The FDTD Method and Its Relation to Fibonacci Polynomials

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Abstract — **In this paper we show that the Finite-Difference Time-Domain method (FDTD method) follows the recurrence relation for Fibonacci polynomials. This observation allows us to easily derive the Courant-Friedrichs-Lewy stability condition by exploiting the connection between Fibonacci polynomials and Chebyshev polynomials of the second kind. In addition, we compare FDTD with the Spectral Lanczos Decomposition method (SLDM) and show that to capture the evolution of the fields in time, SLDM adjust itself to the spectrum of the system matrix, while FDTD takes only extremal eigenvalue information into account.**

1 INTRODUCTION

The Finite-Difference Time-Domain method (FDTD method, see [1] and [2] for example) is a very popular explicit time-stepping method for Maxwell's equations that hardly needs any introduction. It is based on the first-order Maxwell system and simultaneously solves for the electric and magnetic field strengths. The time evolution of the fields is captured by two coupled updating formulas that can be combined into a single recurrence relation involving the product of the discretized Maxwell operator $\mathbf{A}$ and the time step Δt. As it turns out, this recurrence relation is precisely the recurrence relation for Fibonacci polynomials. The FDTD field approximations are therefore Fibonacci polynomials in $\Delta t\mathbf{A}$ acting on the source vector of the problem at hand. Stability and dispersion of FDTD can also be analyzed in terms of these polynomials. We briefly discuss stability in this context and refer to [3] for further details. Finally, a comparison with Lanczos model-order reduction (SLDM, Spectral Lanczos Decomposition Method, [4] – [8]) shows that both solution methods are polynomial methods in the system matrix $\mathbf{A}$. The FDTD recursion generates Fibonacci polynomials, while SLDM generates so-called Lanczos polynomials to approximate the electromagnetic field quantities. Furthermore, SLDM adjusts itself to the spectrum of the system matrix $\mathbf{A}$, while FDTD takes only the spectral radius of the system matrix into account via the Courant-Friedrichs-Lewy (CFL) stability condition. There is no automatic spectral adjustment as in SLDM.

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2 BASIC EQUATIONS

After a standard spatial finite-difference discretization procedure on a Yee-grid (see, for example, [1] and [2]), we obtain the semidiscrete Maxwell system

$$(\mathbf{D} + \mathbf{M}\partial_t)\mathbf{f} = -w(t)\mathbf{q}. \tag{1}$$

In this equation, the field vector $\mathbf{f}$ is given by $\mathbf{f} = [\mathbf{e}^T, \mathbf{h}^T]^T$, where $\mathbf{e}$ and $\mathbf{h}$ contain all time-dependent finite-difference approximations of the electric and magnetic field strength, respectively. Furthermore, $\mathbf{q} = [(\mathbf{j}^{\text{ext}})^T, (\mathbf{k}^{\text{ext}})^T]^T$ is the source vector, where $\mathbf{j}^{\text{ext}}$ and $\mathbf{k}^{\text{ext}}$ are the external finite-difference electric and magnetic current density vectors. The scalar time-dependent function $w(t)$ is called the source wavelet or source signature.

The spatial differentiation matrix is given by

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{D}_{\text{h}} \\ \mathbf{D}_{\text{e}} & \mathbf{0} \end{pmatrix}, \tag{2}$$

where $\mathbf{D}_{\text{h}}$ is the discretized curl operator (including a minus sign) acting on the magnetic field strength and $\mathbf{D}_{\text{e}}$ is the discretized curl operator acting on the electric field strength. Matrix $\mathbf{D}$ is skew-symmetric with respect to a diagonal and positive definite step size matrix $\mathbf{W}$ (see [9] and [10]). More precisely, we have

$$\mathbf{D}^T\mathbf{W} = -\mathbf{W}\mathbf{D}. \tag{3}$$

Finally, the medium matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{\varepsilon} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mu} \end{pmatrix}, \tag{4}$$

where $\mathbf{M}_{\varepsilon}$ and $\mathbf{M}_{\mu}$ are diagonal and positive definite medium matrices with (averaged) permittivity and permeability values on the diagonal.

To obtain an explicit expression for the solution of Eq. (1), we premultiply this equation by the inverse of the medium matrix and obtain

$$\partial_t\mathbf{f} = \mathbf{A}\mathbf{f} - w(t)\mathbf{s}, \tag{5}$$

where $\mathbf{s} = \mathbf{M}^{-1}\mathbf{q}$ and where we have introduced the system matrix as $\mathbf{A} = -\mathbf{M}^{-1}\mathbf{D}$. The solution of Eq. (1) is now essentially given by a temporal

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convolution of source wavelet and the evolution operator $\exp(\mathbf{A}t)$.

The system matrix $\mathbf{A}$ is skew-symmetric with respect to the energy inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{W} \mathbf{M} \mathbf{x},$$

that is, it satisfies $\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = -\langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The reason for calling the above inner product an energy inner product is that $\frac{1}{2}\|\mathbf{f}\|^2 = \frac{1}{2}\langle \mathbf{f}, \mathbf{f} \rangle$ is a finite-difference approximation of the stored electromagnetic energy in the system.

Finally, we introduce the matrices $\mathbf{P}_\mathrm{e}$ and $\mathbf{P}_\mathrm{h}$ as

$$\mathbf{P}_\mathrm{e} = \begin{pmatrix} \mathbf{I}_{n_\mathrm{e}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{P}_\mathrm{h} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_\mathrm{h}} \end{pmatrix},$$

where n_e and n_h denote the total number of electric and magnetic field strength unknowns, respectively. We say that a vector $\mathbf{u}$ is of the electric-type if it satisfies $\mathbf{u} = \mathbf{P}_\mathrm{e}\mathbf{u}$ and a vector $\mathbf{u}$ is said to be of the magnetic-type if it satisfies $\mathbf{u} = \mathbf{P}_\mathrm{h}\mathbf{u}$. From this moment on, we consider electric-type source vectors only (no external magnetic current densities are present), since the analysis for magnetic-type source vectors runs along similar lines. A configuration in which both electric and magnetic external current densities are present can be handled using the superposition principle.

3 THE FINITE-DIFFERENCE TIME-DOMAIN METHOD AND FIBONACCI POLYNOMIALS

Introducing a time step $\Delta t > 0$ and the time instances $t_k = k\Delta t$, we obtain after a leap-frog time discretization of Eq. (1) the finite-difference time stepping equations

$$\mathbf{h}(t_{k+1/2}) = \mathbf{h}(t_{k-1/2}) - \Delta t \mathbf{M}_\mu^{-1} \mathbf{D}_\mathrm{e} \mathbf{e}(t_k) \tag{6}$$

and

$$\mathbf{e}(t_{k+1}) = \mathbf{e}(t_k) - \Delta t \mathbf{M}_\varepsilon^{-1} \mathbf{D}_\mathrm{h} \mathbf{h}(t_{k+1/2}). \tag{7}$$

Notice that the magnetic field strength computed in Eq. (6) is needed in the update equation for the electric field strength (Eq. (7)). The above time stepping equations are well known and can be found in any book on the FDTD method ([1] and [2], for example).

With the help of the FDTD vectors

$$\mathbf{g}_k = \begin{pmatrix} \mathbf{0} \\ \mathbf{h}(t_{\frac{k+1}{2}}) \end{pmatrix} \quad \text{for } k = 0, 2, 4, ... \tag{8}$$

and

$$\mathbf{g}_k = \begin{pmatrix} \mathbf{e}(t_{\frac{k+1}{2}}) \\ \mathbf{0} \end{pmatrix} \quad \text{for } k = 1, 3, 5, ..., \tag{9}$$

we can write the FDTD update equations as

$$\mathbf{g}_{k+1} = \Delta t \mathbf{A} \mathbf{g}_k + \mathbf{g}_{k-1}, \tag{10}$$

and comparing this with the recurrence formula for Fibonacci polynomials

$$F_{k+1}(x) = xF_k(x) + F_{k-1}(x), \tag{11}$$

with $F_0(x) = 0$ and $F_1(x) = 1$, we observe that

$$\mathbf{g}_k = F_k(\Delta t \mathbf{A})\mathbf{g}_1. \tag{12}$$

In other words, the FDTD vectors are Fibonacci polynomials in $\Delta t\mathbf{A}$ acting on the starting (source) vector $\mathbf{g}_1$. Note that $\deg(F_k) = k - 1$ and $F_k(1) = f_k$, where the f_k are the Fibonacci numbers.

Stability and dispersion of FDTD can be analyzed in terms Fibonacci polynomials. We briefly discuss stability and refer to [3] for further details.

Since matrix $\mathbf{A}$ is skew-symmetric with respect to the energy inner product, there exists a matrix $\mathbf{Q}$ such that

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\boldsymbol{\Lambda} \quad \text{with } \mathbf{Q}^H \mathbf{W}\mathbf{M}\mathbf{Q} = \mathbf{I}, \tag{13}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on the diagonal. Since these eigenvalues are all located on the imaginary axis, we also write $\boldsymbol{\Lambda} = \mathrm{i}\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma} = \mathrm{Im}(\boldsymbol{\Lambda})$. Now using the relation

$$F_k(2\mathrm{i}x) = \mathrm{i}^{k-1} U_{k-1}(x), \tag{14}$$

where $U_k(x)$ is the Chebyshev polynomial of the second kind and order k, we have

$$\mathbf{g}_k = \mathrm{i}^{k-1} \mathbf{Q} U_{k-1}\left(\frac{\Delta t}{2}\boldsymbol{\Sigma}\right) \mathbf{Q}^T \mathbf{W}\mathbf{M}\mathbf{g}_1. \tag{15}$$

The Chebyshev polynomials $U_k(x)$ remain bounded for $x \in (-1, 1)$ and become unbounded if $x \notin (-1, 1)$ as $k \to \infty$. Using this property, we conclude from Eq. (15) that the FDTD vectors remain bounded if and only if

$$\Delta t < \frac{2}{\rho(\mathbf{A})}, \tag{16}$$

where $\rho(\mathbf{A})$ is the spectral radius of matrix $\mathbf{A}$. Equation (16) is the famous CFL stability condition for FDTD [3].

4 THE SPECTRAL LANCZOS DECOMPOSITION METHOD

As mentioned above, the system matrix $\mathbf{A}$ is skew-symmetric with respect to the energy inner product and therefore Lanczos reduction for skew-symmetric matrices is possible. This reduction is carried out via the recursion

$$
\begin{aligned}
\mathbf{r}_{i+1} &= \mathbf{A}\mathbf{v}_i + \beta_i \mathbf{v}_{i-1}, \\
\beta_{i+1} &= \|\mathbf{r}_{i+1}\|, \\
\mathbf{v}_{i+1} &= \mathbf{r}_{i+1}/\beta_{i+1},
\end{aligned}
\tag{17}
$$

with $\mathbf{v}_0 = \mathbf{0}$, $\beta_1 = \|\mathbf{s}\|$, and $\mathbf{v}_1 = \mathbf{s}/\beta_1$ is the normalized source vector (normalized means having an energy norm equal to one). The vectors $\mathbf{v}_i$ are known as Lanczos vectors and the vectors $\mathbf{r}_i$ are often called residual vectors. Lanczos vectors are normalized residual vectors.

After m iterations of the above Lanczos algorithm, the Lanczos vectors $\mathbf{v}_i$ form an orthonormal basis (with respect to the energy inner product) of the Krylov space $\mathbb{K}_m = \text{span}\{\mathbf{s}, \mathbf{A}\mathbf{s}, ..., \mathbf{A}^{m-1}\mathbf{s}\}$. Electromagnetic field approximations are taken from this space. Explicitly, the mth order approximation is given by

$$
\mathbf{f}_m(t) = c_1(t)\mathbf{v}_1 + c_2(t)\mathbf{v}_2 + ... + c_m(t)\mathbf{v}_m = \mathbf{V}_m\mathbf{c}, \tag{18}
$$

where the Lanczos vectors form the columns of matrix $\mathbf{V}_m$ and $\mathbf{c} = [c_1, c_2, ..., c_m]^T$ is a vector of time-dependent expansion coefficients. Notice that

$$
\mathbf{v}_i = p_{i-1}(\mathbf{A})\mathbf{v}_1 = \beta_1 p_{i-1}(\mathbf{A})\mathbf{s}, \tag{19}
$$

where p_{i-1} is a so-called Lanczos polynomial of degree $i-1$. Furthermore, the expansion coefficients follow from a Galerkin procedure as

$$
\mathbf{c}(t) = -\beta_1 \int_{\tau=0}^{t} w(\tau) \exp[\mathbf{T}_m(t-\tau)]\,\mathrm{d}\tau\,\mathbf{e}_1, \tag{20}
$$

where $\mathbf{e}_1$ is the first column of the m-by-m identity matrix $\mathbf{I}_m$ and $\mathbf{T}_m$ is a tridiagonal and skew-symmetric matrix of order m containing the recurrence coefficients and is given by $\mathbf{T}_m = \text{tridiag}(\beta_i, 0, -\beta_{i+1})$. Substituting the expansion coefficients in Eq. (18), we obtain the mth order SLDM field approximation

$$
\mathbf{f}_m(t) = -\beta_1 \mathbf{V}_m \int_{\tau=0}^{t} w(\tau) \exp[\mathbf{T}_m(t-\tau)]\,\mathrm{d}\tau\,\mathbf{e}_1. \tag{21}
$$

Notice that the time coordinate is not discretized in SLDM and

$$
\mathbf{f}_m(t) = \tilde{p}_{m-1}(\mathbf{A})\mathbf{s}, \tag{22}
$$

where $\tilde{p}_{m-1}$ is a polynomial of degree $m-1$ with time-dependent coefficients.

5 RELATION BETWEEN THE FDTD METHOD AND SLDM

From Eqs. (12), (19), and (22) we observe that FDTD and SLDM are both polynomial time-integration methods in the system matrix $\mathbf{A}$. The FDTD method generates Fibonacci polynomials, while SLDM generates Lanczos polynomials to approximate the electromagnetic field. Now in both methods the polynomials act on the source vector and this vector is of the electric-type (as mentioned before, a magnetic-type current vector can be handled similarly). Furthermore, we also have $\mathbf{A}\mathbf{P}_\mathrm{e} = \mathbf{P}_\mathrm{h}\mathbf{A}$ and from the Lanczos recursion (17) it then follows that the residual vectors satisfy

$$
\mathbf{r}_i = \begin{cases} \text{of the electric-type if } i \text{ is odd,} \\ \text{of the magnetic-type if } i \text{ is even.} \end{cases}
$$

We make this explicit by introducing the vectors $\tilde{\mathbf{e}}_k$ and $\tilde{\mathbf{h}}_{k+1/2}$ through the relations

$$
\begin{pmatrix} \tilde{\mathbf{e}}_k \\ \mathbf{0} \end{pmatrix} = \mathbf{r}_{2k-1} \quad \text{and} \quad \begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{h}}_{k+1/2} \end{pmatrix} = \mathbf{r}_{2k}, \tag{23}
$$

for $k = 1, 2, \ldots$. Clearly, $\tilde{\mathbf{e}}_k$ equals the electric-part of an odd-numbered residual vector, while $\tilde{\mathbf{h}}_{k+1/2}$ equals the magnetic-part of an even numbered residual vector. Since the residual (and Lanczos) vectors have this particular form, we always carry our an even number of Lanczos iterations. The electric and magnetic field strengths are then equally updated.

In the Lanczos recursion, we now first take i odd. With $i = 2k-1$ we get in terms of the residual vectors

$$
\mathbf{r}_{2k} = \frac{1}{\beta_{2k-1}}\mathbf{A}\mathbf{r}_{2k-1} + \frac{\beta_{2k-1}}{\beta_{2k-2}}\mathbf{r}_{2k-2}. \tag{24}
$$

Second, we take i even. With $i = 2k$, we obtain

$$
\mathbf{r}_{2k+1} = \frac{1}{\beta_{2k}}\mathbf{A}\mathbf{r}_{2k} + \frac{\beta_{2k}}{\beta_{2k-1}}\mathbf{r}_{2k-1}. \tag{25}
$$

Using Eq. (23), these two equations can be written as

$$
\begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{h}}_{k+1/2} \end{pmatrix} = \frac{1}{\beta_{2k-1}}\mathbf{A}\begin{pmatrix} \tilde{\mathbf{e}}_k \\ \mathbf{0} \end{pmatrix} + \frac{\beta_{2k-1}}{\beta_{2k-2}}\begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{h}}_{k-1/2} \end{pmatrix} \tag{26}
$$

and

$$
\begin{pmatrix} \tilde{\mathbf{e}}_{k+1} \\ \mathbf{0} \end{pmatrix} = \frac{1}{\beta_{2k}}\mathbf{A}\begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{h}}_{k+1/2} \end{pmatrix} + \frac{\beta_{2k}}{\beta_{2k-1}}\begin{pmatrix} \tilde{\mathbf{e}}_k \\ \mathbf{0} \end{pmatrix}. \tag{27}
$$

Substituting the definition of the system matrix in the above expressions, the magnetic (bottom) part of Eq. (26) gives

$$\tilde{\mathbf{h}}_{k+1/2} = \frac{\beta_{2k-1}}{\beta_{2k-2}}\tilde{\mathbf{h}}_{k-1/2} - \frac{1}{\beta_{2k-1}}\mathbf{M}_\mu^{-1}\mathbf{D}_\mathrm{e}\tilde{\mathbf{e}}_k \quad (28)$$

and the electric (upper) part of Eq. (27) gives

$$\tilde{\mathbf{e}}_{k+1} = \frac{\beta_{2k}}{\beta_{2k-1}}\tilde{\mathbf{e}}_k - \frac{1}{\beta_{2k}}\mathbf{M}_\varepsilon^{-1}\mathbf{D}_\mathrm{h}\tilde{\mathbf{h}}_{k+1/2}. \quad (29)$$

Comparing these expressions with the FDTD update equations (6) and (7), we observe that the FDTD method and the Lanczos algorithm both use the same recurrence relation. Only the recurrence coefficients are different. Specifically, computing the residual vector of Eq. (17) using $\beta_i = 1/\Delta t$ for all i results in the FDTD method, while the Lanczos algorithm is obtained if we normalize the residual vectors by $\beta_i = \|\mathbf{r}_i\|$ at every iteration. To summarize, computing the residual vector $\mathbf{r}_i$ with

$$\beta_i = \begin{cases} 1/\Delta t & \text{results in the FDTD algorithm,} \\ \|\mathbf{r}_i\| & \text{results in the Lanczos algorithm.} \end{cases}$$

Since the Lanczos recurrence coefficients are contained in the tridiagonal matrix $\mathbf{T}_m$ and since this matrix approximates the eigenvalues of the system matrix $\mathbf{A}$, we observe that SLDM automatically adjusts itself to the spectrum of the system matrix. In FDTD, however, only extremal eigenvalue information is taken into account via the CFL stability condition. Also note that if the coefficients β_i in the Lanczos algorithm approach a constant $\beta > 0$ then the residual vectors of the Lanczos algorithm turn into the field vectors of FDTD with time step $1/\beta$.

6 CONCLUSIONS

The FDTD method and SLDM are both polynomial method in the system matrix $\mathbf{A}$. We have shown that FDTD generates Fibonacci polynomials, while SLDM generates Lanczos polynomials to approximate the electromagnetic field. In the latter method there is no time step selection (the time coordinate remains continuous) and SLDM adjusts itself automatically to the spectrum of the system matrix when time integrating Maxwell's equations. In contrast, FDTD takes only extremal eigenvalue information into account via the CFL stability condition. This condition can be obtained by exploiting the connection between Fibonacci polynomials and Chebyshev polynomials of the second kind.

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References

[1] A. Taflove and S. C. Hagness, *Computational Electromagnetics: The Finite-Difference Time-Domain Method*, 3rd ed. Norwood, MA: Artech House, 2005.

[2] D. B. Davidson, *Computational Electromagnetics for RF and Microwave Engineering*, Cambridge, UK: Cambridge University Press, 2005.

[3] R. F. Remis, "On the relation between FDTD and Fibonacci polynomials," *J. Comput. Phys.*, vol. 230, no. 4, pp. 1382 – 1386, 2011.

[4] V. L. Druskin and L. A. Knizhnerman, "Two polynomial methods for calculating functions of symmetric matrices," *U.S.S.R. Comput. Math. Math. Phys.*, vol. 29, pp. 112 – 121, 1989.

[5] R. F. Remis and P. M. van den Berg, "A modified Lanczos algorithm for the computation of transient electromagnetic wavefields," *IEEE Trans. Microwave Theory Tech.*, vol. 45, pp. 2139 – 2149, Dec. 1997.

[6] R. F. Remis, "Low-frequency model-order reduction of electromagnetic fields without matrix factorization," *IEEE Trans. Microwave Theory Tech.*, vol. 52, no. 9, pp. 2298 – 2304, Sept. 2004.

[7] A. C. Cangellaris and L. Zhao, "Rapid FDTD simulation without time stepping," *IEEE Microw. and Guided W.*, vol. 9, no. 1, pp. 4 – 6, 1999.

[8] A. C. Cangellaris, M. Celik, S. Pasha, and L. Zhao, "Electromagnetic model order reduction for system-level modeling," *IEEE Trans. Microwave Theory Tech.*, vol. 47, pp. 840 – 850, June 1999.

[9] B. Denecker, L. Knockaert, F. Olyslager, and D. De Zutter, "A new state-space-based algorithm to assess the stability of the finite-difference time-domain method for 3D finite inhomogeneous problems," *AEU-Int. J. Electron. C.*, vol. 58, pp. 339 – 348, 2004.

[10] R. F. Remis, "Global form of the 2d finite-difference maxwell system and its symmetry properties," in Proc. 2002 IEEE Antennas and Propagation Symp., San Antonio Texas, pp. 624 – 627, 2002.