 149
 J-external, 229
 J-Gram operator, 219
 J-Gramian, 228, 236
 J-isometric, 149, 212
 J-observability, 221
 J-orthogonal, 217, 223, 225
 J-reachability, 221
 J-unitary, 9, 172, 202, 208–210

 Kailath, 3
 Kalman, 27, 42, 43, 129, 132, 133, 135, 138, 174, 177, 180
 Krein, 212, 217, 235

 left-outer/causal-J-isometric, 212, 227, 228
 left-outer/inner, 237
 left-outer/isometric, 212, 228
 linear systems, 3
 local dimension, 213
 Lyapunov-Stein, 8–10, 26, 39, 41, 43, 47, 49, 60, 67, 68, 72, 75, 82, 89, 92, 94, 97, 98, 102, 103, 110, 118, 119, 126, 127, 137, 139, 146, 154, 156–158, 163, 169, 173, 177, 184, 187, 188, 193, 202, 211, 213, 219–221, 231

 maximal-phase, 8, 9, 39, 69–74, 76–78, 81–83, 85–88, 90, 100, 101, 103, 105, 108, 110, 115–119, 121, 178, 182, 183, 185–188, 202, 204, 206, 208, 212
 maximal-phase-star, 185, 186, 206
 maximal-phase-star/maximal-phase, 202
 maximal-phase/anti-inner, 119
 maximal-phase/inner, 7, 8
 maximal-phase/maximal-phase-star, 202, 208
 minimal, 6–10, 22, 24, 25, 30, 33–37, 39–45, 49, 51–53, 58–64, 66–69, 72–74, 76–78, 80–83, 85–91, 95–101, 103, 105–110, 115, 116, 119, 121, 125, 126, 136, 140, 145, 164, 168, 169, 171–174, 178, 179, 183–185, 187, 196, 201–203, 205–208, 210–212, 218, 221, 222, 227, 228, 230–232
 minimal factorization, 172
 minimal-phase, 55
 mixed causality, 6–10, 22, 24, 28, 30, 36, 37, 47, 49, 51–53, 63, 85, 125, 126, 128, 146, 172, 173, 182, 187, 202
 mixed partial fraction lemma, 29, 37, 52, 53, 73, 75, 79, 126, 173, 187
 Moore-Penrose, 49, 54, 66, 68, 96, 103–105, 117

 Neumann, 22
 neutral subspace, 217
 non-linear, 3, 4, 200
 non-uniform, 5, 11, 13, 19, 20, 30, 34, 127, 168, 172, 202, 212–215, 217
 nonempty, 217

- observability, 25, 29, 30, 34, 37–39, 44, 77, 172
- one-pass algorithm, 8–10, 34, 76, 147, 169
- outer, 20, 22, 35–37, 39, 40, 42, 45, 49, 51–53, 57, 62, 63, 67, 208
- outer/causal- (J_x, J_2) -unitary, 230
- outer/causal-J-factorization, 9
- outer/causal-J-isometric, 228
- outer/causal-J-unitary, 227, 228, 230
- outer/inner, 9, 34, 61, 62, 65, 70, 73, 74, 81–83, 85–88, 90, 91, 95, 98–100, 119, 121, 208, 210, 211, 227, 230, 231, 235, 236
- outer/isometric, 223, 224
- outer/J-inner, 235
- outer/unitary, 34
- partially observable, 129, 135
- partially reachable, 129, 135
- polar, 9
- poles, 3, 4, 36, 172
- Pontryagin, 9, 235
- pseudo-inverse, 49, 54, 103
- QR-decomposition, 7, 10, 26, 35, 37, 38, 40, 74, 77, 82, 83, 96, 97, 101, 104, 109, 111, 115, 126, 134, 138
- quasi-separable, 9, 53, 61
- reachability, 25, 29, 30, 34–36, 56, 62, 64, 172
- realization theory, 172
- representation, 212
- Riccati, 55, 56
- right outer, 34
- RQ-decomposition, 47–49
- s-dim sequences, 13, 24, 213, 214, 218
- Schur, 36, 78, 158
- semi-separable, 7, 9, 10, 21, 33, 55, 125, 168
- signature, 211, 213–215, 217, 218, 225, 231–233, 235
- Smith-McMillan, 7, 10
- spectral radius, 22
- square-root, 74–76, 81, 82, 96, 99–101, 103, 104, 108–110, 115, 116, 119, 120
- standard bases, 213
- state-space, 6–10, 20–24, 26, 28, 33, 39–42, 44, 45, 48–50, 53–55, 57, 61, 64–66, 73, 74, 77, 78, 81, 89, 100, 125, 127, 129, 130, 132–135, 137, 138, 141, 142, 147, 149, 150, 152, 153, 159, 161, 164, 165, 168, 169, 172, 174–178, 180, 188, 199, 203, 206, 210, 212
- subspaces, 21, 34, 213, 221, 224, 225
- SVD-decomposition, 57, 117, 209
- SVD-decompositions, 180
- Sylvester, 8, 126, 152, 159, 167, 169
- time-invariant, 3, 4, 6, 25, 26, 172, 211
- time-varying, 3–6, 8, 10, 11, 25, 30, 33, 55, 61, 62, 71, 110, 211
- Toeplitz, 5
- transfer function, 126
- UL-decomposition, 191–193, 195, 196, 206
- uniformly $J_{\mathcal{M}}$ -negative, 216
- uniformly $J_{\mathcal{M}}$ -positive, 216
- unitary operator, 17
- unitary embedding, 146, 147
- unitary operator, 147, 148, 159, 160, 167–169, 215
- URV-decomposition, 8, 10, 34, 53
- zeros, 3, 4, 172, 211, 213



Summary

Inversion and factorizations of operators specified by sequences of state-space equations form an important issue in both systems theory and control-engineering practice. Although the standard control literature is mainly concerned with transfer operators of single causality, consideration of mixed-causality operators frequently appears in fields such as the modeling with finite element methods and data mining arising from oil-winning technology. On the other side, simplification and better understanding of transfer operators take place after first decomposing them into products of simpler factors.

This thesis treats not only the inversion and the factorizations of such operators but also the numerical computability of involved algorithms. The latter issue becomes important when working with additive realizations since these often necessitate numerical recursions running in opposite time directions (increasing viz. decreasing indices). Recasting the recursions in a 'one-pass' fashion however is not always possible and depends on the mechanism underlying a given problem.

In this thesis, a stable one-pass algorithm for inverting a doubly infinite matrix given by an additive uniformly exponentially stable realization has been developed. The evaluation of the inverse based on a maximal-phase operator description was considered too and a closed formula for the inverse has been derived. Unitary embedding and minimal causal/anti-causal factorizations of isometric operators have been investigated as well and one-pass algorithms to compute them have been established. Connections between unitary extension and solutions of time-varying Sylvester equations have been pointed out and a closed formula relating solutions of Sylvester equations and Lyapunov-Stein equations corresponding to the data from unitary embedded isometric operator has been obtained.

As minimal realizations of minimal degree maximal-phase operators appear to have nice algebraic properties, they have been used in solving the minimal factorization problem of a mixed causality operators. In particular, a minimal causal/anti-causal factorization of a J-unitary matrix have been considered and necessary and sufficient conditions for the factorization have been stated. Finally, causal operators acting between Pontryagin spaces were considered because of their importance in interpolation theory and H_∞ -control. Necessary and sufficient conditions for an indefinite polar decomposition of such operators were stated. In addition, a stable one-pass algorithm

for obtaining the factors from an indefinite factorization of a left-invertible causal operator was given.

Samenvatting

Inversie en factorizatie van operatoren die gegeven zijn door middel van toestandsvergelijkingen vormen een belangrijk onderwerp in zowel de systeemtheorie als de regeltechniek, en inmiddels ook de theorie van gestructureerde stelsels vergelijkingen bekend als 'quasi-separabel'. Al bekommert de standaard regeltechnische literatuur zich hoofdzakelijk om causale overdrachtsoperatoren, komt het beschouwen van operatoren van gemengde causaliteit vaak voor in gebieden zoals het modeleren met de eindig elementen methode en de databewerking in de olie-winning technologie. Bovendien ontstaat beter begrip van overdrachtsoperatoren na decompositie in eenvoudigere factoren.

Deze thesis gaat niet alleen over de inversie en de factorizatie van gemengde operatoren, maar ook over hun numerische berekendbaarheid. Dit laatste wordt belangrijk als additieve realisaties gehanteerd worden, omdat dan meestal dubbele recursies noodzakelijk zijn - een voorwaartse en een achterwaartse. Het vertalen van de recursies in één richting is echter meestal niet mogelijk op een stabiele manier, zodat andere methoden vereist zijn.

In deze thesis wordt een stabiel één-richtingsalgoritme afgeleid voor de inversie van een dubbel oneindige matrix die gegeven is door uniforme, exponentieel stabiele realisaties. Een licht verschillende andere aanpak met een soortgelijk resultaat waarbij het berekenen van de inversie plaatsvindt door middel van maximaal-fase operatoren wordt ook onder de loep genomen en een gesloten formule voor de inverse wordt verkregen. Unitaire extensies en minimale factorizaties van isometrische operatoren van gemengde causaliteit worden ook aangepakt en een één-richtingsalgoritmen wordt daarvoor geformuleerd. Het verband tussen unitaire extensies en oplossingen van tijdsvariërende Sylvester vergelijkingen wordt vastgesteld en een gesloten formula gegeven die de oplossingen van Sylvester en Lyapunov-Stein vergelijkingen relateert - overigens in bijzondere omstandigheden.

Aangezien minimale realisaties van maximale fase operatoren van minimale graad mooie algebraïsche eigenschappen bezitten, worden zij in de thesis gebruikt voor het oplossen van het minimaal factorizatieprobleem van operatoren van gemengde causaliteit. In het bijzonder wordt een minimale causaal/anti-causale decompositie van een J -unitaire matrix beschouwd en worden nodige en voldoende voorwaarden voor het bestaan van de decompositie afgeleid. Als laatste hoofdstuk volgt een

studie van causale operatoren acterend tussen Pontryagin ruimten. Deze theorie heeft toepassingen in interpolatie theorie en robuuste controle. Nodige en voldoende voorwaarden voor het bestaan van een indefinite polaire decompositie van zo'n operator worden gegeven. In addendum wordt verder een stabiel één-richtingsalgoritme voor het verkrijgen van de factoren bij indefinite factorizatie van een links-inverteerbare causale operator gegeven.

About the Author

Edin Alijagić was born in Trebinje, Bosnia and Herzegovina, on Augustus 10, 1969. After receiving his secondary school diploma in 1989, he started his study of electrical engineering at Sarajevo University (SU), Bosnia and Herzegovina. Due to the Bosnian war 1992-1995 his studies were interrupted and, after moving to The Netherlands as a refuge, he continued them 1996 at Delft University of Technology (DUT). The topic of his master's thesis was in field of time-varying discrete linear systems and entitled as *Robust Control of Time-Varying Plants*. He graduated cum laude in December 2000. Attracted by the beauty of this topic, he joined Networks Systems section in January 2001 as a research assistant. There he was working towards his Ph.D. degree till April 2005 under the supervision of Prof. dr. ir. P.M. Dewilde.

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